Fair Division

Whether it is two kids sharing a candy bar or a couple splitting assets during a divorce, there are times in life where items of value need to be divided between two or more parties. While some cases can be handled through mutual agreement or mediation, in others the parties are adversarial or cannot reach a decision all feel is fair. In these cases, fair division methods can be utilized.

Fair Division Method

A fair division method is a procedure that can be followed that will result in a division of items in a way so that each party feels they have received their fair share. For these methods to work, we have to make a few assumptions:

- The parties are non-cooperative, so the method must operate without communication between the parties.
- The parties have no knowledge of what the other players like (their valuations).
- The parties act rationally, meaning they act in their best interest, and do not make emotional decisions.
- The method should allow the parties to make a fair division without requiring an outside arbitrator or other intervention.

With these methods, each party will be entitled to some **fair share**. When there are N parties equally dividing something, that fair share would be $1/N$. For example, if there were 4 parties, each would be entitled to a fair share of ¼ = 25% of the whole. More specifically, they are entitled to a share that *they* value as 25% of the whole.

Fair Share

When N parties divide something equally, each party's **fair share** is the amount they entitled to. As a fraction, it will be $1/N$

It should be noted that a fair division method simply needs to guarantee that each party will receive a share they view as fair. A basic fair division does not need to be envy free; an **envy-free** division is one in which no party would prefer another party's share over their own. A basic fair division also does not need to be Pareto optimal; a **Pareto optimal** division is one in which no other division would make a participant better off without making someone else worse off. Nor does fair division have to be **equitable**; an equitable division is one in which the proportion of the whole each party receives, judged by their own valuation, is the same. Basically, a simple fair division doesn't have to be the best possible division – it just has to give each party their fair share.

Example 1

Suppose that 4 classmates are splitting equally a $12 pizza that is half pepperoni, half veggie that someone else bought them. What is each person's fair share?

Since they all are splitting the pizza equally, each person's fair share is $3, or pieces they value as 25% of the pizza.

It is important to keep in mind that each party might value portions of the whole differently. For example, a vegetarian would probably put zero value on the pepperoni half of the pizza.

Example 2

Suppose that 4 classmates are splitting equally a \$12 pizza that is half pepperoni, half veggie. Steve likes pepperoni twice as much as veggie. Describe a fair share for Steve.

He would value the veggie half as being worth \$4 and the pepperoni half as \$8, twice as much. If the pizza was divided up into 4 pepperoni slices and 4 veggie slices, he would value a pepperoni slice as being worth \$2, and a veggie slice as being worth \$1.

If we weren't able to guess the values, we could take a more algebraic approach. If Steve values a veggie slice as worth x dollars, then he'd value a pepperoni slice as worth $2x$ dollars – twice as much. Four veggie slices would be worth $4 \cdot x = 4x$ dollars, and 4 pepperoni slices would be worth $4 \cdot 2x = 8x$ dollars. Altogether, the eight slices would be worth $4x + 8x = 12x$ dollars. Since the total value of the pizza was \$12, then $12x = \$12$. Solving we get $x = \$1$; the value of a veggie slice is \$1, and the value of a pepperoni slice is $2x = \$2$.

A fair share for Steve would be one pepperoni slice and one veggie slice (\$2 + \$1 = \$3 value), 1½ pepperoni slices (1½ · \$2 = \$3 value), 3 veggie slices (3 · \$1 = \$3 value), or a variety of more complicated possibilities.

Try it Now 1

Suppose Kim is another classmate splitting the pizza, but Kim is vegetarian, so won't eat pepperoni. Describe a fair share for Kim.

You will find that many examples and exercises in this topic involve dividing food – dividing candy, cutting cakes, sharing pizza, etc. This may make this topic seem somewhat trivial, but instead of cutting a cake, we might be drawing borders dividing Germany after WWII. Instead of splitting a bag of candy, siblings might be dividing belongings from an inheritance. Mathematicians often characterize very important and contentious issues in terms of simple items like cake to separate any emotional influences from the mathematical method.

Because of this, our requirement that the players not communicate about their preferences can seem silly. After all, why wouldn't four classmates talk about what kind of pizza they like if they're splitting a pizza? Just remember that in issues of politics, business, finance, divorce settlements, etc. the players are usually less cooperative and more concerned about the other players trying to get more than their fair share.

There are two broad classifications of fair division methods: those that apply to **continuously divisible** items, and those that apply to **discretely divisible** items. Continuously divisible items are things that can be divided into pieces of any size, like dividing a candy bar into two pieces or drawing borders to split a piece of land into smaller plots. Discretely divisible items are when you are dividing several items that cannot be broken apart easily, such as assets in a divorce (house, car, furniture, etc).

Divider-Chooser

The first method we will look at is a method for continuously divisible items. This method will be familiar to many parents - it is the "You cut, I choose" method. In this method, one party is designated the **divider** and the other the **chooser**, perhaps with a coin toss. The method works as follows:

Divider-Chooser Method

1. The divider cuts the item into two pieces that are, in his eyes, equal in value.
2. The chooser selects either of the two pieces
3. The divider receives the remaining piece

Notice that the divider-chooser method is specific to a two-party division. Examine why this method guarantees a fair division: since the divider doesn't know which piece he will receive, the rational action for him to take would be to divide the whole into two pieces he values equally. There is no incentive for the divider to attempt to "cheat" since he doesn't know which piece he will receive. Since the chooser can pick either piece, she is guaranteed that one of them is worth at least 50% of the whole in her eyes. The chooser is guaranteed a piece she values as at least 50%, and the divider is guaranteed a piece he values at 50%.

Example 3

Two retirees, Fred and Martha, buy a vacation beach house in Florida together, with the agreement that they will split the year into two parts.

Fred is chosen to be the divider, and splits the year into two pieces: November – February and March – October. Even though the first piece is 4 months and the second is 8 months, Fred places equal value on both pieces since he really likes to be in Florida during the winter. Martha gets to pick whichever piece she values more. Suppose she values all months equally. In this case, she would choose the March – October time, resulting in a piece that she values as 8/12 = 66.7% of the whole. Fred is left with the November – February slot which he values as 50% of the whole.

Of course, in this example, Fred and Martha probably could have discussed their preferences and reached a mutually agreeable decision. The divider-chooser method is more necessary in cases where the parties are suspicious of each other's motives, or are unable to communicate effectively, such as two countries drawing a border, or two children splitting a candy bar.

Try it Now 2

Dustin and Quinn were given an apple pie and a chocolate cake, and need to divide them. Dustin values the apple pie at $6 and the chocolate cake at $4. Quinn values the apple pie as $4 and the chocolate cake at $10. Describe a fair division if Quinn is dividing, and specify which "half" Dustin will choose.

Things quickly become more complicated when we have more than two parties involved. We will look at three different approaches. But first, let us look at one that doesn't work.

How <u>not</u> to divide with 3 parties

When first approaching the question of 3-party fair division, it is very tempting to propose this method: Randomly designate one participant to be the divider, and designate the rest choosers. Proceed as follows:

1) Have the divider divide the item into 3 pieces
2) Have the first chooser select any of the three pieces they feel is worth a fair share
3) Have the second chooser select either of the remaining pieces
4) The divider gets the piece left.

Example 4. Don't do this – it is bad!

Suppose we have three people splitting a cake. We can immediately see that the divider will receive a fair share as long as they cut the cake fairly at the beginning. The first chooser certainly will also receive a fair share. What about the second chooser? Suppose each person values the three pieces like this:

	Piece 1	**Piece 2**	**Piece 3**
Chooser 1	40%	30%	30%
Chooser 2	45%	30%	25%
Divider	33.3%	33.3%	33.3%

Since the first chooser will clearly select Piece 1, the second chooser is left to select between Piece 2 and Piece 3, neither of which she values as a fair share (1/3 or about 33.3%). This example shows that this method does *not* guarantee a fair division.

To handle division with 3 or more parties, we'll have to take a more clever approach.

Lone Divider

The lone divider method works for any number of parties – we will use *N* for the number of parties. One participant is randomly designated the divider, and the rest of the participants are designated as choosers.

Lone Divider Method

The Lone Divider method proceeds as follows:

1) The divider divides the item into N pieces, which we'll label s_1, s_2, …, s_N.
2) Each of the choosers will separately list which pieces they consider to be a fair share. This is called their **declaration**, or **bid**.
3) The lists are examined. There are two possibilities:
    a. If it is possible to give each party a piece they declared then do so, and the divider gets the remaining piece.
    b. If two or more parties both want the same pieces and no others, then give a non-contested piece to the divider. The rest of the pieces are combined and repeat the entire procedure with the remaining parties. If there are only two parties left, they can use divider-chooser.

Example 5

Consider the example from earlier, in which the pieces were valued as:

	Piece 1	Piece 2	Piece 3
Chooser 1	40%	30%	30%
Chooser 2	45%	30%	25%
Divider	33.3%	33.3%	33.3%

Each chooser makes a declaration of which pieces they value as a fair share. In this case,

Chooser 1 would make the declaration: Piece 1
Chooser 2 would make the declaration: Piece 1

Since both choosers want the same piece, we cannot immediately allocate the pieces. The lone divider method specifies that we give a non-contested piece to the divider. Both pieces 2 and 3 are uncontested, so we flip a coin and give Piece 2 to the divider. Piece 1 and 3 are then recombined to make a piece worth 70% to Chooser 1, and 70% to Chooser 2. Since there are only two players left, they can divide the recombined pieces using divider-chooser. Each is guaranteed a piece they value as at least 35%, which is a fair share.

Try it Now 3

Use the Lone Divider method to complete the fair division given the values below.

	Piece 1	Piece 2	Piece 3
Chooser 1	40%	30%	30%
Chooser 2	40%	35%	25%
Divider	33.3%	33.3%	33.3%

Example 6

Suppose that Abby, Brian, Chris, and Dorian are dividing a plot of land. Dorian was selected to be the divider through a coin toss. Each person's valuation of each piece is shown below.

	Piece 1	Piece 2	Piece 3	Piece 4
Abby	15%	30%	20%	35%
Brian	30%	35%	10%	25%
Chris	20%	45%	20%	15%
Dorian	25%	25%	25%	25%

Based on this, their declarations should be:
Abby: Piece 2, Piece 4
Brian: Piece 1, Piece 2, Piece 4
Chris: Piece 2

This case can be settled simply – by awarding Piece 2 to Chris, Piece 4 to Abby, Piece 1 to Brian, and Piece 3 to Dorian. Each person receives a piece that they value as at least a fair share (25% value).

Example 7

Suppose the valuations in the previous problem were:

	Piece 1	Piece 2	Piece 3	Piece 4
Abby	15%	30%	20%	35%
Brian	20%	35%	10%	35%
Chris	20%	45%	20%	15%
Dorian	25%	25%	25%	25%

The declarations would be:
Abby: Piece 2, Piece 4
Brian: Piece 2, Piece 4
Chris: Piece 2

Notice in this case that there is no simple settlement. So, the piece no one else declared, Piece 3, is awarded to the original divider Dorian, and the procedure is repeated with the remaining three players.

Suppose that on the second round of this method Brian is selected to be the divider, three new pieces are cut, and the valuations are as follows:

	Piece 1	Piece 2	Piece 3
Abby	40%	30%	30%
Brian	33.3%	33.3%	33.3%
Chris	50%	20%	30%

The declarations here would be:
Abby: Piece 1
Chris: Piece 1

Once again we have a standoff. Brian can be awarded either of Piece 2 or Piece 3, and the remaining pieces can be recombined. Since there are only two players left, they can divide the remaining land using the basic divider-chooser method.

Try it Now 4

Four investors are dividing a piece of land valued at \$320,000. One was chosen as the divider, and their values of the division (in thousands) are shown below. Who was the divider? Describe the outcome of the division.

	Piece 1	Piece 2	Piece 3	Piece 4
Sonya	\$90	\$70	\$80	\$80
Cesar	\$80	\$80	\$80	\$80
Adrianna	\$60	\$70	\$100	\$90
Raquel	\$70	\$50	\$90	\$110

Last Diminisher

The Last Diminisher method is another approach to division among 3 or more parties.

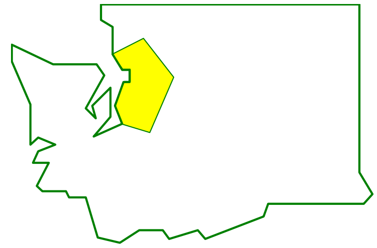
Last Diminisher Method

In this method, the parties are randomly assigned an order, perhaps by pulling names out of a hat. The method then proceeds as follows:

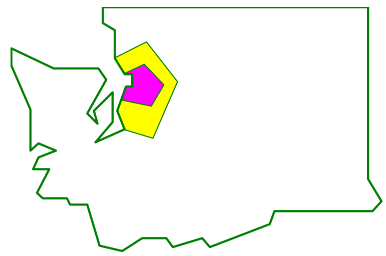
1) The first person cuts a slice they value as a fair share.
2) The second person examines the piece
    a. If they think it is worth less than a fair share, they then pass on the piece unchanged.
    b. If they think the piece is worth more than a fair share, they trim off the excess and lay claim to the piece. The trimmings are added back into the to-be-divided pile.
3) Each remaining person, in turn, can either pass or trim the piece
4) After the last person has made their decision, the last person to trim the slice receives it. If no one has modified the slice, then the person who cut it receives it.
5) Whoever receives the piece leaves with their piece and the process repeats with the remaining people. Continue until only 2 people remain; they can divide what is left with the divider-chooser method.

Example 8

Suppose that four salespeople are dividing up Washington State into sales regions; each will get one region to work in. They pull names from a hat to decide play order.

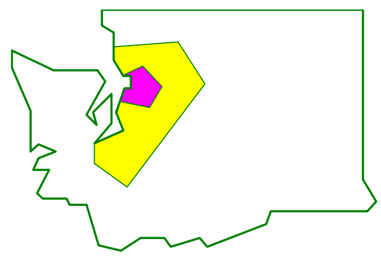


Round 1. The first salesman, Bob, draws a region around Seattle, the most populous area of the state. The piece Bob cuts and automatically lays claim to is shown in yellow.

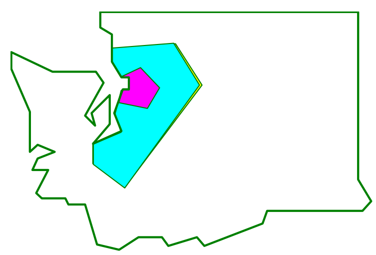


The second salesman, Henry, felt that this region was worth more than 25%, each player's fair share. Because of this, Henry opts to trim this piece. The new piece is shown in pink. The trimmings (in yellow) return to the to-be-divided portion of the state. Henry automatically lays claim to this smaller piece since he trimmed it.

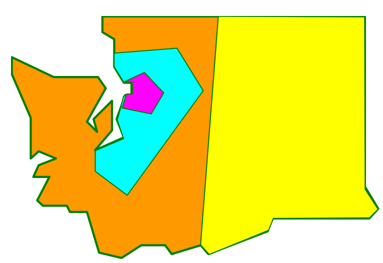
The third saleswoman, Marjo, feels this piece is worth less than 25% and passes, as does the fourth saleswoman, Beth. Since both pass, the last person to trim it, Henry, receives the piece.



Round 2. The second round begins with Bob laying claim to a piece, shown again in yellow. Henry already has a piece, so is out of the process now. Marjo passes on this piece, feeling it is worth less than a fair share.



Beth, on the other hand, feels the piece as currently drawn is worth 35%. Beth is in an advantageous position, being the last to make a decision. Even though Beth values this piece at 35%, she can cut a very small amount and still lay claim to it. So Beth barely cuts the piece, resulting in a piece (blue) that is essentially worth 35% to her. Since she is the last to trim, she receives the piece.



Round 3. At this point, Bob and Marjo are the only players without a piece. Since there are two of them, they can finish the division using the divider-chooser method. They flip a coin, and determine that Marjo will be the divider. Marjo draws a line dividing the remainder of the state into two pieces. Bob chooses the Eastern piece, leaving Marjo with the Western half.

Notice that in this division, Henry and Marjo ended up with pieces that they feel are worth exactly 25% - a fair share. Beth was able to receive a piece she values as more than a fair share, and Bob may feel the piece he received is worth more than 25%.

Example 9

Marcus, Abby, Julian, and Ben are splitting a pizza that is 4 slices of cheese and 4 slices of veggie with total value \$12. Marcus and Ben like both flavors equally, Abby only likes cheese, and Julian likes veggie twice as much as cheese. They divide the pizza using last diminisher method, playing in the order Marcus, Abby, Ben, then Julian.

Notice Ben and Marcus both value any slice of pizza at \$1.50.
Abby values each slice of cheese at \$3, and veggie at \$0.
Julian values each slice of cheese at \$1, and each slice of veggie at \$2. (see Example 2)
A fair share for any player is \$3.

In the first round, suppose Marcus cuts out 2 slices of cheese, which he values at \$3.
Abby only likes cheese, so will value this cut at \$6. She will trim it to 1 slice of cheese, which she values as her fair share of \$3.
Ben will view this piece as less than a fair share, and will pass.
Julian will view this piece as less than a fair share, and will pass.
Abby receives the piece.

In the second round, suppose Marcus cuts a slice that is 2 slices of veggie.
Abby already received a slice so is out.
Ben will view this piece as having value \$3. He can barely trim it and lay claim to it.
Julian will value this piece as having value \$4, so will barely trim it and claim it.

Marcus and Ben can then split the remaining 3 slices of cheese and 2 slices of veggie using the divider-chooser method.

In the second round, both Ben and Julian will make tiny trims (pulling off a small crumb) in order to lay claim to the piece without practically reducing the value. The piece Julian receives is still essentially worth \$4 to him; we don't worry about the value of that crumb.

Try it Now 5

Five players are dividing a \$20 cake. In the first round, Player 1 makes the initial cut and claims the piece. For each of the remaining players, the value of the current piece (which may have been trimmed) at the time it is their turn is shown below. Describe the outcome of the first round.

	P_2	**P_3**	**P_4**	**P_5**
Value of the current piece	\$3	\$5	\$3.50	\$3

In the second round, Player 1 again makes the initial cut and claims the piece, and the current values are shown again. Describe the outcome of the second round.

	P_2	**P_4**	**P_5**
Value of the current piece	\$7	\$3	\$5

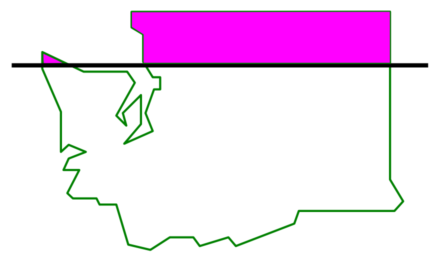
Moving Knife

A somewhat different approach to continuous fair division is called the moving knife procedure.

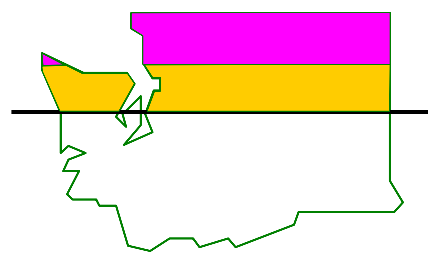
> **Moving Knife Method**
> In this method, applied to a cake,
> 1) A referee starts moving a knife from left to right across a cake.
> 2) As soon as any player feels the piece to the left of the knife is worth a fair share, they shout "STOP." The referee then cuts the cake at the current knife position and the player who called stop gets the piece to the left of the knife.
> 3) This procedure continues until there is only one player left. The player left gets the remaining cake.

Example 10

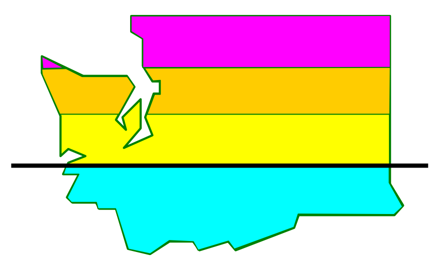
Suppose that our four salespeople from above decided to use this approach to divide Washington. Rather than move the "knife" from left to right, they decide to move it from top to bottom.



The referee starts moving a line down a map of the state. Henry is the first to call STOP when the knife is at the position shown, giving him the portion of the state above the line.



Marjo is the next to call STOP when the knife is at the position shown, giving her the second portion of the state.



Bob is the next to call STOP, leaving Beth with the southernmost portion of the state

While this method guarantees a fair division, it clearly results in some potentially silly divisions in a case like this. The method is probably better suited to situations like dividing an actual cake.

Sealed Bids Method

The Sealed Bids method provides a method for *discrete* fair division, allowing for the division of items that cannot be split into smaller pieces, like a house or a car. Because of this, the method requires that all parties have a large amount of cash at their disposal to balance out the difference in item values.

Sealed Bids Method

The method begins by compiling a list of items to be divided. Then:

1) Each party involved lists, in secret, a dollar amount they value each item to be worth. This is their sealed bid.
2) The bids are collected. For each party, the value of all the items is totaled, and divided by the number of parties. This defines their fair share.
3) Each item is awarded to the highest bidder.
4) For each party, the value of all items received is totaled. If the value is more than that party's fair share, they pay the difference into a holding pile. If the value is less than that party's fair share, they receive the difference from the holding pile. This ends the initial allocation.
5) In most cases, there will be a surplus, or leftover money, in the holding pile. The surplus is divided evenly between all the players. This produces the final allocation.

While the assumptions we made for fair division methods specified that an arbitrator should not be necessary, it is common for an independent third party to collect the bids and announce the outcome. While not technically necessary, since the method can be executed without a third party involved, this protects the secrecy of the bids, which can sometimes help avoid resentment or bad feelings between the players.

Example 11

Sam and Omar have cohabitated for the last 3 years, during which time they shared the expense of purchasing several items for their home. Sam has accepted a job in another city, and now they find themselves needing to divide their shared assets.

Each records their value of each item, as shown below.

	Sam	**Omar**
Couch	\$150	\$100
TV	\$200	\$250
Video game system	\$250	\$150
Surround sound system	\$50	\$100

Sam's total valuation of the items is \$150+\$200+\$250+\$50 = \$650, making a fair share for Sam \$650/2 = \$325.

Omar's total valuation of the items is $100+$250+$150+$100 = $600, making a fair share for Omar $600/2 = $300.

Each item is now awarded to the highest bidder. Sam will receive the couch and video game system, providing $150+$250 = $400 of value to Sam. Since this exceeds his fair share, he has to pay the difference, $75, into a holding pile.

Omar will receive the TV and surround sound system, providing $250+$100=$350 in value. This is more than his fair share, so he has to pay the difference, $50, into the holding pile.

Thus, in the initial allocation, Sam receives the couch and video game system and pays $75 into the holding pile. Omar receives the TV and surround sound system and pays $50 into the holding pile. At this point, both players would feel they have received a fair share.

There is now $125 remaining in the holding pile. This is the surplus from the division. This is now split evenly, and both Sam and Omar are given back $62.50. Since Sam had paid in $75, the net effect is that he paid $12.50. Since Omar had originally paid in $50, the net effect is that he receives $12.50.

Thus, in the final allocation, Sam receives the couch and video game system and pays $12.50 to Omar. Omar receives the TV and surround sound system and receives $12.50. At this point, both players feel they have received more than a fair share.

Example 12

Four small graphic design companies are merging operations to become one larger corporation. In this merger, there are a number of issues that need to be settled. Each company is asked to place a monetary value (in thousands of dollars) on each issue:

	Super Designs	**DesignByMe**	**LayoutPros**	**Graphix**
Company name	$5	$3	$3	$6
Company location	$8	$9	$7	$6
CEO	$10	$5	$6	$7
Chair of the board	$7	$6	$6	$8

We can then calculate for each company:

	Super Designs	**DesignByMe**	**LayoutPros**	**Graphix**
Total value of all issues	$30	$23	$22	$27
Fair share	$7.50	$5.75	$5.50	$6.75

The items would then be allocated to the company that bid the most for each.
Company name would be awarded to Graphix
Company location would be awarded to DesignByMe
CEO would be awarded to Super Designs
Chair of the board would be awarded to Graphix

For each company, we calculate the total value of the items they receive, and how much they get or pay in the initial allocation

	Super Designs	DesignByMe	LayoutPros	Graphix
Total value of issues awarded	$10	$9	$0	$14
Amount they pay/get	$7.50-$10 Pay $2.50	$5.75 - $9 Pay $3.25	$5.50 - $0 Get $5.50	$6.75 - $14 Pay $7.25

After the initial allocation, there is a total of $2.50+$3.25-$5.50+$7.25 = $7.50 in surplus. Dividing that evenly, each company gets $1.875 (approximately)

	Super Designs	DesignByMe	LayoutPros	Graphix
Amount they pay/get	Pay $2.50 Get $1.875	Pay $3.25 Get $1.875	Get $5.50 Get $1.875	Pay $7.25 Get $1.875
After surplus	Pay $0.625	Pay $1.375	Get $7.375	Pay $5.375

So in the final allocation,
Super Designs wins the CEO, and pays $625 ($0.625 thousand)
DesignByMe wins the company location and pays $1,375 ($1.375 thousand)
LayoutPros wins no issues, but receives $7,375 in compensation
Graphix wins the company name and chair of the board, and pays $5,375.

Try it Now 6

Jamal, Maggie, and Kendra are dividing an estate consisting of a house, a vacation home, and a small business. Their valuations (in thousands) are shown below. Determine the final allocation.

	Jamal	Maggie	Kendra
House	$250	$300	$280
Vacation home	$170	$180	$200
Small business	$300	$255	$270

Example 13

Fair division does not always have to be used for items of value. It can also be used to divide undesirable items. Suppose Chelsea and Mariah are sharing an apartment, and need to split the chores for the household. They list the chores, assigning a *negative* dollar value to each item; in other words, the amount they would pay for someone else to do the chore (a per week amount). We will assume, however, that they are committed to doing all the chores themselves and not hiring a maid.

	Chelsea	Mariah
Vacuuming	-\$10	-\$8
Cleaning bathroom	-\$14	-\$20
Doing dishes	-\$4	-\$6
Dusting	-\$6	-\$4

We can then calculate fair share:

	Chelsea	Mariah
Total value	-\$34	-\$38
Fair Share	-\$17	-\$19

We award to the person with the largest bid. For example, we award vacuuming to Mariah since she dislikes it less (remember -8 > -10).
Chelsea gets cleaning the bathroom and doing dishes. Value: -\$18
Mariah gets vacuuming and dusting. Value: -\$12

Notice that Chelsea's fair share is -\$17 but she is doing chores she values at -\$18. She should get \$1 to bring her to a fair share. Mariah is doing chores valued at -\$12, but her fair share is -\$19. She needs to pay \$7 to bring her to a fair share.

This creates a surplus of \$6, which will be divided between the two. In the final allocation:
Chelsea gets cleaning the bathroom and doing dishes, and receives \$1 + \$3 = \$4/week.
Mariah gets vacuuming and dusting, and pays \$7 - \$3 = \$4/week.

Try it Now Answers

1. Kim will value the veggie half of the pizza at the full value, \$12, and the pepperoni half as worth \$0. Since a fair share is 25%, a fair share for Kim is one slice of veggie, which she'll value at \$3. Of course, Kim only getting one slice doesn't really seem very fair, but if every player had the same valuation as Kim, this would be the only fair outcome. Luckily, if the classmates splitting the pizza are friends, they are probably cooperative, and will talk about what kind of pizza they like.

2. There are a lot of possible fair divisions Quinn could make. Since she values the two desserts at \$14 together, a fair share in her eyes is \$7. Notice since Dustin values the desserts at \$10 together, a fair share in his eyes is \$5 of value. A couple possible divisions:

Piece 1	Piece 2	Dustin would choose
7/10 chocolate (\$7) No pie (\$0)	3/10 chocolate (\$3) All pie (\$4)	Piece 2. Value in his eyes: \$6 + 3/10·\$4 = \$7.20
½ chocolate (\$5) ½ pie (\$2)	½ chocolate (\$5) ½ pie (\$2)	Either. Value in his eyes: ½ ·\$6 + ½ ·\$4 = \$5

3. Chooser 1 would make the declaration: Piece 1
Chooser 2 would make the declaration: Piece 1, Piece 2
We can immediately allocate the pieces, giving Piece 2 to Chooser 2, Piece 1 to Chooser 1, and Piece 3 to the Divider. All players receive a piece they value as a fair share.

Try it Now Answers Continued

4. A fair share would be $80,000. Cesar was the divider.
Their declarations would be:
Sonya: Pierce 1, Piece 3, Piece 4.
Adrianna: Piece 3, Piece 4
Raquel: Piece 3, Piece 4

Sonya would receive Piece 1 and Cesar would receive the uncontested Piece 2.
Since Adrianna and Raquel both want Piece 3 or Piece 4, a coin would be flipped to allocate those pieces between them.

5. In the first round, Player 1 will cut a piece he values as a fair share of $4. Player 2 values the piece as $3, so will pass. Player 3 values the piece as $5, so will claim it and trim it to something she values as $4. Player 4 receives that piece and values it as $3.50 so will pass. Player 5 values the piece at $3 and will also pass. Player 3 receives the trimmed piece she values at $4.

In the second round, Player 1 will again cut a piece he values as a fair share of $4. Player 2 values the piece as $7, so will claim it and trim it to something he values as $4. Player 4 values the trimmed piece at $3 and passes. Player 5 values the piece at $5, so will claim it. Since Player 5 is the last player, she has an advantage and can claim then barely trim the piece. Player 5 receives a piece she values at $5.

6. Jamal's total value is $250 + $170 + $300 = $720. His fair share is $240 thousand.
Maggie's total value is $300 + $180 + $255 = $735. Her fair share is $245 thousand.
Kendra's total value is$280 + $200 + $270 = $750. Her fair share is $250 thousand.

In the initial allocation,
Jamal receives the business, and pays $300 - $240 = $60 thousand into holding.
Maggie receives the house, and pays $300 - $245 = $55 thousand into holding.
Kendra receives the vacation home, and gets $250 - $200 = $50 thousand from holding.

There is a surplus of $60 + $55 - $50 = $65 thousand in holding, so each person will receive $21,667 from surplus. In the final allocation,
Jamal receives the business, and pays $38,333.
Maggie receives the house, and pays $33,333.
Kendra receives the vacation home, and gets $71,667.

Exercises

Skills

1. Chance and Brianna buy a pizza for $10 that is half pepperoni and half veggie. They cut the pizza into 8 slices.

   If Chance likes veggie three times as much as pepperoni, what is the value of a slice that is half pepperoni, half veggie?

2. Ahmed and Tiana buy a cake for $14 that is half chocolate and half vanilla. They cut the cake into 8 slices.

   If Ahmed likes chocolate four times as much as vanilla, what is the value of a slice that is half chocolate, half vanilla?

3. Erin, Catherine, and Shannon are dividing a large bag of candy. They randomly split the bag into three bowls. The values of the entire bag and each of the three bowls in the eyes of each of the players are shown below. For each player, identify which bowls they value as a fair share.

	Whole Bag	**Bowl 1**	**Bowl 2**	**Bowl 3**
Erin	$5	$2.75	$1.25	$1.00
Catherine	$4	$0.75	$2.50	$0.75
Shannon	$8	$1.75	$2.25	$4.00

4. Jenna, Tatiana, and Nina are dividing a large bag of candy. They randomly split the bag into three bowls. The values of the entire bag and each of the three bowls in the eyes of each of the players are shown below. For each player, identify which bowls they value as a fair share.

	Whole Bag	**Bowl 1**	**Bowl 2**	**Bowl 3**
Jenna	$8	$4.50	$0.75	$2.75
Tatiana	$4	$1.00	$1.00	$2.00
Nina	$6	$1.75	$2.50	$1.75

5. Dustin and Kendra want to split a bag of fun-sized candy, and decide to use the divider-chooser method. The bag contains 100 Snickers, 100 Milky Ways, and 100 Reese's, which Dustin values at $1, $5, and $2 respectively. (This means Dustin values the 100 Snickers together at $1, or $0.01 for 1 Snickers).

   If Kendra is the divider, and in one half puts:
   25 Snickers, 20 Milky Ways, and 60 Reese's
   a. What is the value of this half in Dustin's eyes?
   b. Does Dustin consider this a fair share?
   c. If Dustin was a divider, find a possible division that is consistent with his value system.

6. Dustin and Kendra want to split a bag of fun-sized candy, and decide to use the divider-chooser method. The bag contains 100 Snickers, 100 Milky Ways, and 100 Reese's, which Dustin values at $1, $3, and $5 respectively. (This means Dustin values the 100 Snickers together at $1, or $0.01 for 1 Snickers).

   If Kendra is the divider, and in one half puts:
   30 Snickers, 40 Milky Ways, and 66 Reese's
   a. What is the value of this half in Dustin's eyes?
   b. Does Dustin consider this a fair share?
   c. If Dustin was a divider, find a possible division that is consistent with his value system.

7. Maggie, Meredith, Holly, and Zoe are dividing a piece of land using the lone-divider method. The values of the four pieces of land in the eyes of the each player are shown below.

	Piece 1	Piece 2	Piece 3	Piece 4
Maggie	21%	27%	32%	20%
Meredith	27%	29%	22%	22%
Holly	23%	14%	41%	22%
Zoe	25%	25%	25%	25%

   a. Who was the divider?
   b. If playing honestly, what will each player’s declaration be?
   c. Find the final division.

8. Cody, Justin, Ahmed, and Mark are going to share a vacation property. The year will be divided into 4 time slots using the lone-divider method. The values of each time slot in the eyes of the each player are shown below.

	Time 1	Time 2	Time 3	Time 4
Cody	10%	35%	34%	21%
Justin	25%	25%	25%	25%
Ahmed	19%	24%	30%	27%
Mark	23%	31%	22%	24%

   a. Who was the divider?
   b. If playing honestly, what will each player’s declaration be?
   c. Find the final division.

9. A 6-foot sub valued at $30 is divided among five players (P_1, P_2, P_3, P_4, P_5) using the last-diminisher method. The players play in a fixed order, with P_1 first, P_2 second, and so on. In round 1, P_1 makes the first cut and makes a claim on a piece. For each of the remaining players, the value of the *current* claimed piece at the time it is their turn is given in the following table:

	P_2	P_3	P_4	P_5
Value of the current claimed piece	$6.00	$8.00	$7.00	$6.50

a. Which player gets his or her share at the end of round 1?
b. What is the value of the share to the player receiving it?
c. How would your answer change if the values were:

	P_2	P_3	P_4	P_5
Value of the current claimed piece	$6.00	$8.00	$7.00	$4.50

10. A huge collection of low-value baseball cards appraised at $100 is being divided by 5 kids (P_1, P_2, P_3, P_4, P_5) using the last-diminisher method. The players play in a fixed order, with P_1 first, P_2 second, and so on. In round 1, P_1 makes the first selection and makes a claim on a pile of cards. For each of the remaining players, the value of the *current* pile of cards at the time it is their turn is given in the following table:

	P_2	P_3	P_4	P_5
Value of the current pile of cards	$15.00	$22.00	$18.00	$19.00

a. Which player gets his or her share at the end of round 1?
b. What is the value of the share to the player receiving it?
c. How would your answer change if the values were:

	P_2	P_3	P_4	P_5
Value of the current pile of cards	$15.00	$22.00	$18.00	$21.00

11. Four heirs (A, B, C, and D) must fairly divide an estate consisting of two items - a desk and a vanity - using the method of sealed bids. The players' bids (in dollars) are:

	A	**B**	**C**	**D**
Desk	320	240	300	260
Vanity	220	140	200	180

a. What is A's fair share?
b. Find the initial allocation.
c. Find the final allocation.

12. Three heirs (A, B, C) must fairly divide an estate consisting of three items - a house, a car, and a coin collection - using the method of sealed bids. The players' bids (in dollars) are:

	A	B	C
House	180,000	210,000	220,000
Car	12,000	10,000	8,000
Coins	3,000	6,000	2,000

a. What is A's fair share?
b. Find the initial allocation.
c. Find the final allocation.

13. As part of an inheritance, four children, Abby, Ben and Carla, are dividing four vehicles using Sealed Bids. Their bids (in thousands of dollars) for each item is shown below. Find the final allocation.

	Abby	Ben	Carla
Motorcycle	10	9	8
Car	10	11	9
Tractor	4	1	2
Boat	7	6	4

14. As part of an inheritance, four children, Abby, Ben, Carla, and Dan, are dividing four vehicles using Sealed Bids. Their bids (in thousands of dollars) for each item is shown below. Find the final allocation.

	Abby	Ben	Carla	Dan
Motorcycle	6	7	11	8
Car	8	13	10	11
Tractor	3	1	5	4
Boat	7	6	3	8

15. After living together for a year, Sasha and Megan have decided to go their separate ways. They have several items they bought together to divide, as well as some moving-out chores. The values each place are shown below. Find the final allocation.

	Sasha	Megan
Couch	120	80
TV	200	250
Stereo	40	50
Detail cleaning	-40	-60
Cleaning carpets	-50	-40

16. After living together for a year, Emily, Kayla, and Kendra have decided to go their separate ways. They have several items they bought together to divide, as well as some moving-out chores. The values each place are shown below. Find the final allocation.

	Emily	Kayla	Kendra
Dishes	20	30	40
Vacuum cleaner	100	120	80
Dining table	100	80	130
Detail cleaning	-70	-40	-50
Cleaning carpets	-30	-60	-50

Exploration

17. This question explores how bidding dishonestly can end up hurting the cheater. Four partners are dividing a million-dollar property using the lone-divider method. Using a map, Danny divides the property into four parcels s_1, s_2, s_3, and s_4. The following table shows the value of the four parcels in the eyes of each partner (in thousands of dollars):

	s_1	s_2	s_3	s_4
Danny	\$250	\$250	\$250	\$250
Brianna	\$460	\$180	\$200	\$160
Carlos	\$260	\$310	\$220	\$210
Greedy	\$330	\$300	\$270	\$100

a. Assuming all players bid honestly, which piece will Greedy receive?
b. Assume Brianna and Carlos bid honestly, but Greedy decides to bid only for s1, figuring that doing so will get him s1. In this case there is a standoff between Brianna and Greedy. Since Danny and Carlos are not part of the standoff, they can receive their fair shares. Suppose Danny gets s3 and Carlos gets s2, and the remaining pieces are put back together and Brianna and Greedy will split them using the basic divider-chooser method. If Greedy gets selected to be the divider, what will be the value of the piece he receives?
c. Extension: Create a Sealed Bids scenario that shows that sometimes a player can successfully cheat and increase the value they receive by increasing their bid on an item, but if they increase it too much, they could end up receiving less than their fair share.

18. Explain why divider-chooser method with two players will always result in an envy-free division.

19. Will the lone divider method always result in an envy-free division? If not, will it ever result in an envy-free division?

20. The Selfridge-Conway method is an envy-free division method for three players. Research how the method works and prepare a demonstration for the class.

21. Suppose that two people are dividing a \$12 pizza that is half pepperoni, half cheese. Steve likes both equally, but Maria likes cheese twice as much as pepperoni. As divider, Steve divides the pizza so that one piece is 1/3 cheese and 2/3 pepperoni, and the second piece is 1/3 pepperoni and 2/3 cheese.
    a. Describe the value of each piece to each player
    b. Since the value to each player is not the same, this division is not equitable. Find a division that would be equitable. Is it still envy-free?
    c. The original division is not Pareto optimal. To show this, find another division that would increase the value to one player without decreasing the value to the other player. Is this division still envy-free?
    d. Would it be possible with this set of preferences to find a division that is both equitable and Pareto optimal? If so, find it. If not, explain why.

22. Is the Sealed Bids method Pareto optimal when used with two players? If not, can you adjust the method to be so?

23. Is the Sealed Bids method envy-free when used with two players? If not, can you adjust the method to be so?

24. Is the Sealed Bids method equitable when used with two players? If not, can you adjust the method to be so?

25. All the problems we have looked at in this chapter have assumed that all participants receive an equal share of what is being divided. Often, this does not occur in real life. Suppose Fred and Maria are going to divide a cake using the divider-chooser method. However, Fred is only entitled to 30% of the cake, and Maria is entitled to 70% of the cake (maybe it was a \$10 cake, and Fred put in \$3 and Maria put in \$7). Adapt the divider-choose method to allow them to divide the cake fairly.

    Assume (as we have throughout this chapter) that different parts of the cake may have different values to Fred and Maria, and that they don't communicate their preferences/values with each other. You goal is to come up with a method of fair division, meaning that although the participants may not receive equal shares, they should be *guaranteed* their fair share. Your method needs to be designed so that each person will always be guaranteed a share that they value as being worth at least as much as they're entitled to.

The last few questions will be based on the Adjusted Winner method, described here:
For discrete division between two players, there is a method called Adjusted Winner that produces an outcome that is always equitable, envy-free, and Pareto optimal. It does, however, require that items can be split or shared. The method works like this:

1) Each player gets 100 points that they assign to the items to be divided based on their relative worth.
2) In the initial allocation, each item is given to the party that assigned it more points. If there were any items with both parties assigned the same number of points, they'd go to the person with the fewest points so far.
3) If the assigned point values are not equal, then begin transferring items from the person with more points to the person with fewer points. Start with the items that have the smallest point ratio, calculated as (higher valuation/lower valuation).
4) If transferring an entire item would move too many points, only transfer a portion of the item.

Example: A couple is attempting to settle a contentious divorce[1]. They assign their 100 points to the issues in contention:

	Mike	**Carol**
Custody of children	25	65
Alimony	60	25
House	15	10

In the initial allocation, Mike gets his way on alimony and house, and Carol gets custody of the children. In the initial allocation, Mike has 75 points and Carol has 65 points. To decide what to transfer, we calculate the point ratios.

	Mike	**Carol**	**Point ratio**
Custody of children	25	65	65/25 = 2.6
Alimony	60	25	60/25 = 2.4
House	15	10	15/10 = 1.5

Since the house has the smallest point ratio, the house will be the item we work with first. Since transferring the entire house would give Carol too many points, we instead need to transfer some fraction, p, of the house to that Carol and Mike will end up with the same point values. If Carol receives a fraction p of the house, then Mike will give up $(1-p)$ of the house. The value Carol will receive is $10p$: the fraction p of the 10 points Carol values the house at. The value Mike will get is $15(1-p)$. We set their point totals equal to solve for p:

$65 + 10p = 60 + 15(1-p)$
$65 + 10p = 60 + 15 - 15p$
$25p = 10$
$p = 10/25 = 0.4 = 40\%$. So Carol should receive 40% of the house.

[1] From *Negotiating to Settlement in Divorce*, 1987

26. Apply the Adjusted Winner method to settle a divorce where the couples have assigned the point values below

	Sandra	Kenny
Home	20	30
Summer home	15	10
Retirement account	50	40
Investments	10	10
Other	5	10

27. In 1974, the United States and Panama negotiated over US involvement and interests in the Panama Canal. Suppose that these were the issues and point values assigned by each side[2]. Apply the Adjusted Winner method.

	United States	Panama
US defense rights	22	9
Use rights	22	15
Land and water	15	15
Expansion rights	14	3
Duration	11	15
Expansion routes	6	5
Jurisdiction	2	7
US military rights	2	7
Defense role of Panama	2	13

[2] Taken from *The Art and Science of Negotiation*, 1982