

Sets

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

- a) The set of all even numbers
- b) The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

- a) $\{1, 3, 9, 12\}$
- b) $\{\text{red, orange, yellow, green, blue, indigo, purple}\}$

A set simply specifies the contents; order is not important. The set represented by $\{1, 2, 3\}$ is equivalent to the set $\{3, 1, 2\}$.

Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol \in means “is an element of”.

A set that contains no elements, $\{ \}$, is called the **empty set** and is notated \emptyset

Example 2

Let $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Subset

A **subset** of a set A is another set that contains only elements from the set A , but may not contain all the elements of A .

If B is a subset of A , we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If B is a proper subset of A , we write $B \subset A$

Example 3

Consider these three sets

$A =$ the set of all even numbers $B = \{2, 4, 6\}$ $C = \{2, 3, 4, 6\}$

Here $B \subset A$ since every element of B is also an even number, so is an element of A .

More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.

It is also true that $B \subset C$.

C is not a subset of A , since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the plays “Much Ado About Nothing”, “MacBeth”, and “A Midsummer’s Night Dream”. What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Try it Now 1

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).

The union is notated $A \cup B$.

More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated $A \cap B$.

More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$

The **complement** of a set A contains everything that is *not* in the set A .

The complement is notated A' , or A^c , or sometimes $\sim A$.

Example 5

Consider the sets: $A = \{\text{red, green, blue}\}$ $B = \{\text{red, yellow, orange}\}$
 $C = \{\text{red, orange, yellow, green, blue, purple}\}$

a) Find $A \cup B$

The union contains all the elements in either set: $A \cup B = \{\text{red, green, blue, yellow, orange}\}$
 Notice we only list red once.

b) Find $A \cap B$

The intersection contains all the elements in both sets: $A \cap B = \{\text{red}\}$

c) Find $A^c \cap C$

Here we're looking for all the elements that are *not* in set A and are also in C .
 $A^c \cap C = \{\text{orange, yellow, purple}\}$

Try it Now 2

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

Notice that in the example above, it would be hard to just ask for A^c , since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A .

Example 6

- a) If we were discussing searching for books, the universal set might be all the books in the library.
- b) If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
- c) If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7

Suppose the universal set is $U =$ all whole numbers from 1 to 9. If $A = \{1, 2, 4\}$, then $A^c = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8

Suppose $H = \{\text{cat, dog, rabbit, mouse}\}$, $F = \{\text{dog, cow, duck, pig, rabbit}\}$
 $W = \{\text{duck, rabbit, deer, frog, mouse}\}$

- a) Find $(H \cap F) \cup W$

We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$

Now we union that result with W : $(H \cap F) \cup W = \{\text{dog, duck, rabbit, deer, frog, mouse}\}$

- b) Find $H \cap (F \cup W)$

We start with the union: $F \cup W = \{\text{dog, cow, rabbit, duck, pig, deer, frog, mouse}\}$

Now we intersect that result with H : $H \cap (F \cup W) = \{\text{dog, rabbit, mouse}\}$

- c) Find $(H \cap F)^c \cap W$

We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$

Now we want to find the elements of W that are *not* in $H \cap F$

$(H \cap F)^c \cap W = \{\text{duck, deer, frog, mouse}\}$

Venn Diagrams

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

Venn Diagram

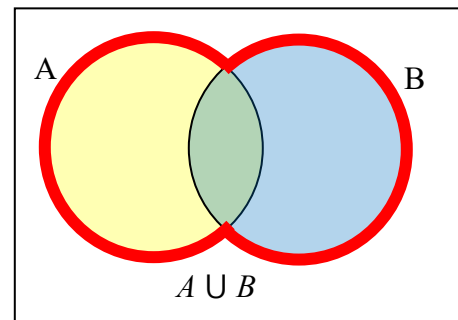
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

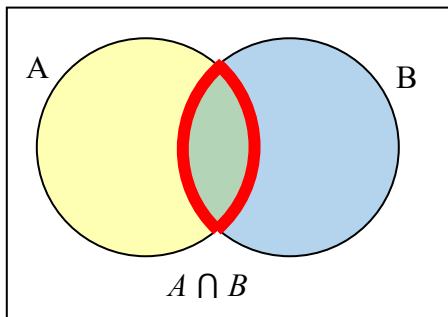
Example 9

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

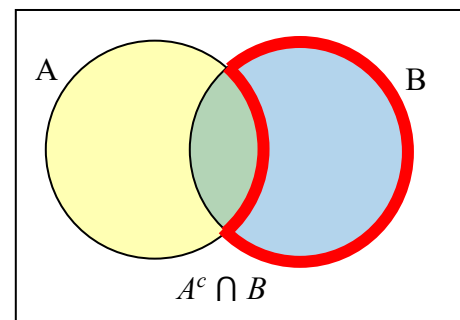
$A \cup B$ contains all elements in *either* set.



$A \cap B$ contains only those elements in both sets – in the overlap of the circles.



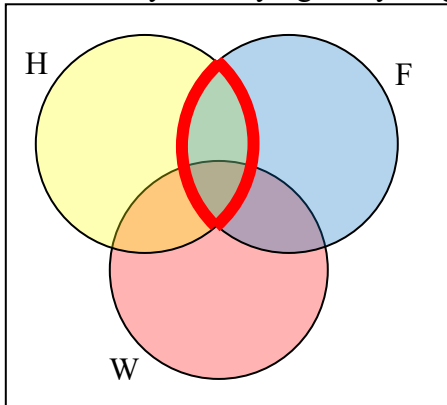
A^c will contain all elements *not* in the set A . $A^c \cap B$ will contain the elements in set B that are not in set A .



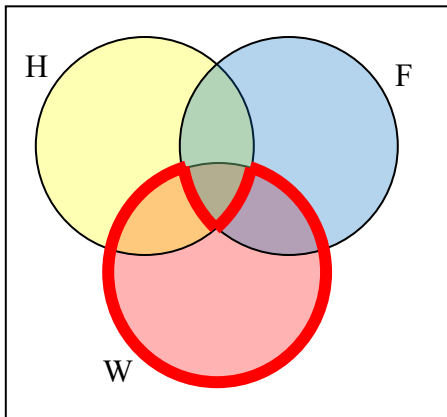
Example 10

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

We'll start by identifying everything in the set $H \cap F$



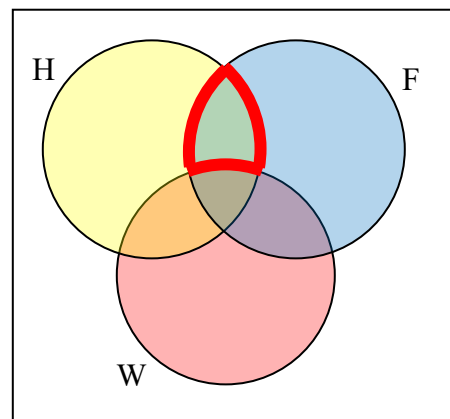
Now, $(H \cap F)^c \cap W$ will contain everything *not* in the set identified above that is also in set W .



Example 11

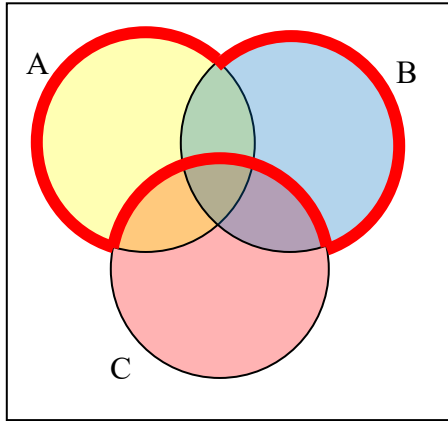
Create an expression to represent the outlined part of the Venn diagram shown.

The elements in the outlined set *are* in sets H and F , but are not in set W . So we could represent this set as $H \cap F \cap W^c$



Try it Now 3

Create an expression to represent the outlined portion of the Venn diagram shown

**Cardinality**

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Cardinality

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or $n(A)$

Example 12

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

What is the cardinality of B ? $A \cup B$, $A \cap B$?

The cardinality of B is 4, since there are 4 elements in the set.

The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.

The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 13

What is the cardinality of $P =$ the set of English names for the months of the year?

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

Example 14

A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:

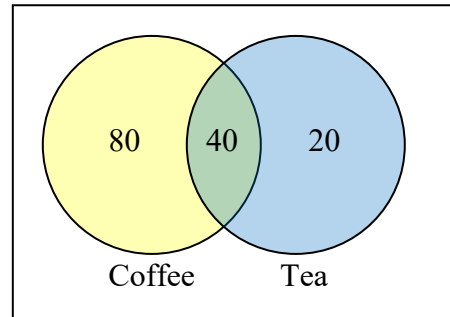
- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$200 - 20 - 80 - 40 = 60$ people who drink neither.



Example 15

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply $70\% + 40\%$, since that would count those who use both services twice. To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T , then subtract those in intersection that we've counted twice. In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$

$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$. Since the universal set contains 100% of people and the cardinality of $F \cup T = 90\%$, the cardinality of $(F \cup T)^c$ must be the other 10%.

The previous example illustrated two important properties

Cardinality properties

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A^c) = n(U) - n(A)$$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Example 16

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course

26 were taking a HM course

19 were taking a NS course

9 were taking SS and HM

7 were taking SS and NS

10 were taking HM and NS

3 were taking all three

7 were taking none

How many students are only taking a SS course?

It might help to look at a Venn diagram.

From the given data, we know that there are 3 students in region e and 7 students in region h .

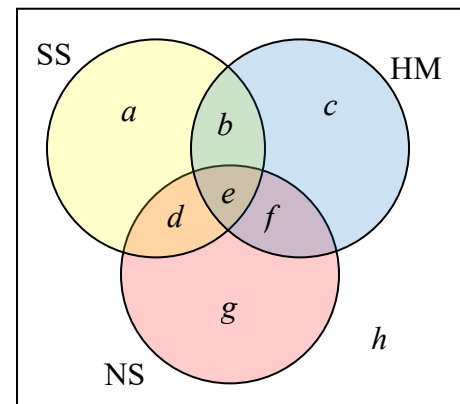
Since 7 students were taking a SS and NS course, we know that $n(d) + n(e) = 7$. Since we know there are 3 students in region e , there must be $7 - 3 = 4$ students in region d .

Similarly, since there are 10 students taking HM and NS, which includes regions e and f , there must be $10 - 3 = 7$ students in region f .

Since 9 students were taking SS and HM, there must be $9 - 3 = 6$ students in region b .

Now, we know that 21 students were taking a SS course. This includes students from regions a , b , d , and e . Since we know the number of students in all but region a , we can determine that $21 - 6 - 4 - 3 = 8$ students are in region a .

8 students are taking only a SS course.



Try it Now 4

One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs	44 believed in ghosts
25 believed in Bigfoot	10 believed in UFOs and ghosts
8 believed in ghosts and Bigfoot	5 believed in UFOs and Bigfoot
2 believed in all three	

How many people surveyed believed in at least one of these things?

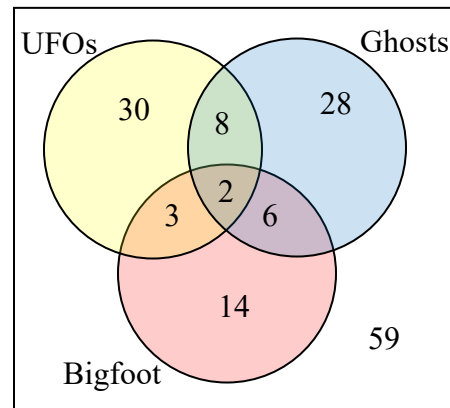
Try it Now Answers

1. There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

2. $A \cup C = \{\text{red, orange, yellow, green, blue, purple}\}$
 $B^c \cap A = \{\text{green, blue}\}$

3. $A \cup B \cap C^c$

4. Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves $150 - 91 = 59$ who believe in none.



Exercises

- List out the elements of the set “The letters of the word Mississippi”
- List out the elements of the set “Months of the year”
- Write a verbal description of the set $\{3, 6, 9\}$
- Write a verbal description of the set $\{a, i, e, o, u\}$
- Is $\{1, 3, 5\}$ a subset of the set of odd integers?
- Is $\{A, B, C\}$ a subset of the set of letters of the alphabet?

For problems 7-12, consider the sets below, and indicate if each statement is true or false.
 $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$ $U = \{\text{numbers from 0 to 10}\}$

7. $3 \in B$ 8. $5 \in C$ 9. $B \subset A$ 10. $C \subset A$ 11. $C \subset B$ 12. $C \subset U$

Using the sets from above, and treating U as the Universal set, find each of the following:

13. $A \cup B$ 14. $A \cup C$ 15. $A \cap C$ 16. $B \cap C$ 17. A^c 18. B^c

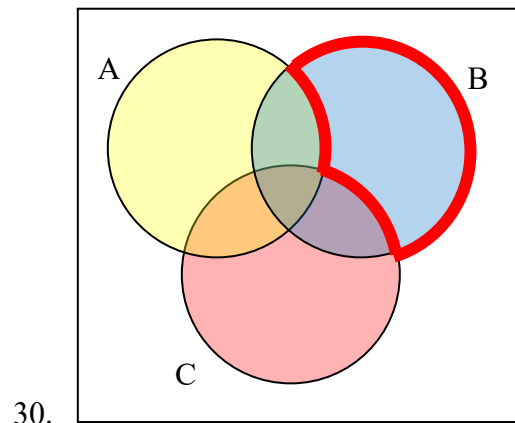
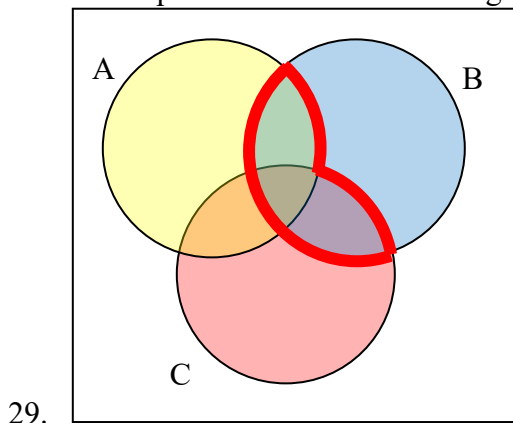
Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:

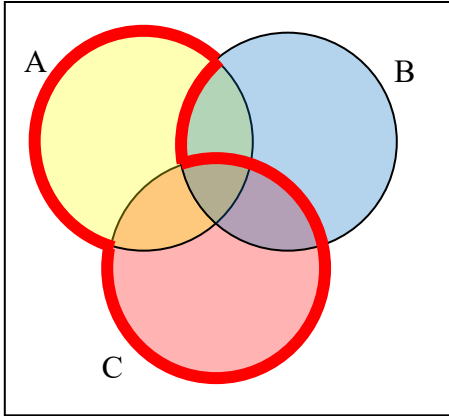
19. $D^c \cap E$ 20. $F^c \cap D$ 21. $(D \cap E) \cup F$ 22. $D \cap (E \cup F)$
 23. $(F \cap E)^c \cap D$ 24. $(D \cup E)^c \cap F$

Create a Venn diagram to illustrate each of the following:

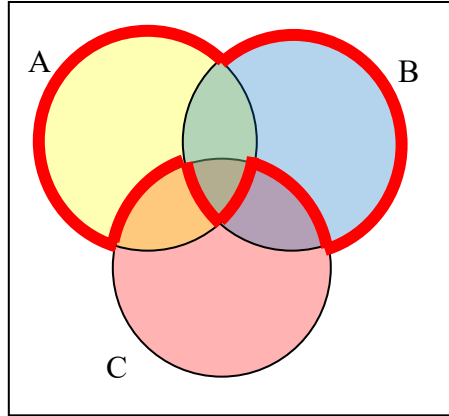
25. $(F \cap E) \cup D$ 26. $(D \cup E)^c \cap F$
 27. $(F^c \cap E^c) \cap D$ 28. $(D \cup E) \cup F$

Write an expression for the shaded region.





31.



32.

Let $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$. Find the cardinality of the given set.

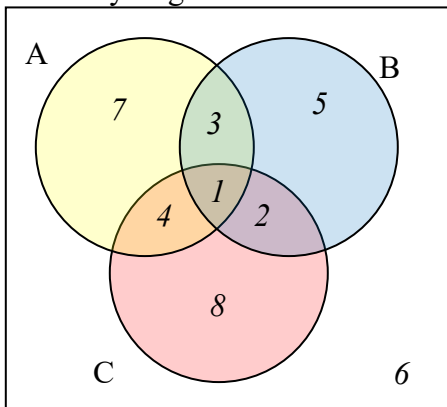
33. $n(A)$

34. $n(B)$

35. $n(A \cup C)$

36. $n(A \cap C)$

The Venn diagram here shows the cardinality of each set. Use this in 37-40 to find the cardinality of given set.



37. $n(A \cap C)$ 38. $n(B \cup C)$ 39. $n(A \cap B \cap C^c)$ 40. $n(A \cap B^c \cap C)$

41. If $n(G) = 20$, $n(H) = 30$, $n(G \cap H) = 5$, find $n(G \cup H)$

42. If $n(G) = 5$, $n(H) = 8$, $n(G \cap H) = 4$, find $n(G \cup H)$

43. A survey was given asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.

52 only use Netflix

62 only use Redbox

24 only use a video store

16 use only a video store and Redbox

48 use only Netflix and Redbox

30 use only a video store and Netflix

10 use all three

25 use none of these

44. A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color	8 only said size
16 only said brand	20 said only color and size
42 said only color and brand	53 said only size and brand
102 said all three	20 said none of these

45. Use the given information to complete a Venn diagram, then determine: a) how many students have seen exactly one of these movies, and b) how many had seen only *Star Wars*.

18 had seen <i>The Matrix</i> (<i>M</i>)	24 had seen <i>Star Wars</i> (<i>SW</i>)
20 had seen <i>Lord of the Rings</i> (<i>LotR</i>)	10 had seen <i>M</i> and <i>SW</i>
14 had seen <i>LotR</i> and <i>SW</i>	12 had seen <i>M</i> and <i>LotR</i>
6 had seen all three	

46. A survey asked people what alternative transportation modes they use. Using the data to complete a Venn diagram, then determine: a) what percent of people only ride the bus, and b) how many people don't use any alternate transportation.

30% use the bus	20% ride a bicycle
25% walk	5% use the bus and ride a bicycle
10% ride a bicycle and walk	12% use the bus and walk
2% use all three	

