

Apportionment

Apportionment is the problem of dividing up a fixed number of things among groups of different sizes. In politics, this takes the form of allocating a limited number of representatives amongst voters. This problem, presumably, is older than the United States, but the best-known ways to solve it have their origins in the problem of assigning each state an appropriate number of representatives in the new Congress when the country was formed. States also face this apportionment problem in defining how to draw districts for state representatives. The apportionment problem comes up in a variety of non-political areas too, though. We face several restrictions in this process:

Apportionment rules

1. The things being divided up can exist only in whole numbers.
2. We must use all of the things being divided up, and we cannot use any more.
3. Each group must get at least one of the things being divided up.
4. The number of things assigned to each group should be at least approximately proportional to the population of the group. (Exact proportionality isn't possible because of the whole number requirement, but we should try to be close, and in any case, if Group A is larger than Group B, then Group B shouldn't get more of the things than Group A does.)

In terms of the apportionment of the United States House of Representatives, these rules imply:

1. We can only have whole representatives (a state can't have 3.4 representatives)
2. We can only use the (currently) 435 representatives available. If one state gets another representative, another state has to lose one.
3. Every state gets at least one representative
4. The number of representatives each state gets should be approximately proportional to the state population. This way, the number of constituents each representative has should be approximately equal.

We will look at four ways of solving the apportionment problem. Three of them (Lowndes's method is the exception) have been used at various times to apportion the U.S. Congress, although the method currently in use (the Huntington-Hill method) is significantly more complicated.

Hamilton's Method

Alexander Hamilton proposed the method that now bears his name. His method was approved by Congress in 1791, but was vetoed by President Washington. It was later adopted in 1852 and used through 1911. He begins by determining, to several decimal places, how many things each group should get. Since he was interested in the question of Congressional representation, we'll use the language of states and representatives, so he determines how many representatives each state should get. He follows these steps:

Hamilton's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.

Since we can only allocate whole representatives, Hamilton resolves the whole number problem, as follows:

3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are called the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives (and the "or equal to" part happens only in very specific circumstances that are incredibly unlikely to turn up).
4. Assuming that the total from Step 3 was less than the total number of representatives, assign the remaining representatives, one each, to the states whose decimal parts of the quota were largest, until the desired total is reached.

Make sure that each state ends up with at least one representative!

Note on rounding: Today we have technological advantages that Hamilton (and the others) couldn't even have imagined. Take advantage of them, and keep several decimal places.

Example 1

The state of Delaware has three counties: Kent, New Castle, and Sussex. The Delaware state House of Representatives has 41 members. If Delaware wants to divide this representation along county lines (which is *not* required, but let's pretend they do), let's use Hamilton's method to apportion them. The populations of the counties are as follows (from the 2010 Census):

<u>County</u>	<u>Population</u>
Kent	162,310
New Castle	538,479
Sussex	197,145
Total	897,934

1. First, we determine the divisor: $897,934/41 = 21,900.82927$
2. Now we determine each county's quota by dividing the county's population by the divisor:

County	Population	Quota
Kent	162,310	7.4111
New Castle	538,479	24.5872
Sussex	197,145	9.0017
Total	897,934	

3. Removing the decimal parts of the quotas gives:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	897,934		40

4. We need 41 representatives, and this only gives 40. The remaining one goes to the county with the largest decimal part, which is New Castle:

County	Population	Quota	Initial	Final
Kent	162,310	7.4111	7	7
New Castle	538,479	24.5872	24	25
Sussex	197,145	9.0017	9	9
Total	897,934		40	41

Example 2

Use Hamilton’s method to apportion the 75 seats of Rhode Island’s House of Representatives among its five counties.

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	1,052,567

1. The divisor is $1,052,567/75 = 14,034.22667$

2. Determine each county’s quota by dividing its population by the divisor:

County	Population	Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	1,052,567	

3. Remove the decimal part of each quota:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

4. We need 75 representatives, and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport, Kent, and Providence:

County	Population	Quota	Initial	Final
Bristol	49,875	3.5538	3	3
Kent	166,158	11.8395	11	12
Newport	82,888	5.9061	5	6
Providence	626,667	44.6528	44	45
Washington	126,979	9.0478	9	9
Total	1,052,567		72	75

Note that even though Bristol County's decimal part is greater than .5, it isn't big enough to get an additional representative, because three other counties have greater decimal parts.

Hamilton's method obeys something called the Quota Rule. The Quota Rule isn't a law of any sort, but just an idea that some people, including Hamilton, think is a good one.

Quota Rule

The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, that means that each state should either go up to the next whole number above its quota, or down to the next whole number below its quota.

Controversy

After seeing Hamilton's method, many people find that it makes sense, it's not that difficult to use (or, at least, the difficulty comes from the numbers that are involved and the amount of computation that's needed, not from the method), and they wonder why anyone would want another method. The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives.

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method and the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives. The assumption was that some states would gain another representative and others would stay with the same number they already had (since there weren't enough new representatives being added to give one more to every state). The paradox is that Alabama ended up *losing* a representative in the process, even though no populations were changed and the total number of representatives increased.

The **New States Paradox** happened when Oklahoma became a state in 1907. Oklahoma had enough population to qualify for five representatives in Congress. Those five representatives would need to come from somewhere, though, so five states, presumably, would lose one representative each. That happened, but another thing also happened: Maine gained a representative (from New York).

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative *to Maine*.

Jefferson's Method

Thomas Jefferson proposed a different method for apportionment. After Washington vetoed Hamilton's method, Jefferson's method was adopted, and used in Congress from 1791 through 1842. Jefferson, of course, had political reasons for wanting his method to be used rather than Hamilton's. Primarily, his method favors larger states, and his own home state of Virginia was the largest in the country at the time. He would also argue that it's the ratio of people to representatives that is the critical thing, and apportionment methods should be based on that. But the paradoxes we saw also provide mathematical reasons for concluding that Hamilton's method isn't so good, and while Jefferson's method might or might not be the best one to replace it, at least we should look for other possibilities.

The first steps of Jefferson's method are the same as Hamilton's method. He finds the same divisor and the same quota, and cuts off the decimal parts in the same way, giving a total number of representatives that is less than the required total. The difference is in how Jefferson resolves that difference. He says that since we ended up with an answer that is too small, our divisor must have been too big. He changes the divisor by making it smaller, finding new quotas with the new divisor, cutting off the decimal parts, and looking at the new total, until we find a divisor that produces the required total.

Jefferson's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are the **lower quotas**. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 3

We'll return to Delaware and apply Jefferson's method. We begin, as we did with Hamilton's method, by finding the quotas with the original divisor, 21,900.82927:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	897,934		40

We need 41 representatives, and this divisor gives only 40. We must reduce the divisor until we get 41 representatives. Let's try 21,500 as the divisor:

County	Population	Quota	Initial
Kent	162,310	7.5493	7
New Castle	538,479	25.0455	25
Sussex	197,145	9.1695	9
Total	897,934		41

This gives us the required 41 representatives, so we're done. If we had fewer than 41, we'd need to reduce the divisor more. If we had more than 41, we'd need to choose a divisor less than the original but greater than the second choice.

Notice that with the new, lower divisor, the quota for New Castle County (the largest county in the state) increased by much more than those of Kent County or Sussex County.

Example 4

We'll apply Jefferson's method for Rhode Island. The original divisor of 14,034.22667 gave these results:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

We need 75 representatives and we only have 72, so we need to use a smaller divisor. Let's try 13,500:

County	Population	Quota	Initial
Bristol	49,875	3.6944	3
Kent	166,158	12.3080	12
Newport	82,888	6.1399	6
Providence	626,667	46.4198	46
Washington	126,979	9.4059	9
Total	1,052,567		76

We've gone too far. We need a divisor that's greater than 13,500 but less than 14,034.22667. Let's try 13,700:

County	Population	Quota	Initial
Bristol	49,875	3.6405	3
Kent	166,158	12.1283	12
Newport	82,888	6.0502	6
Providence	626,667	45.7421	45
Washington	126,979	9.2685	9
Total	1,052,567		75

This works.

Notice, in comparison to Hamilton's method, that although the results were the same, they came about in a different way, and the outcome was almost different. Providence County (the largest) almost went up to 46 representatives before Kent (which is much smaller) got to 12. Although that didn't happen here, it can. Divisor-adjusting methods like Jefferson's are not guaranteed to follow the quota rule!

Webster's Method

Daniel Webster (1782-1852) proposed a method similar to Jefferson's in 1832. It was adopted by Congress in 1842, but replaced by Hamilton's method in 1852. It was then adopted again in 1901. The difference is that Webster rounds the quotas to the nearest whole number rather than dropping the decimal parts. If that doesn't produce the desired results at the beginning, he says, like Jefferson, to adjust the divisor until it does. (In Jefferson's case, at least the first adjustment will always be to make the divisor smaller. That is not always the case with Webster's method.)

Webster's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Round all the quotas to the nearest whole number (but don't forget what the decimals were). Add up the remaining whole numbers.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 3 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 5

Again, Delaware, with an initial divisor of 21,900.82927:

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	25
Sussex	197,145	9.0017	9
Total	897,934		41

This gives the required total, so we're done.

Example 6

Again, Rhode Island, with an initial divisor of 14,034.22667:

County	Population	Quota	Initial
Bristol	49,875	3.5538	4
Kent	166,158	11.8395	12
Newport	82,888	5.9061	6
Providence	626,667	44.6528	45
Washington	126,979	9.0478	9
Total	1,052,567		76

This is too many, so we need to increase the divisor. Let's try 14,100:

County	Population	Quota	Initial
Bristol	49,875	3.5372	4
Kent	166,158	11.7843	12
Newport	82,888	5.8786	6
Providence	626,667	44.4445	44
Washington	126,979	9.0056	9
Total	1,052,567		75

This works, so we're done.

Like Jefferson's method, Webster's method carries a bias in favor of states with large populations, but rounding the quotas to the nearest whole number greatly reduces this bias. (Notice that Providence County, the largest, is the one that gets a representative trimmed because of the increased quota.) Also like Jefferson's method, Webster's method does not always follow the quota rule, but it follows the quota rule much more often than Jefferson's method does. (In fact, if Webster's method had been applied to every apportionment of Congress in all of American history, it would have followed the quota rule every single time.)

In 1980, two mathematicians, Peyton Young and Mike Balinski, proved what we now call the Balinski-Young Impossibility Theorem.

Balinski-Young Impossibility Theorem

The Balinski-Young Impossibility Theorem shows that any apportionment method which always follows the quota rule will be subject to the possibility of paradoxes like the Alabama, New States, or Population paradoxes. In other words, we can choose a method that avoids those paradoxes, but only if we are willing to give up the guarantee of following the quota rule.

Huntington-Hill Method

In 1920, no new apportionment was done, because Congress couldn't agree on the method to be used. They appointed a committee of mathematicians to investigate, and they recommended the Huntington-Hill Method. They continued to use Webster's method in 1931, but after a second report recommending Huntington-Hill, it was adopted in 1941 and is the current method of apportionment used in Congress.

The Huntington-Hill Method is similar to Webster's method, but attempts to minimize the percent differences of how many people each representative will represent.

Huntington-Hill Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **standard divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off the decimal part of the quota to obtain the lower quota, which we'll call n . Compute $\sqrt{n(n+1)}$, which is the **geometric mean** of the lower quota and one value higher.
4. If the quota is larger than the geometric mean, round up the quota; if the quota is smaller than the geometric mean, round down the quota. Add up the resulting whole numbers to get the **initial allocation**.
5. If the total from Step 4 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 4 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 4 is equal to the total number of representatives. The divisor we end up using is called the **modified divisor** or **adjusted divisor**.

Example 7

Again, Delaware, with an initial divisor of 21,900.82927:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Kent	162,310	7.4111	7	7.48	7
New Castle	538,479	24.5872	24	24.49	25
Sussex	197,145	9.0017	9	9.49	9
Total	897,934				41

This gives the required total, so we're done.

Example 8

Again, Rhode Island, with an initial divisor of 14,034.22667:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5538	3	3.46	4
Kent	166,158	11.8395	11	11.49	12
Newport	82,888	5.9061	5	5.48	6
Providence	626,667	44.6528	44	44.50	45
Washington	126,979	9.0478	9	9.49	9
Total	1,052,567				76

This is too many, so we need to increase the divisor. Let's try 14,100:

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5372	3	3.46	4
Kent	166,158	11.7943	11	11.49	12
Newport	82,888	5.8786	5	5.48	6
Providence	626,667	44.4445	44	44.50	44
Washington	126,979	9.0056	9	9.49	9
Total	1,052,567				75

This works, so we're done.

In both these cases, the apportionment produced by the Huntington-Hill method was the same as those from Webster's method.

Example 9

Consider a small country with 5 states, two of which are much larger than the others. We need to apportion 70 representatives. We will apportion using both Webster's method and the Huntington-Hill method.

State	Population
A	300,500
B	200,000
C	50,000
D	38,000
E	21,500

1. The total population is 610,000. Dividing this by the 70 representatives gives the divisor: 8714.286
2. Dividing each state's population by the divisor gives the quotas

State	Population	Quota
A	300,500	34.48361
B	200,000	22.95082
C	50,000	5.737705
D	38,000	4.360656
E	21,500	2.467213

Webster's Method

3. Using Webster's method, we round each quota to the nearest whole number

State	Population	Quota	Initial
A	300,500	34.48361	34
B	200,000	22.95082	23
C	50,000	5.737705	6
D	38,000	4.360656	4
E	21,500	2.467213	2

4. Adding these up, they only total 69 representatives, so we adjust the divisor down.

Adjusting the divisor down to 8700 gives an updated allocation totaling 70 representatives

State	Population	Quota	Initial
A	300,500	34.54023	35
B	200,000	22.98851	23
C	50,000	5.747126	6
D	38,000	4.367816	4
E	21,500	2.471264	2

Huntington-Hill Method

3. Using the Huntington-Hill method, we round down to find the lower quota, then calculate the geometric mean based on each lower quota. If the quota is less than the geometric mean, we round down; if the quota is more than the geometric mean, we round up.

State	Population	Quota	Lower Quota	Geom Mean	Initial
A	300,500	34.48361	34	34.49638	34
B	200,000	22.95082	22	22.49444	23
C	50,000	5.737705	5	5.477226	6
D	38,000	4.360656	4	4.472136	4
E	21,500	2.467213	2	2.44949	3

These allocations add up to 70, so we're done.

Notice that this allocation is different than that produced by Webster's method. In this case, state E got the extra seat instead of state A.

Lowndes' Method

William Lowndes (1782-1822) was a Congressman from South Carolina (a small state) who proposed a method of apportionment that was more favorable to smaller states. Unlike the methods of Hamilton, Jefferson, and Webster, Lowndes's method has never been used to apportion Congress.

Lowndes believed that an additional representative was much more valuable to a small state than to a large one. If a state already has 20 or 30 representatives, getting one more doesn't matter very much. But if it only has 2 or 3, one more is a big deal, and he felt that the additional representatives should go where they could make the most difference.

Like Hamilton's method, Lowndes's method follows the quota rule. In fact, it arrives at the same quotas as Hamilton and the rest, and like Hamilton and Jefferson, it drops the decimal parts. But in deciding where the remaining representatives should go, we divide the decimal part of each state's quota by the whole number part (so that the same decimal part with a smaller whole number is worth more, because it matters more to that state).

Lowndes's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the **divisor**.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the **quota**.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). Add up the remaining whole numbers.
4. Assuming that the total from Step 3 was less than the total number of representatives, divide the decimal part of each state's quota by the whole number part. Assign the remaining representatives, one each, to the states whose **ratio** of decimal part to whole part were largest, until the desired total is reached.

Example 10

We'll do Delaware again. We begin in the same way as with Hamilton's method:

<u>County</u>	<u>Population</u>	<u>Quota</u>	<u>Initial</u>
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	897,934		40

We need one more representative. To find out which county should get it, Lowndes says to divide each county's decimal part by its whole number part, with the largest result getting the extra representative:

Kent: $0.4111/7 \approx 0.0587$
 New Castle: $0.5872/24 \approx 0.0245$
 Sussex: $0.0017/9 \approx 0.0002$

The largest of these is Kent's, so Kent gets the 41st representative:

County	Population	Quota	Initial	Ratio	Final
Kent	162,310	7.4111	7	0.0587	8
New Castle	538,479	24.5872	24	0.0245	24
Sussex	197,145	9.0017	9	0.0002	9
Total	897,934		40		41

Example 11

Rhode Island, again beginning in the same way as Hamilton:

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	1,052,567		72

We divide each county's quota's decimal part by its whole number part to determine which three should get the remaining representatives:

Bristol: $0.5538/3 \approx 0.1846$
 Kent: $0.8395/11 \approx 0.0763$
 Newport: $0.9061/5 \approx 0.1812$
 Providence: $0.6528/44 \approx 0.0148$
 Washington: $0.0478/9 \approx 0.0053$

The three largest of these are Bristol, Newport, and Kent, so they get the remaining three representatives:

County	Population	Quota	Initial	Ratio	Final
Bristol	49,875	3.5538	3	0.1846	4
Kent	166,158	11.8395	11	0.0763	12
Newport	82,888	5.9061	5	0.1812	6
Providence	626,667	44.6528	44	0.0148	44
Washington	126,979	9.0478	9	0.0053	9
Total	1,052,567		72		75

As you can see, there is no “right answer” when it comes to choosing a method for apportionment. Each method has its virtues and favors different sized states.

Apportionment of Legislative Districts

In most states, there are a fixed number of representatives to the state legislature. Rather than apportioning each county a number of representatives, legislative districts are drawn so that each legislator represents a district. The apportionment process, then, comes in the drawing of the legislative districts, with the goal of having each district include approximately the same number of constituents. Because of this goal, a geographically small city may have several representatives, while a large rural region may be represented by one legislator.

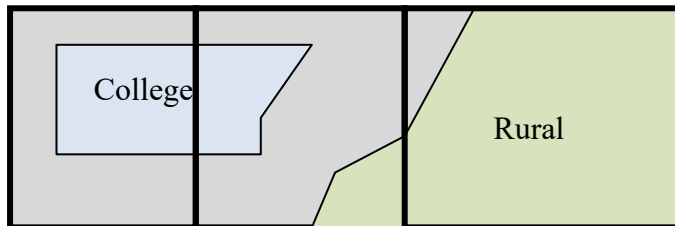
When populations change, it becomes necessary to redistrict the regions each legislator represents (Incidentally, this also occurs for the regions that federal legislators represent). The process of redistricting is typically done by the legislature itself, so not surprisingly it is common to see **gerrymandering**.

Gerrymandering

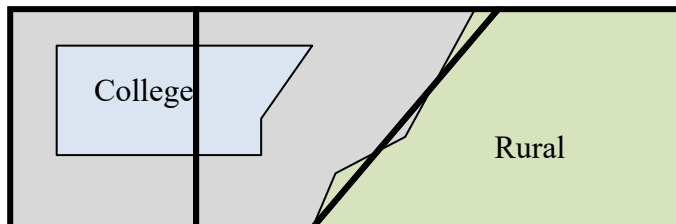
Gerrymandering is when districts are drawn based on the political affiliation of the constituents to the advantage of those drawing the boundary.

Example 12

Consider three districts, simplified to the three boxes below. On the left there is a college area that typically votes Democratic. On the right is a rural area that typically votes Republican. The rest of the people are more evenly split. The middle district has been voting 50% Democratic and 50% Republican.

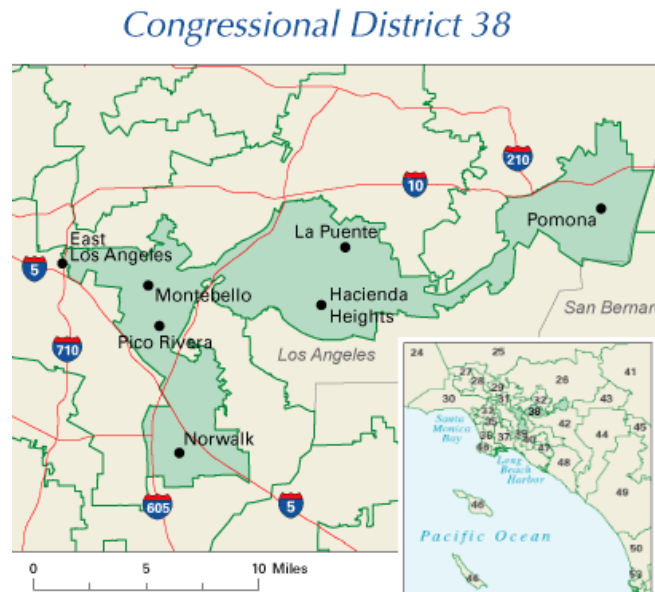


As part of a redistricting, a Democratic led committee could redraw the boundaries so that the middle district includes less of the typically Republican voters, thereby making it more likely that their party will win in that district, while increasing the Republican majority in the third district.

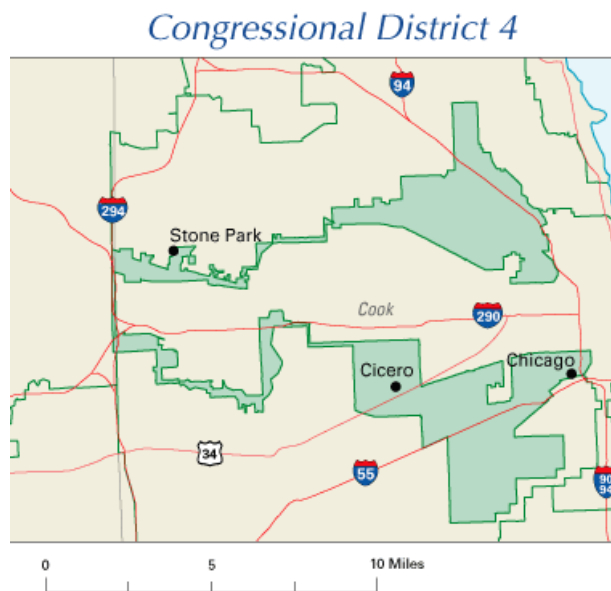


Example 13

The map to the right shows the 38th congressional district in California in 2004¹. This district was created through a bi-partisan committee of incumbent legislators. This gerrymandering leads to districts that are not competitive; the prevailing party almost always wins with a large margin.



The map to the right shows the 4th congressional district in Illinois in 2004.² This district was drawn to contain the two predominantly Hispanic areas of Chicago. The largely Puerto Rican area to the north and the southern Mexican areas are only connected in this districting by a piece of the highway to the west.



¹ http://en.wikipedia.org/wiki/File:California_District_38_2004.png

² http://en.wikipedia.org/wiki/File:Illinois_District_4_2004.png

Exercises

In exercises 1-8, determine the apportionment using

- a. Hamilton’s Method
 - b. Jefferson’s Method
 - c. Webster’s Method
 - d. Huntington-Hill Method
 - e. Lowndes’ method
1. A college offers tutoring in Math, English, Chemistry, and Biology. The number of students enrolled in each subject is listed below. If the college can only afford to hire 15 tutors, determine how many tutors should be assigned to each subject.

Math: 330 English: 265 Chemistry: 130 Biology: 70

2. Reapportion the previous problem if the college can hire 20 tutors.
3. The number of salespeople assigned to work during a shift is apportioned based on the average number of customers during that shift. Apportion 20 salespeople given the information below.

Shift	Morning	Midday	Afternoon	Evening
Average number of customers	95	305	435	515

4. Reapportion the previous problem if the store has 25 salespeople.
5. Three people invest in a treasure dive, each investing the amount listed below. The dive results in 36 gold coins. Apportion those coins to the investors.
 Alice: \$7,600 Ben: \$5,900 Carlos: \$1,400
6. Reapportion the previous problem if 37 gold coins are recovered.
7. A small country consists of five states, whose populations are listed below. If the legislature has 119 seats, apportion the seats.
 A: 810,000 B: 473,000 C: 292,000 D: 594,000 E: 211,000
8. A small country consists of six states, whose populations are listed below. If the legislature has 200 seats, apportion the seats.
 A: 3,411 B: 2,421 C: 11,586 D: 4,494 E: 3,126 F: 4,962
9. A small country consists of three states, whose populations are listed below.
 A: 6,000 B: 6,000 C: 2,000
 - a. If the legislature has 10 seats, use Hamilton’s method to apportion the seats.
 - b. If the legislature grows to 11 seats, use Hamilton’s method to apportion the seats.
 - c. Which apportionment paradox does this illustrate?

10. A state with five counties has 50 seats in their legislature. Using Hamilton's method, apportion the seats based on the 2000 census, then again using the 2010 census. Which apportionment paradox does this illustrate?

County	2000 Population	2010 Population
Jefferson	60,000	60,000
Clay	31,200	31,200
Madison	69,200	72,400
Jackson	81,600	81,600
Franklin	118,000	118,400

11. A school district has two high schools: Lowell, serving 1715 students, and Fairview, serving 7364. The district could only afford to hire 13 guidance counselors.
- Determine how many counselors should be assigned to each school using Hamilton's method.
 - The following year, the district expands to include a third school, serving 2989 students. Based on the divisor from above, how many additional counselors should be hired for the new school?
 - After hiring that many new counselors, the district recalculates the reapportionment using Hamilton's method. Determine the outcome.
 - Does this situation illustrate any apportionment issues?
12. A small country consists of four states, whose populations are listed below. If the legislature has 116 seats, apportion the seats using Hamilton's method. Does this illustrate any apportionment issues?
 A: 33,700 B: 559,500 C: 141,300 D: 89,100

Exploration

13. Explore and describe the similarities, differences, and interplay between weighted voting, fair division (if you've studied it yet), and apportionment.
14. In the methods discussed in the text, it was assumed that the number of seats being apportioned was fixed. Suppose instead that the number of seats could be adjusted slightly, perhaps 10% up or down. Create a method for apportioning that incorporates this additional freedom, and describe why you feel it is the best approach. Apply your method to the apportionment in Exercise 7.
15. Lowndes felt that small states deserved additional seats more than larger states. Suppose you were a legislator from a larger state, and write an argument refuting Lowndes.
16. Research how apportionment of legislative seats is done in other countries around the world. What are the similarities and differences compared to how the United States apportions congress?
17. Adams's method is similar to Jefferson's method, but rounds quotas *up* rather than *down*. This means we usually need a modified divisor that is smaller than the standard divisor. Rework problems 1-8 using Adam's method. Which other method are the results most similar to?