Section 1: The Definite Integral

Distance from Velocity

Example 1

Suppose a car travels on a straight road at a constant speed of 40 miles per hour for two hours. See the graph of its velocity below. How far has it gone?

We all remember **distance = rate × time**, so this one is easy. The car has gone 40 miles per hour × 2 hours = 80 miles.

Example 2

Now suppose that a car travels so that its speed increases steadily from 0 to 40 miles per hour, for two hours. (Just be grateful you weren't stuck behind this car on the highway.) See the graph of its velocity in below. How far has this car gone?
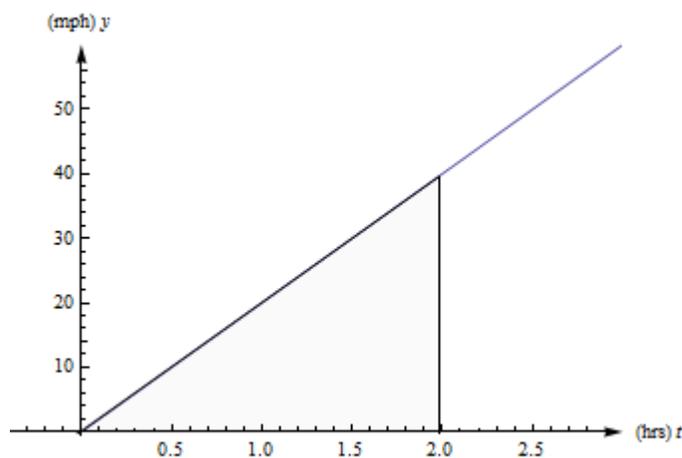

The trouble with our old reliable **distance = rate × time** relationship is that it only works if the rate is constant. If the rate is changing, there isn't a good way to use this formula.

But look at graph from the last example again. Notice that **distance = rate × time** also describes the area between the velocity graph and the t-axis, between t = 0 and t = 2 hours. The **rate** is the height of the rectangle, the **time** is the length of the rectangle, and the **distance** is the **area** of the rectangle. This is the way we can extend our simple formula to handle more complicated velocities: And this is the way we can answer the second example.

The distance the car travels is the area between its velocity graph, the t-axis, t = 0 and t = 2. This region is a triangle, so its area is ½bh = ½(2 hours)(40 miles per hour) = 40 miles. So the car travels 40 miles during its annoying trip.

In our distance/velocity examples, the function represented a **rate** of travel (miles per hour), and the area represented the **total** distance traveled. This principle works more generally:

For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.
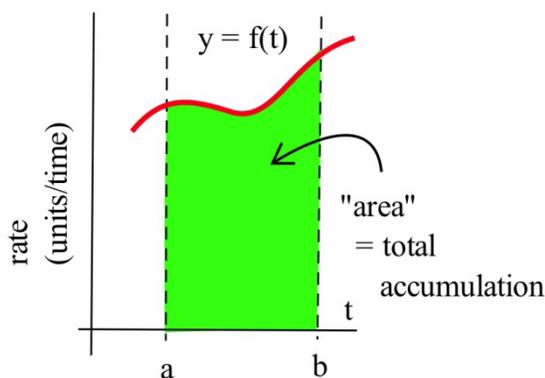
Example 3

The graph below shows the flow rate (cubic feet per second) of water in the Skykomish river at the town of Goldbar in Washington state.
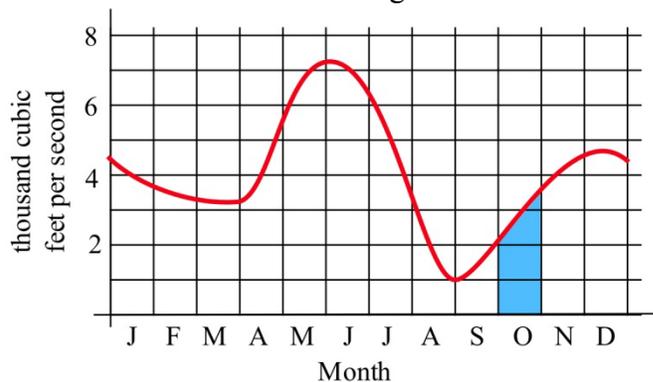
The area of the shaded region represents the total volume (cubic feet) of water flowing past the town during the month of October. We can approximate this area to approximate the total water by thinking of the shaded region as a rectangle with a triangle on top.

Total water   = total area ≈ area of rectangle + area of the "triangle"

$$\approx (2000 \text{ cubic feet/sec})(30 \text{ days}) + \frac{1}{2}(1500 \text{ cf/s})(30 \text{ days}) = (2750 \text{ cubic feet/sec})(30 \text{ days})$$

Note that we need to convert the units to make sense of our result:
Total water ≈ (2750 cubic feet/sec)(30 days) = (2750 cubic feet/sec)(2,592,000 sec)

$$\approx 7.128 \times 10^9 \text{ cubic feet.}$$

About 7 billion cubic feet of water flowed past Goldbar in October.

Approximating with Rectangles

How do we approximate the area if the rate curve is, well, curvy? We could use rectangles and triangles, like we did in the last example. But it turns out to be more useful (and easier) to simply use rectangles. The more rectangles we use, the better our approximation is.
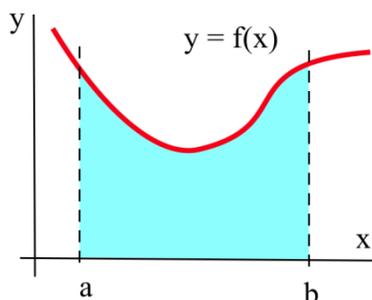

Suppose we want to calculate the area between the graph of a positive function f and the x–axis on the interval [a, b] (graphed above). The **Riemann Sum method** is to build several rectangles with bases on the interval [a, b] and sides that reach up to the graph of f (see below). Then the areas of the rectangles can be calculated and added together to get a number called a Riemann Sum of f on [a, b]. The area of the region formed by the rectangles is an **approximation** of the area we want.
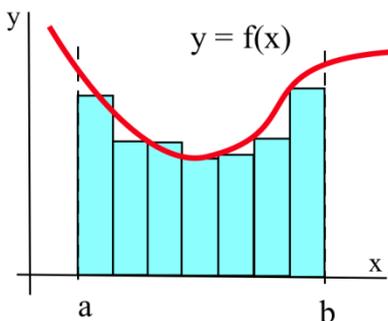
Example 4

Approximate the area in the graph on the left between the graph of f and the x–axis on the interval [2, 5] by summing the areas of the rectangles in the graph on the right.

The total area of rectangles is (2)(3) + (1)(5) = 11 square units.

(a)   (b)
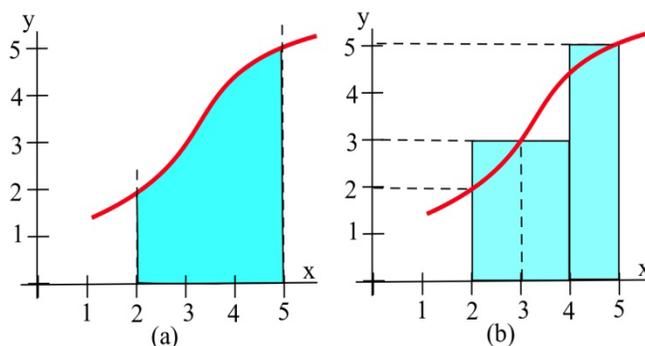
Example 5

Let A be the region bounded by the graph of f(x) = 1/x, the x–axis, and vertical lines at x = 1 and x = 5. We can't find the area exactly (with what we know now), but we can approximate it using rectangles.

When we make our rectangles, we have a lot of choices. We could pick any (non-overlapping) rectangles whose bottoms lie within the interval on the x-axis, and whose tops intersect with the curve somewhere. But it's easiest to choose rectangles that – (a) have all the same width, and (b) take their heights from the function at one edge. Below you'll see two ways to use four rectangles to approximate this area. In the first graph, we used left-endpoints; the height of each rectangle comes from the function value at its left edge. In the second graph on the next page, we used right-hand endpoints.

Left-hand endpoints: The area is approximately the sum of the areas of the rectangles. Each rectangle gets its height from the function $f(x) = \dfrac{1}{x}$ and each rectangle has width = 1.
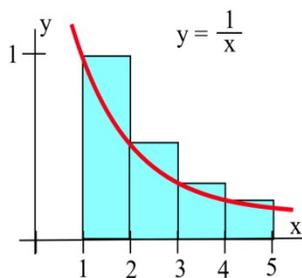

You can find the area of each rectangle using area = height × width. So the total area of the rectangles, the left-hand estimate of the area under the curve, is

$$f(1)(1) + f(2)(1) + f(3)(1) + f(4)(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \cong 2.08$$

Notice that because this function is decreasing, all the left endpoint rectangles stick out above the region we want – using left-hand endpoints will overestimate the area.

Right-hand endpoints: The right-hand estimate of the area is

$$f(2)(1) + f(3)(1) + f(4)(1) + f(5)(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} \cong 1.28$$
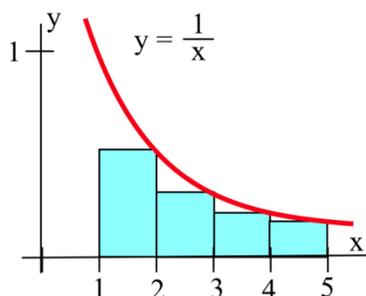
All the right-hand rectangles lie completely under the curve, so this estimate will be an underestimate.

We can see that the true area is actually in between these two estimates. So we could take their average:

Average: $\dfrac{25/12+77/60}{2} = \dfrac{101}{60} \cong 1.68$

In general, the average of the left-hand and right-hand estimates will be closer to the real area than either individual estimate.

My estimate of the area under the curve is about 1.68. (The actual area is about 1.61.)

If we wanted a better answer, we could use even more, even narrower rectangles. But there's a limit to how much work we want to do by hand. In practice, it's probably best to choose a manageable number of rectangles. We'll have better methods to get more accurate answers before long.

These sums of areas of rectangles are called **Riemann sums**. You may see a shorthand notation used when people talk about sums.   We won't use it much in this book, but you should know what it means.

> **Riemann sum:** A Riemann sum for a function f(x) over an interval [a, b] is a sum of areas of rectangles that approximates the area under the curve. Start by dividing the interval [a, b] into n subintervals; each subinterval will be the base of one rectangle. We usually make all the rectangles the same width Δx. The height of each rectangle comes from the function evaluated at some point in its sub interval. Then the Riemann sum is:
>
> $$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$
>
> **Sigma Notation:** The upper-case Greek letter Sigma Σ is used to stand for Sum. Sigma notation is a way to compactly represent a sum of many similar terms, such as a Riemann sum.
>
> Using the Sigma notation, the Riemann sum can be written $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$.
>
> This is read aloud as "the sum as i = 1 to n of f of x sub i Delta x." The "i" is a counter, like you might have seen in a programming class.

Definition of the Definite Integral

Because the area under the curve is so important, it has a special vocabulary and notation.

The Definite Integral:

The **definite integral** of a positive function f(x) over an interval [a, b] is the area
  between f, the x-axis, x = a and x = b.
The **definite integral** of a positive function f(x) from a to b is the area under the curve
  between a and b.

If f(t) represents a positive rate (in y-units per t-units), then the **definite integral** of f from a
  to b is the total y-units that accumulate between t = a and t = b.

Notation for the Definite Integral:
The definite integral of f from a to b is written

$$\int_a^b f(x)\, dx$$

The $\int$ symbol is called an **integral sign**; it's an elongated letter S, standing for sum.

(The $\int$ corresponds to the Σ from the Riemann sum)

The dx on the end must be included; The dx tells what the variable is – in this example, the
  variable is x. (The dx corresponds to the Δx from the Riemann sum)
The function f is called the **integrand**.
The a and b are called the limits **of integration**.

Verb forms:
We **integrate**, or **find the definite integral** of a function. This process is called
  integration.

Formal Algebraic Definition: $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.

Practical Definition:
  The definite integral can be approximated with a Riemann sum (dividing the area into
  rectangles where the height of each rectangle comes from the function, computing the
  area of each rectangle, and adding them up). The more rectangles you use, the narrower
  the rectangles are, the better your approximation will be.

Looking Ahead:
  We will have methods for computing exact values of some definite integrals from
  formulas soon. In many cases, including when the function is given to you as a table or
 graph, you will still need to approximate the definite integral with rectangles.

Example 6

The graph shows y = r(t), the number of telephone calls made per hours on a Tuesday. Approximately how many calls were made between 9 pm and 11 pm? Express this as a definite integral and approximate with a Riemann sum.
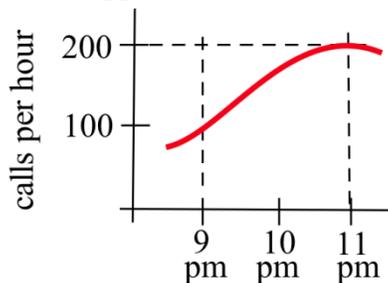
We know that the accumulated calls will be the area under this rate graph over that two-hour period, the definite integral of this rate from t = 9 to t = 11.

The total number of calls will be $\int_9^{11} r(t)\,dt$.

The top here is a curve, so we can't get an exact answer. But we can approximate the area using rectangles. I'll choose to use 4 rectangles, and I'll choose left-endpoints:
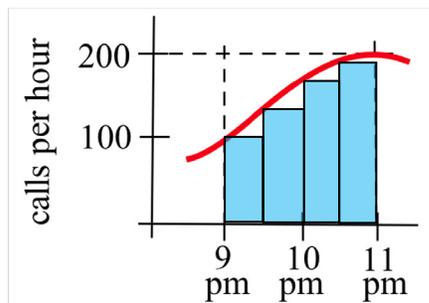
$$\int_9^{11} r(t)\,dt \cong 100(.5)+150(.5)+180(.5)+195(.5)=312.5.$$

The units are *calls per hour × hours = calls*. My estimate is that about 312 calls were made between 9 pm and 11 pm. Is this an under-estimate or an over-estimate?

Example 7

Describe the area between the graph of f(x) = 1/x, the x–axis, and the vertical lines at x = 1 and x = 5 as a definite integral.

This is the same area we estimated to be about 1.68 before. Now we can use the notation of the definite integral to describe it. Our estimate of $\int_1^5 \frac{1}{x}\,dx$ was 1.68. The true value of $\int_1^5 \frac{1}{x}\,dx$ is about 1.61.

Example 8

Using the idea of area, determine the value of $\int_1^3 (1+x)\,dx$.

$\int_1^3 (1+x)\,dx$ represents the area between the graph of $f(x) = 1+x$, the x–axis, and the vertical lines at 1 and 3.

Since this area can be broken into a rectangle and a triangle, we can find the area exactly. The area equals
$4 + \frac{1}{2}(2)(2) = 6$ square units.
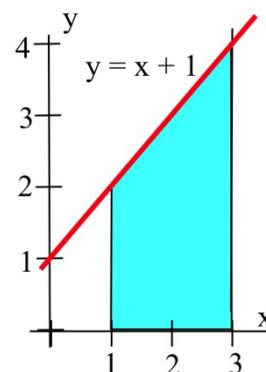
Example 9

The table shows rates of population growth for Berrytown for several years. Use this table to estimate the total population growth from 1970 to 2000:

Year (t)	1970	1980	1990	2000
Rate of population growth R(t) (thousands of people per year)	1.5	1.9	2.2	2.4

The definite integral of this rate will give the total change in population over the thirty-year period. We only have a few pieces of information, so we can only estimate. Even though I haven't made a graph, we're still approximating the area under the rate curve, using rectangles. How wide are the rectangles? I have information every 10 years, so the rectangles have a width of 10 years. How many rectangles? Be careful here – this is a thirty-year span, so there are three rectangles.

Using left-hand endpoints: $(1.5)(10) + (1.9)(10) + (2.2)(10) = 56$
Using right-hand endpoints: $(1.9)(10) + (2.2)(10) + (2.4)(10) = 65$

Taking the average of these two: $\dfrac{56+65}{2} = 60.5$

Our best estimate of the total population growth from 1970 to 2000 is 60.5 thousand people.

Signed Area

You may have noticed that until this point, we've insisted that the integrand (the function we're integrating) be positive. That's because we've been talking about area, which is always positive. If the "height" (from the function) is a negative number, then multiplying it by the width doesn't give us actual area, it gives us the area with a negative sign.

But it turns out to be useful to think about the possibility of negative area. We'll expand our idea of a definite integral now to include integrands that might not always be positive. The "heights" of the rectangles, the values from the function, now might not always be positive.

The Definite Integral and Signed Area:

> The **definite integral** of a function f(x) over an interval [a, b] is the **signed area** between f, the x-axis, x = a and x = b.
>
> The **definite integral** of a function f(x) from a to b is the **signed area** under the curve between a and b.
>
> If the function is positive, the signed area is positive, as before (and we can call it area.)
> If the function dips below the x-axis, the areas of the regions below the x-axis come in with a negative sign. In this case, we cannot call it simply "area." These negative areas take away from the definite integral.
>
> $$\int_a^b f(x)\,dx = (\text{Area above x-axis}) - (\text{Area below x-axis}).$$
>
> If f(t) represents a positive rate (in y-units per t-units), then the **definite integral** of f from a to b is the **total** y-units that accumulate between t = a and t = b.
> If f(t) represents any rate (in y-units per t-units), then the **definite integral** of f from a to b is the **net** y-units that accumulate between t = a and t = b.

Example 10

Find the definite integral of of $f(x) = -2$ on the interval $[1,4]$.
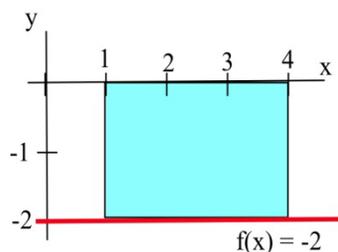
$\int\limits_{1}^{4} -2\ dx$ is the signed area of the region shown to the right. The region lies below the x-axis,

so the area, 6, comes in with a negative sign. So the definite integral is $\int\limits_{1}^{4} -2\ dx = -6$.

Negative rates indicate that the amount is decreasing. For example, if f(t) is the velocity of a car in the positive direction along a straight line at time t (miles/hour) , then negative values of f indicate that the car is traveling in the negative direction, backwards. The definite integral of f is the change in position of the car during the time interval. If the velocity is positive, positive distance accumulates. If the velocity is negative, distance in the negative direction accumulates.

This is true of any rate. For example, if f(t) is the rate of population change (people/year) for a town, then negative values of f would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would be the **change** in the population, a decrease, during the time interval.

Example 11

In 1980 there were 12,000 ducks nesting around a lake, and the **rate** of population change (in ducks per year) is shown in Fig. 15. Write a definite integral to represent the total change in the duck population from 1980 to 1990, and estimate the population in 1990.
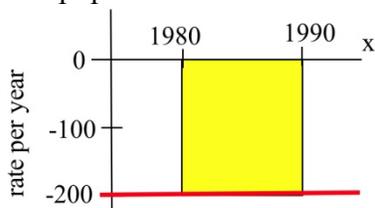

The **change** in population

$= \int\limits_{1980}^{1990} f(t)\ dt = -$ (area between f and axis)

$\approx -$ (200 ducks/year)·(10 years) $= -$ 2000 ducks.

Then (1990 duck population) = (1980 population) + (change from 1980 to 1990)
$= (12{,}000) + (-2000) = $ 10,000 ducks.

Example 12

A bug starts at the location $x = 12$ on the x–axis at 1 pm walks along the axis with the velocity $v(x)$ shown in the graph. How far does the bug travel between 1 pm and 3 pm, and where is the bug at 3 pm?
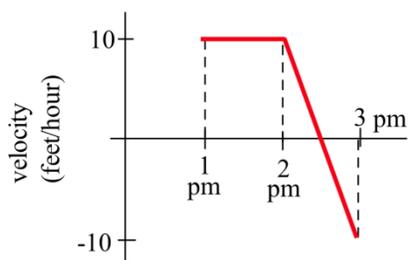
Note that the velocity is positive from 1 until 2:30, then becomes negative. So the bug moves in the positive direction from 1 until 2:30, then turns around and moves back toward where it started. The area under the velocity curve from 1 to 2:30 shows the total distance traveled by the bug in the positive direction; the bug moved 12.5 feet in the positive direction. The **area** between the velocity curve and the x-axis, between 2:30 and 3, shows the total distance traveled by the bug in the negative direction, back toward home; the bug traveled 2.5 feet in the negative direction. The definite integral of the velocity curve, $\int_{1}^{3} v(t)\,dt$, shows the net change in distance:
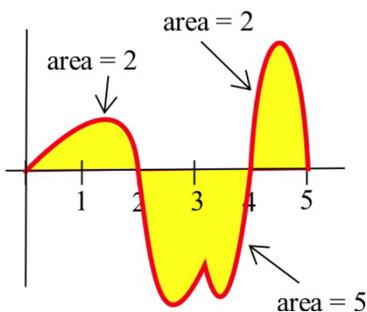
$$\int_{1}^{3} v(t)\,dt = 12.5 - 2.5 = 10$$

The bug ended up 10 feet further in the positive direction than he started. At 3 pm, the bug is at x = 22.

Example 13

Use the graph to calculate $\int_{0}^{2} f(x)\,dx$, $\int_{2}^{4} f(x)\,dx$, $\int_{4}^{5} f(x)\,dx$, and $\int_{0}^{5} f(x)\,dx$.

Using the given areas, $\int_{0}^{2} f(x)\,dx = 2,$ $\int_{2}^{4} f(x)\,dx = -\mathbf{5},$ $\int_{4}^{5} f(x)\,dx = 2,$ and

$\int_{0}^{5} f(x)\,dx = $ (area above) − (area below) = (2+2) − (5) = −1.

Approximating with Technology

If your function is given as a graph or table, you will still have to approximate definite integrals using areas, usually of rectangles. But if your function is given as a formula, you can turn to technology to get a better approximate answer. For example, most graphing calculators have some kind of numerical integration tool built in. You can also find many online tools that can do this; type numerical integration into any search engine to see a selection of these.

Most numerical integration tools use rectangles to estimate the signed area, just as you would do by hand. But they use many more rectangles than you would have the patience for, so they get a better answer. Some of them use computer algebra systems to find exact answers; we will learn how to do this ourselves later in this chapter.

When you turn to technology to find the value of a definite integral, be careful. Not every tool will be able to give you a correct answer for every integral. I have had good luck with my TI 84. You should make an estimate of the answer yourself first so you can judge whether the answer you get makes sense.

Example 14

Use technology to approximate the definite integral $\int_{1}^{5} \frac{1}{x}\, dx$. (This is the same definite integral we approximated with rectangles before.)

I used my TI-84; the answer it gave me was 1.609437912. This agrees with the exact answer for all the decimal digits displayed. WebMath said the answer was 1.60944, which is accurate for all the decimal digits displayed. Microsoft Math said the answer was ln(5); that's exactly correct. Wolfram|Alpha says the answer is log(5); that's not how everyone writes the natural log, so that might trick you into writing the wrong answer.

Example 15

Use technology to approximate the definite integral $\int_{1}^{2} e^{x^2+x}\, dx$

I asked WebMath, and it said the answer was zero – I know this is not correct, because the function here is positive, so there must be some area under the curve here. I asked Microsoft Math, and it simply repeated the definite integral; that's because there isn't an algebraic way to find the exact answer. I asked my TI-84, and it said the answer was 86.83404047; that makes sense with what I expected. Wolfram|Alpha also says the answer is about 86.834. So I believe: $\int_{1}^{2} e^{x^2+x}\, dx \cong 86.834$.

Accumulation in Real Life

We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if t is a measure of time in seconds and $f(t)$ is a velocity with units feet/second, then the definite integral has units (feet/second) · (seconds) = feet.

In general, the units for the definite integral $\int_a^b f(x)\,dx$ are (y-units) · (x-units). A quick check of the units can help avoid errors in setting up an applied problem.

In previous examples, we looked at a function represented a **rate** of travel (miles per hour); in that case, the area represented the **total** distance traveled. For functions representing other **rates** such as the production of a factory (bicycles per day), or the flow of water in a river (gallons per minute) or traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week), the area will still represent the **total** amount of something.

Example 16

Suppose MR(q) is the marginal revenue in dollars/item for selling q items. What does $\int_0^{150} MR(q)\,dq$ represent?

$\int_0^{150} MR(q)\,dq$ has units (dollars/item) · (items) = dollars, and represents the accumulated dollars for selling from 0 to 150 items. That is, $\int_0^{150} MR(q)\,dq = TR(150)$, the total revenue from selling 150 items.

Example 17

Suppose $r(t)$, in centimeters per year, represents how the diameter of a tree changes with time. What does $\int_{T_1}^{T_2} r(t)\,dt$ represent?

$\int_{T_1}^{T_2} r(t)\,dt$ has units of (centimeters per year) · (years) = centimeters, and represents the accumulated growth of the tree's diameter from T_1 to T_2. That is, $\int_{T_1}^{T_2} r(t)\,dt$ is the change in the diameter of the tree over this period of time.

3.1 Exercises
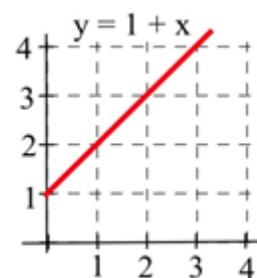
1. Let A(x) represent the area bounded by the graph and the
    horizontal axis and vertical lines at $t=0$ and $t=x$ for the graph
   shown. Evaluate A(x) for $x = 1, 2, 3, 4$, and 5.
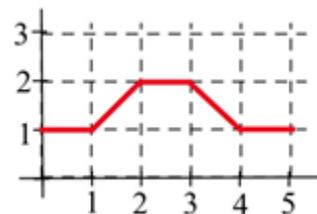
2. Let B(x) represent the area bounded by the graph and the
    horizontal axis and vertical lines at $t=0$ and $t=x$ for the
   graph shown. Evaluate B(x) for $x = 1, 2, 3, 4$, and 5.
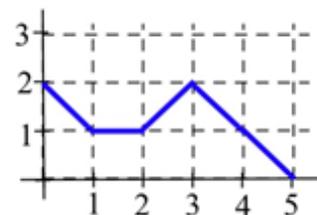
3. Let C(x) represent the area bounded by the graph and the
    horizontal axis and vertical lines at $t=0$ and $t=x$ for the graph
    shown. Evaluate C(x) for $x = 1, 2$, and 3 and find a formula for
    C(x).

    $y = 1 + x$

4. Let A(x) represent the area bounded by the graph and the
    horizontal axis and vertical lines at $t=0$ and $t=x$ for the graph
    shown. Evaluate A(x) for $x = 1, 2$, and 3   and find a formula for
   A(x).

    $y = 4 - x$
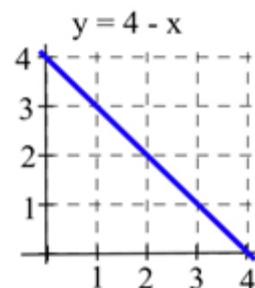
5. A car had the velocity shown in the graph to the right. How far
   did the car travel from $t= 0$ to $t = 30$ seconds?
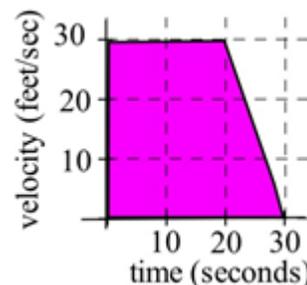
6. A car had the velocity shown below. How afar did the car travel
   from $t = 0$ to $t = 30$ seconds?
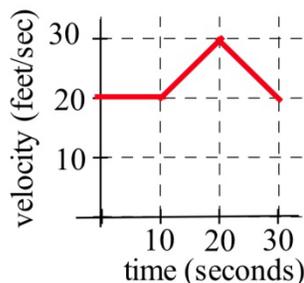
7. The velocities of two cars are shown in the graph.
   (a) From the time the brakes were applied, how many seconds did it take each car to stop?
   (b) From the time the brakes were applied, which car traveled farther until it came to a complete stop?
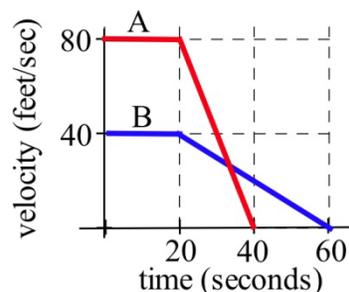
8. You and a friend start off at noon and walk in the same direction along the same path at the rates shown.
    (a) Who is walking faster at 2 pm? Who is ahead at 2 pm?
    (b) Who is walking faster at 3 pm? Who is ahead at 3 pm?
    (c) When will you and your friend be together? (Answer in words.)
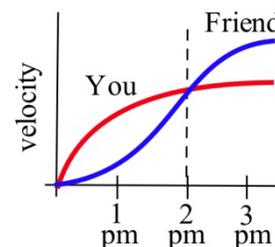
9. Police chase: A speeder traveling 45 miles per hour (in a 25 mph zone) passes a stopped police car which immediately takes off after the speeder. If the police car speeds up steadily to 60 miles/hour in 20 seconds and then travels at a steady 60 miles/hour, **how long** and **how far** before the police car catches the speeder who continued traveling at 45 miles/hour?
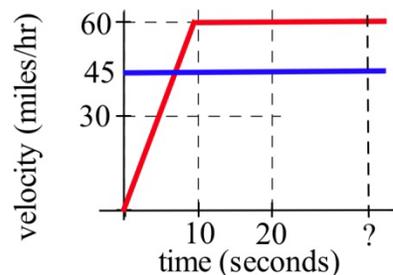
10. Water is flowing into a tub. The table shows the rate at which the water flows, in gallons per minute. The tub is initially empty.

t, in minutes	0	1	2	3	4	5	6	7	8	9	10
Flow rate, in gal/min	0.5	1.0	1.2	1.4	1.7	2.0	2.3	1.8	0.7	0.5	0.2

Use the table to estimate how much water is in the tub after
   a. five minutes
   b. ten minutes

11. The table shows the speedometer readings for a short car trip.

t, in minutes	0	5	10	15	20
Speed, in mph	0	30	40	65	40

a. Use the table to estimate how far the car traveled over the twenty minutes shown.
b. How accurate would you expect your estimate to be?

12. The table shows values of $f(t)$. Use the table to estimate $\int_0^{40} f(t)\,dt$.

t	0	10	20	30	40
$f(t)$	17	22	18	11	35

13. The table shows values of $g(x)$.

x	0	1	2	3	4	5	6
$g(x)$	140	142	144	152	154	165	200

Use the table to estimate

a. $\int_0^3 g(x)\,dx$ b. $\int_3^6 g(x)\,dx$ c. $\int_0^6 g(x)\,dx$

14. What are the units for the "area" of a rectangle with the given
base and height units?

Base units	Height units	"Area" units
miles per second	seconds	
hours	dollars per hour	
square feet	feet	
kilowatts	hours	
houses	people per house	
meals	meals	

In problems 15 – 17 , represent the area of each bounded region as a definite integral, and use geometry to
determine the value of the definite integral.

15. The region bounded by $y = 2x$, the x–axis, the line $x = 1$, and $x = 3$.

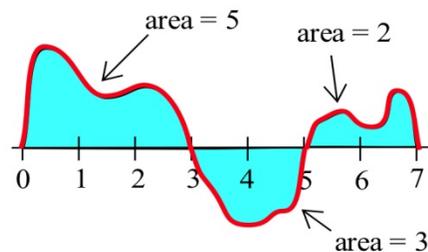16. The region bounded by $y = 4 - 2x$, the x–axis, and the y–axis.

17. The shaded region in the graph to the right.
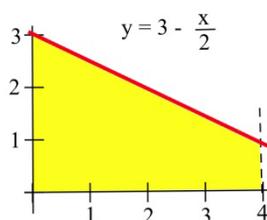
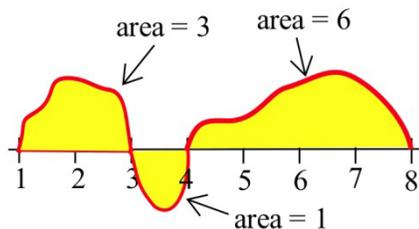18. Using the graph of f shown and the given areas of several regions, evaluate:

(a) $\int_{0}^{3}$ f(x) dx
(b) $\int_{3}^{5}$ f(x) dx
(c) $\int_{5}^{7}$ f(x) dx

area = 5
area = 2
area = 3

19. Using the graph of f shown and the given areas of several regions, evaluate:
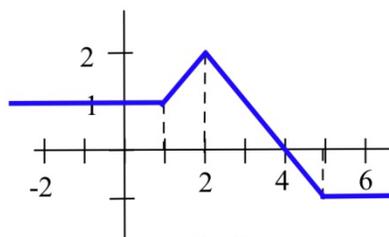
(a) $\int_{1}^{3}$ g(x) dx
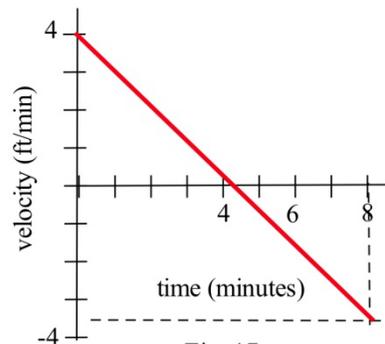(b) $\int_{3}^{4}$ g(x) dx

(c) $\int_{4}^{8}$ g(x) dx
(d) $\int_{1}^{8}$ g(x) dx

area = 3
area = 6
area = 1

20. Use the graph to evaluate:

(a) $\int_{-2}^{1}$ h(x) dx
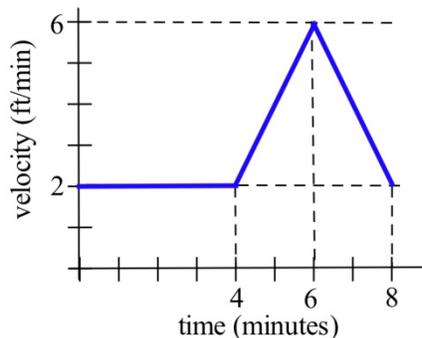(b) $\int_{4}^{6}$ h(x) dx

(c) $\int_{-2}^{6}$ h(x) dx
(d) $\int_{-2}^{4}$ h(x) dx

21. Your velocity along a straight road is shown to the right. How far did you travel in 8 minutes?

22. Your velocity along a straight road is shown below. How many feet did you walk in 8 minutes?

velocity (ft/min)

time (minutes)

velocity (ft/min)

time (minutes)

In problems 23 - 26, the units are given for x and for $f(x)$. Give the units of $\int_a^b f(x)\, dx$.

23. x is time in "seconds", and $f(x)$ is velocity in "meters per second."

24. x is time in "hours", and $f(x)$ is a flow rate in "gallons per hour."

25. x is a position in "feet", and $f(x)$ is an area in "square feet."

26. x is a position in "inches", and $f(x)$ is a density in "pounds per inch."

In problems 27 – 31, represent the area with a definite integral and use technology to find the approximate answer.
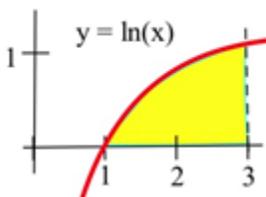
27. The region bounded by $y = x^3$, the x–axis, the line $x = 1$, and $x = 5$.

28. The region bounded by $y = \sqrt{x}$, the x–axis, and the line $x = 9$.

29. The shaded region shown to the right.
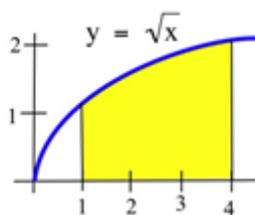
30. The shaded region below.

31. Consider the definite integral $\int_0^3 (3 + x)\, dx$.

   (a) Using six rectangles, find the left-hand Riemann sum for this definite integral.
   (b) Using six rectangles, find the right-hand Riemann sum for this definite integral.
   (c) Using geometry, find the exact value of this definite integral.

32. Consider the definite integral $\int_0^2 x^3\, dx$.

   (a) Using four rectangles, find the left-hand Riemann sum for this definite integral.
   (b) Using four rectangles, find the right-hand Riemann sum for this definite integral.

33. Write the total distance traveled by the car in the graph between 1 pm and 4 pm as a definite integral and estimate the value of the integral.
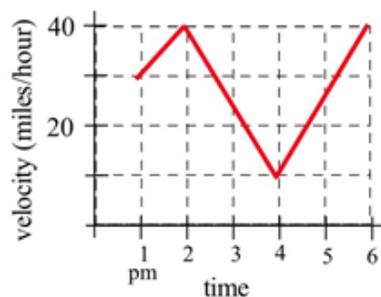
Problems 34 – 41 refer to the graph of f shown. Use the
graph to determine the values of the definite integrals.
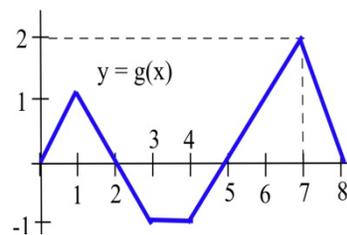(The bold numbers represent the **area** of each region.)
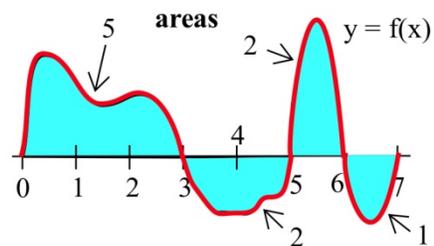
34. $\int_{0}^{3} f(x)\, dx$ 35. $\int_{3}^{5} f(x)\, dx$ 36. $\int_{2}^{2} f(x)\, dx$

37. $\int_{6}^{7} f(w)\, dw$ 38. $\int_{0}^{5} f(x)\, dx$ 39. $\int_{0}^{7} f(x)\, dx$ 40. $\int_{3}^{6} f(t)\, dt$ 41. $\int_{5}^{7} f(x)\, dx$

Problems 42 – 47 refer to the graph of g shown. Use the
graph to evaluate the integrals.

42. $\int_{0}^{2} g(x)\, dx$ 43. $\int_{1}^{3} g(t)\, dt$ 44. $\int_{0}^{5} g(x)\, dx$

45. $\int_{0}^{8} g(s)\, ds$ 46. $\int_{0}^{3} 2g(t)\, dt$ 47. $\int_{5}^{8} 1+g(x)\, dx$