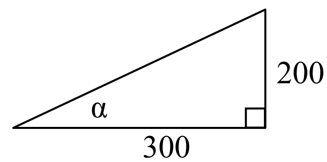
Section 6.5 Modeling with Trigonometric Functions

Solving right triangles for angles
In Section 5.5, we used trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trig functions, we can solve for the angles of a right triangle given two sides.

Example 1

An airplane needs to fly to an airfield located 300 miles east and 200 miles north of its current location. At what heading should the airplane fly? In other words, if we ignore air resistance or wind speed, how many degrees north of east should the airplane fly?



We might begin by drawing a picture and labeling all of the known information. Drawing a triangle, we see we are looking for the angle α. In this triangle, the side opposite the angle α is 200 miles and the side adjacent is 300 miles. Since we know the values for the opposite and adjacent sides, it makes sense to use the tangent function.

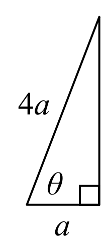
$\tan(\alpha) = \dfrac{200}{300}$ Using the inverse,

$\alpha = \tan^{-1}\left(\dfrac{200}{300}\right) \approx 0.588$, or equivalently about 33.7 degrees.

The airplane needs to fly at a heading of 33.7 degrees north of east.

Example 2

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length[3]. Find the angle such a ladder forms with the ground.



For any length of ladder, the base needs to be one quarter of the distance the foot of the ladder is away from the wall. Equivalently, if the base is a feet from the wall, the ladder can be $4a$ feet long. Since a is the side adjacent to the angle and $4a$ is the hypotenuse, we use the cosine function.

$\cos(\theta) = \dfrac{a}{4a} = \dfrac{1}{4}$ Using the inverse

$\theta = \cos^{-1}\left(\dfrac{1}{4}\right) \approx 75.52$ degrees

The ladder forms a 75.52 degree angle with the ground.

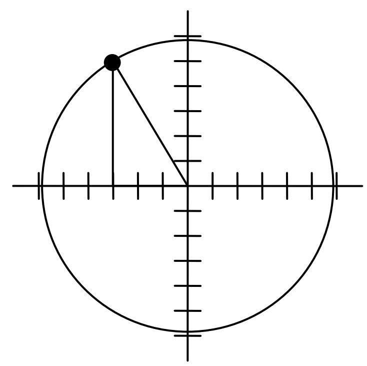
[3] http://www.osha.gov/SLTC/etools/construction/falls/4ladders.html

Try it Now

1. One of the cables that anchor the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground and the second anchor on the ground is 23 meters from the base of the Ferris wheel. What is the angle of elevation (from ground up to the center of the Ferris wheel) and how long is the cable?

Example 3

In a video game design, a map shows the location of other characters relative to the player, who is situated at the origin, and the direction they are facing. A character currently shows on the map at coordinates (-3, 5). If the player rotates counterclockwise by 20 degrees, then the objects in the map will correspondingly rotate 20 degrees clockwise. Find the new coordinates of the character.



To rotate the position of the character, we can imagine it as a point on a circle, and we will change the angle of the point by 20 degrees. To do so, we first need to find the radius of this circle and the original angle.

Drawing a right triangle inside the circle, we can find the radius using the Pythagorean Theorem:

$$(-3)^2 + 5^2 = r^2$$

$$r = \sqrt{9+25} = \sqrt{34}$$

To find the angle, we need to decide first if we are going to find the acute angle of the triangle, the reference angle, or if we are going to find the angle measured in standard position. While either approach will work, in this case we will do the latter. Since for any point on a circle we know $x = r\cos(\theta)$, using our given information we get

$$-3 = \sqrt{34}\cos(\theta)$$

$$\frac{-3}{\sqrt{34}} = \cos(\theta)$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{34}}\right) \approx 120.964°$$

While there are two angles that have this cosine value, the angle of 120.964 degrees is in the second quadrant as desired, so it is the angle we were looking for.

Rotating the point clockwise by 20 degrees, the angle of the point will decrease to 100.964 degrees. We can then evaluate the coordinates of the rotated point

$$x = \sqrt{34}\cos(100.964°) \approx -1.109$$

$$y = \sqrt{34}\sin(100.964°) \approx 5.725$$

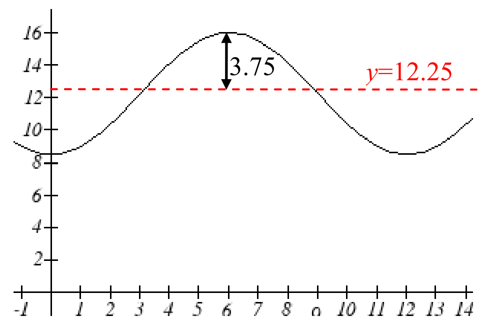
The coordinates of the character on the rotated map will be (-1.109, 5.725).

Modeling with sinusoidal functions
Many modeling situations involve functions that are periodic. Previously we learned that sinusoidal functions are a special type of periodic function. Problems that involve quantities that oscillate can often be modeled by a sine or cosine function and once we create a suitable model for the problem we can use that model to answer various questions.

Example 4
The hours of daylight in Seattle oscillate from a low of 8.5 hours in January to a high of 16 hours in July[4]. When should you plant a garden if you want to do it during a month where there are 14 hours of daylight?

To model this, we first note that the hours of daylight oscillate with a period of 12 months. With a low of 8.5 and a high of 16, the midline will be halfway between these values, at $\frac{16+8.5}{2}=12.25$. The amplitude will be half the difference between the highest and lowest values: $\frac{16-8.5}{2}=3.75$, or equivalently the distance from the midline to the high or low value, 16-12.25=3.75. Letting January be $t = 0$, the graph starts at the lowest value, so it can be modeled as a flipped cosine graph. Putting this together, we get a model:

$$h(t)=-3.75\cos\left(\frac{\pi}{6}t\right)+12.25$$



-cos(t) represents the flipped cosine,
3.75 is the amplitude,
12.25 is the midline,
$\frac{2\pi}{12}=\frac{\pi}{6}$ corresponds to the horizontal stretch, found by using the ratio of the "original period / new period"

h(t) is our model for hours of day light t months after January.

To find when there will be 14 hours of daylight, we solve $h(t) = 14$.

$14=-3.75\cos\left(\frac{\pi}{6}t\right)+12.25$ Isolating the cosine

$1.75=-3.75\cos\left(\frac{\pi}{6}t\right)$ Subtracting 12.25 and dividing by -3.75

$-\frac{1.75}{3.75}=\cos\left(\frac{\pi}{6}t\right)$ Using the inverse

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[4] http://www.mountaineers.org/seattle/climbing/Reference/DaylightHrs.html

$$\frac{\pi}{6}t = \cos^{-1}\left(-\frac{1.75}{3.75}\right) \approx 2.0563$$ multiplying by the reciprocal

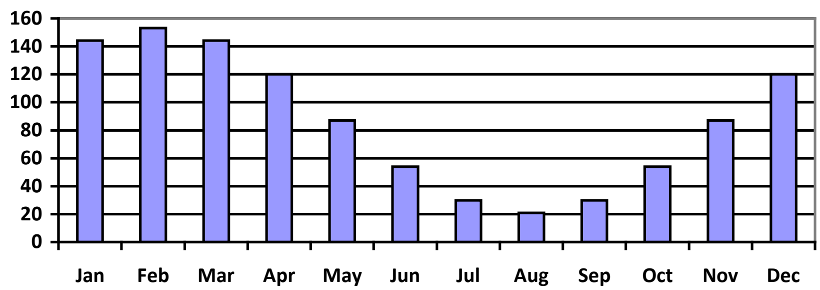
$$t = 2.0563 \cdot \frac{6}{\pi} = 3.927$$ t=3.927 months past January

There will be 14 hours of daylight 3.927 months into the year, or near the end of April.

While there would be a second time in the year when there are 14 hours of daylight, since we are planting a garden, we would want to know the first solution, in spring, so we do not need to find the second solution in this case.

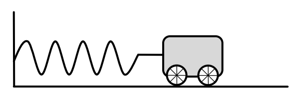
Try it Now

2. The author's monthly gas usage (in therms) is shown here. Find a function to model the data.



Example 6

An object is connected to the wall with a spring that has a natural length of 20 cm. The object is pulled back 8 cm past the natural length and released. The object oscillates 3 times per second. Find an equation for the horizontal position of the object ignoring the effects of friction. How much time during each cycle is the object more than 27 cm from the wall?



If we use the distance from the wall, x, as the desired output, then the object will oscillate equally on either side of the spring's natural length of 20, putting the midline of the function at 20 cm.

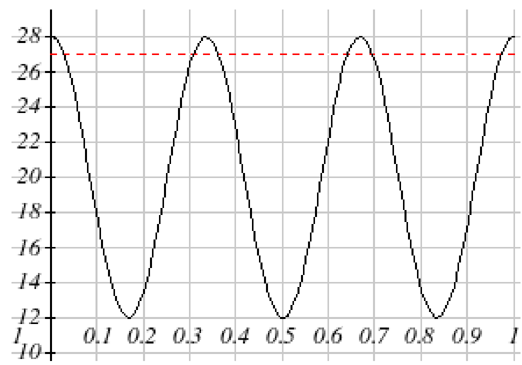
If we release the object 8 cm past the natural length, the amplitude of the oscillation will be 8 cm.

We are beginning at the largest value and so this function can most easily be modeled using a cosine function.

Since the object oscillates 3 times per second, it has a frequency of 3 and the period of one oscillation is 1/3 of second. Using this we find the horizontal compression using the ratios of the periods: $\frac{2\pi}{1/3} = 6\pi$.

Using all this, we can build our model:
$x(t) = 8\cos(6\pi t) + 20$

To find when the object is 27 cm from the wall, we can solve *x(t)* = 27
$27 = 8\cos(6\pi t) + 20$ Isolating the cosine
$7 = 8\cos(6\pi t)$
$\frac{7}{8} = \cos(6\pi t)$ Using the inverse
$6\pi t = \cos^{-1}\left(\frac{7}{8}\right) \approx 0.505$
$t = \frac{0.505}{6\pi} = 0.0268$

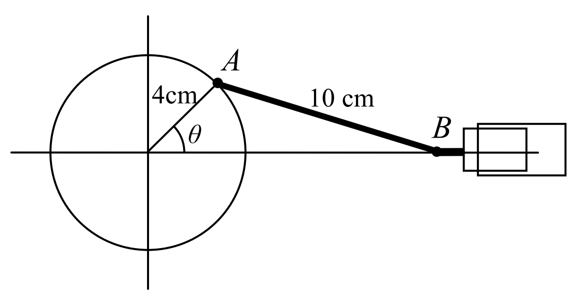


Based on the shape of the graph, we can conclude that the object will spend the first 0.0268 seconds more than 27 cm from the wall. Based on the symmetry of the function, the object will spend another 0.0268 seconds more than 27 cm from the wall at the end of the cycle. Altogether, the object spends 0.0536 seconds each cycle at a distance greater than 27 cm from the wall.

In some problems, we can use trigonometric functions to model behaviors more complicated than the basic sinusoidal function.

Example 7

A rigid rod with length 10 cm is attached to a circle of radius 4cm at point *A* as shown here. The point *B* is able to freely move along the horizontal axis, driving a piston[5]. If the wheel rotates counterclockwise at 5 revolutions per second, find the location of point *B* as a function of time. When will the point *B* be 12 cm from the center of the circle?



To find the position of point *B*, we can begin by finding the coordinates of point *A*. Since it is a point on a circle with radius 4, we can express its coordinates as $(4\cos(\theta), 4\sin(\theta))$, where θ is the angle shown.

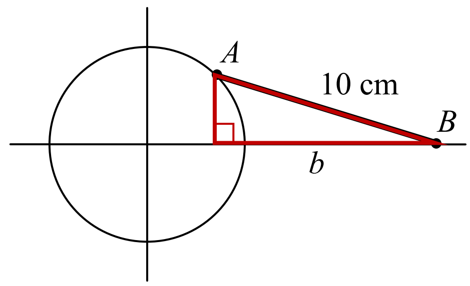
[5] For an animation of this situation, see http://www.mathdemos.org/mathdemos/sinusoidapp/engine1.gif

The angular velocity is 5 revolutions per second, or equivalently 10π radians per second. After t seconds, the wheel will rotate by $\theta = 10\pi t$ radians. Substituting this, we can find the coordinates of A in terms of t.
$(4\cos(10\pi t), 4\sin(10\pi t))$

Notice that this is the same value we would have obtained by observing that the period of the rotation is 1/5 of a second and calculating the stretch/compression factor:

$$\frac{\text{"original"}}{\text{"new"}}\frac{2\pi}{1/5} = 10\pi .$$

Now that we have the coordinates of the point A, we can relate this to the point B. By drawing a vertical line segment from A to the horizontal axis, we can form a right triangle. The height of the triangle is the y coordinate of the point A: $4\sin(10\pi t)$. Using the Pythagorean Theorem, we can find the base length of the triangle:



$$\left(4\sin(10\pi t)\right)^2 + b^2 = 10^2$$
$$b^2 = 100 - 16\sin^2(10\pi t)$$
$$b = \sqrt{100 - 16\sin^2(10\pi t)}$$

Looking at the x coordinate of the point A, we can see that the triangle we drew is shifted to the right of the y axis by $4\cos(10\pi t)$. Combining this offset with the length of the base of the triangle gives the x coordinate of the point B:
$$x(t) = 4\cos(10\pi t) + \sqrt{100 - 16\sin^2(10\pi t)}$$

To solve for when the point B will be 12 cm from the center of the circle, we need to solve $x(t) = 12$.

$12 = 4\cos(10\pi t) + \sqrt{100 - 16\sin^2(10\pi t)}$	Isolate the square root
$12 - 4\cos(10\pi t) = \sqrt{100 - 16\sin^2(10\pi t)}$	Square both sides
$\left(12 - 4\cos(10\pi t)\right)^2 = 100 - 16\sin^2(10\pi t)$	Expand the left side
$144 - 96\cos(10\pi t) + 16\cos^2(10\pi t) = 100 - 16\sin^2(10\pi t)$	Move all terms to the left
$44 - 96\cos(10\pi t) + 16\cos^2(10\pi t) + 16\sin^2(10\pi t) = 0$	Factor out 16
$44 - 96\cos(10\pi t) + 16\left(\cos^2(10\pi t) + \sin^2(10\pi t)\right) = 0$	

At this point, we can utilize the Pythagorean Identity, which tells us that $\cos^2(10\pi t) + \sin^2(10\pi t) = 1$.

Using this identity, our equation simplifies to

$44 - 96\cos(10\pi t) + 16 = 0$	Combine the constants and move to the right side
$-96\cos(10\pi t) = -60$	Divide
$\cos(10\pi t) = \frac{60}{96}$	Make a substitution
$\cos(u) = \frac{60}{96}$	
$u = \cos^{-1}\left(\frac{60}{96}\right) \approx 0.896$	By symmetry we can find a second solution
$u = 2\pi - 0.896 = 5.388$	Undoing the substitution
$10\pi t = 0.896$, so $t = 0.0285$	
$10\pi t = 5.388$, so $t = 0.1715$	

The point B will be 12 cm from the center of the circle 0.0285 seconds after the process begins, 0.1715 seconds after the process begins, and every 1/5 of a second after each of those values.

Important Topics of This Section

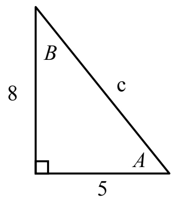
Modeling with trig equations
Modeling with sinusoidal functions
Solving right triangles for angles in degrees and radians

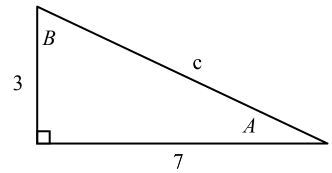
Try it Now Answers

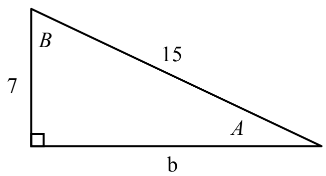
1. Angle of elevation for the cable is 71.69 degrees and the cable is 73.21 m long
2. Approximately $G(t) = 66\cos\left(\frac{\pi}{6}(t-1)\right) + 87$

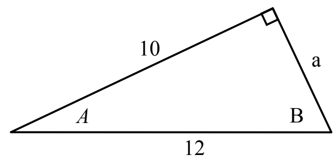
Section 6.5 Exercises

In each of the following triangles, solve for the unknown side and angles.

1. 

2. 

3. 

4. 

Find a possible formula for the trigonometric function whose values are in the following tables.

5.

x	0	1	2	3	4	5	6
y	-2	4	10	4	-2	4	10

6.

x	0	1	2	3	4	5	6
y	1	-3	-7	-3	1	-3	-7

7. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature for the day is 63 degrees and the low temperature of 37 degrees occurs at 5 AM. Assuming t is the number of hours since midnight, find an equation for the temperature, D, in terms of t.

8. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature for the day is 92 degrees and the low temperature of 78 degrees occurs at 4 AM. Assuming t is the number of hours since midnight, find an equation for the temperature, D, in terms of t.

9. A population of rabbits oscillates 25 above and below an average of 129 during the year, hitting the lowest value in January ($t = 0$).
   a. Find an equation for the population, P, in terms of the months since January, t.
   b. What if the lowest value of the rabbit population occurred in April instead?

10. A population of elk oscillates 150 above and below an average of 720 during the year, hitting the lowest value in January ($t = 0$).
    a. Find an equation for the population, P, in terms of the months since January, t.
    b. What if the lowest value of the rabbit population occurred in March instead?

11. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature of 105 degrees occurs at 5 PM and the average temperature for the day is 85 degrees. Find the temperature, to the nearest degree, at 9 AM.

12. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature of 84 degrees occurs at 6 PM and the average temperature for the day is 70 degrees. Find the temperature, to the nearest degree, at 7 AM.

13. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature varies between 47 and 63 degrees during the day and the average daily temperature first occurs at 10 AM. How many hours after midnight does the temperature first reach 51 degrees?

14. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature varies between 64 and 86 degrees during the day and the average daily temperature first occurs at 12 AM. How many hours after midnight does the temperature first reach 70 degrees?

15. A Ferris wheel is 20 meters in diameter and boarded from a platform that is 2 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 6 minutes. How many minutes of the ride are spent higher than 13 meters above the ground?

16. A Ferris wheel is 45 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. How many minutes of the ride are spent higher than 27 meters above the ground?

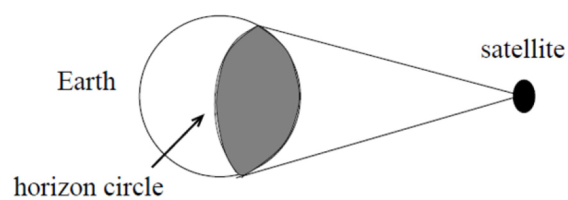
17. The sea ice area around the North Pole fluctuates between about 6 million square kilometers in September to 14 million square kilometers in March. Assuming sinusoidal fluctuation, during how many months are there less than 9 million square kilometers of sea ice?

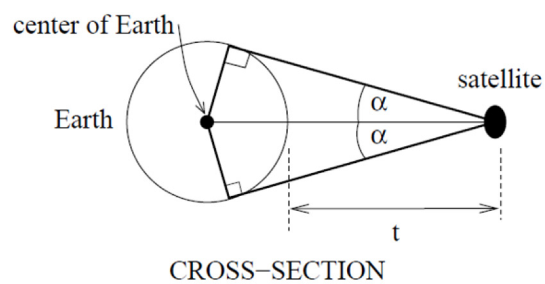
18. The sea ice area around the South Pole fluctuates between about 18 million square kilometers in September to 3 million square kilometers in March. Assuming sinusoidal fluctuation, during how many months are there more than 15 million square kilometers of sea ice?

19. A respiratory ailment called "Cheyne-Stokes Respiration" causes the volume per breath to increase and decrease in a sinusoidal manner, as a function of time. For one particular patient with this condition, a machine begins recording a plot of volume per breath versus time (in seconds). Let $b(t)$ be a function of time t that tells us the volume (in liters) of a breath that starts at time t. During the test, the smallest volume per breath is 0.6 liters and this first occurs for a breath that starts 5 seconds into the test. The largest volume per breath is 1.8 liters and this first occurs for a breath beginning 55 seconds into the test. [UW]
    a. Find a formula for the function $b(t)$ whose graph will model the test data for this patient.
    b. If the patient begins a breath every 5 seconds, what are the breath volumes during the first minute of the test?

20. Suppose the high tide in Seattle occurs at 1:00 a.m. and 1:00 p.m, at which time the water is 10 feet above the height of low tide. Low tides occur 6 hours after high tides. Suppose there are two high tides and two low tides every day and the height of the tide varies sinusoidally. [UW]
    a. Find a formula for the function $y = h(t)$ that computes the height of the tide above low tide at time t. (In other words, $y = 0$ corresponds to low tide.)
    b. What is the tide height at 11:00 a.m.?

21. A communications satellite orbits the earth t miles above the surface. Assume the radius of the earth is 3,960 miles. The satellite can only "see" a portion of the earth's surface, bounded by what is called a horizon circle. This leads to a two-dimensional cross-sectional picture we can use to study the size of the horizon slice: [UW]





    a. Find a formula for α in terms of t.
    b. If t = 30,000 miles, what is α? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?
    c. If t = 1,000 miles, what is α? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?
    d. Suppose you wish to place a satellite into orbit so that 20% of the circumference is covered by the satellite. What is the required distance t?

22. Tiffany is a model rocket enthusiast. She has been working on a pressurized rocket filled with nitrous oxide. According to her design, if the atmospheric pressure exerted on the rocket is less than 10 pounds/sq.in., the nitrous oxide chamber inside the rocket will explode. Tiff worked from a formula $p = 14.7e^{-h/10}$ pounds/sq.in. for the atmospheric pressure h miles above sea level. Assume that the rocket is launched at an angle of α above level ground at sea level with an initial speed of 1400 feet/sec. Also, assume the height (in feet) of the rocket at time t seconds is given by the equation $y(t) = -16t^2 + 1400\sin(\alpha)t$. [UW]
    a. At what altitude will the rocket explode?
    b. If the angle of launch is α = 12°, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?
    c. If the angle of launch is α = 82°, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?
    d. Find the largest launch angle α so that the rocket will not explode.