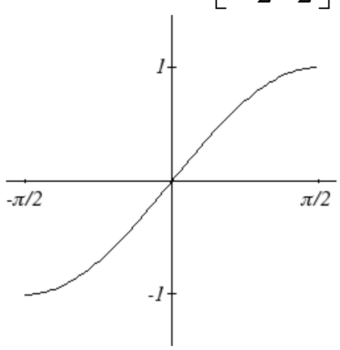
Section 6.3 Inverse Trig Functions

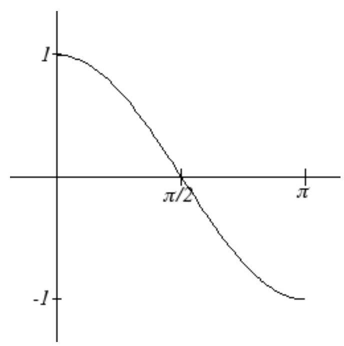
In previous sections we have evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

You probably are already recognizing an issue – that the sine, cosine, and tangent functions are not one-to-one functions. To define an inverse of these functions, we will need to restrict the domain of these functions to yield a new function that is one-to-one. We choose a domain for each function that includes the angle zero.

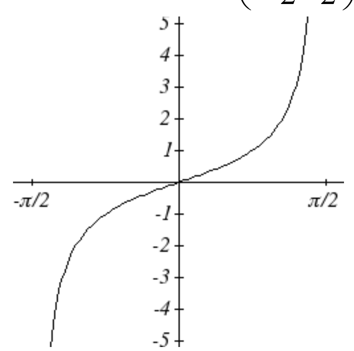
Sine, limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Cosine, limited to $[0, \pi]$



Tangent, limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



On these restricted domains, we can define the inverse sine, inverse cosine, and inverse tangent functions.

Inverse Sine, Cosine, and Tangent Functions

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin(\theta) = a$, then $\sin^{-1}(a) = \theta$

For angles in the interval $[0, \pi]$, if $\cos(\theta) = a$, then $\cos^{-1}(a) = \theta$

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan(\theta) = a$, then $\tan^{-1}(a) = \theta$

$\sin^{-1}(x)$ has domain [-1, 1] and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

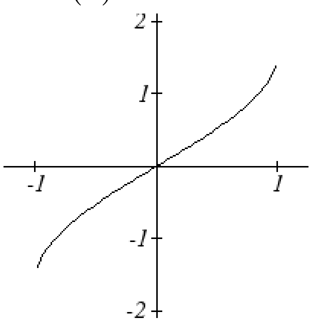
$\cos^{-1}(x)$ has domain [-1, 1] and range $[0, \pi]$

$\tan^{-1}(x)$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

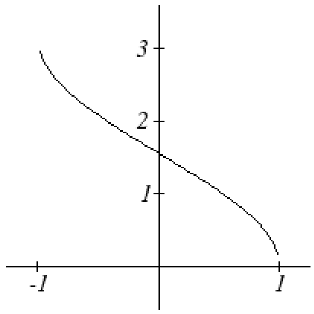
The $\sin^{-1}(x)$ is sometimes called the **arcsine** function, and notated $\arcsin(a)$.
The $\cos^{-1}(x)$ is sometimes called the **arccosine** function, and notated $\arccos(a)$.
The $\tan^{-1}(x)$ is sometimes called the **arctangent** function, and notated $\arctan(a)$.

The graphs of the inverse functions are shown here:

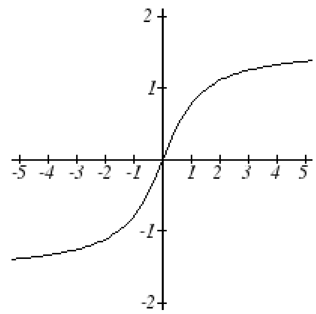
$\sin^{-1}(x)$



$\cos^{-1}(x)$



$\tan^{-1}(x)$



Notice that the output of each of these inverse functions is an *angle*.

Example 1

Evaluate

a) $\sin^{-1}\left(\frac{1}{2}\right)$ b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ d) $\tan^{-1}(1)$

a) Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as asking what angle would have a sine value of $\frac{1}{2}$. In other words, what angle θ would satisfy $\sin(\theta)=\frac{1}{2}$? There are multiple angles that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$. Remember that the inverse is a *function* so for each input, we will get exactly one output.

b) Evaluating $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$.

c) Evaluating $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0,\pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$.

d) Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1)=\frac{\pi}{4}$.

Try It Now

1. Evaluate

a) $\sin^{-1}(-1)$ b) $\tan^{-1}(-1)$ c) $\cos^{-1}(-1)$ d) $\cos^{-1}\left(\frac{1}{2}\right)$

Example 2

Evaluate $\sin^{-1}(0.97)$ using your calculator.

Since the output of the inverse function is an angle, your calculator will give you a degree value if in degree mode, and a radian value if in radian mode.

In radian mode, $\sin^{-1}(0.97)\approx 1.3252$ In degree mode, $\sin^{-1}(0.97)\approx 75.93°$

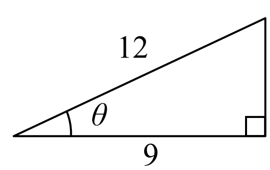
Try it Now

2. Evaluate $\cos^{-1}(-0.4)$ using your calculator.

In Section 5.5, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trig functions, we can solve for the angles of a right triangle given two sides.

Example 3

Solve the triangle for the angle θ.



Since we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$\cos(\theta) = \dfrac{9}{12}$ Using the definition of the inverse,

$\theta = \cos^{-1}\left(\dfrac{9}{12}\right)$ Evaluating

$\theta \approx 0.7227$, or about $41.4096°$

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can find exact values for the resulting expressions

Example 4

Evaluate $\sin^{-1}\left(\cos\left(\dfrac{13\pi}{6}\right)\right)$.

a) Here, we can directly evaluate the inside of the composition.

$$\cos\left(\frac{13\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Try it Now

3. Evaluate $\cos^{-1}\left(\sin\left(-\dfrac{11\pi}{4}\right)\right)$.

Example 5

Find an exact value for $\sin\left(\cos^{-1}\left(\dfrac{4}{5}\right)\right)$.

Beginning with the inside, we can say there is some angle so $\theta = \cos^{-1}\left(\dfrac{4}{5}\right)$, which means $\cos(\theta) = \dfrac{4}{5}$, and we are looking for $\sin(\theta)$. We can use the Pythagorean identity to do this.

$\sin^2(\theta)+\cos^2(\theta)=1$ Using our known value for cosine

$\sin^2(\theta)+\left(\frac{4}{5}\right)^2=1$ Solving for sine

$\sin^2(\theta)=1-\frac{16}{25}$

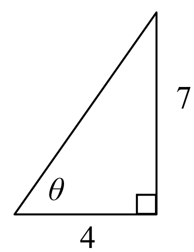
$\sin(\theta)=\pm\sqrt{\frac{9}{25}}=\pm\frac{3}{5}$

Since we know that the inverse cosine always gives an angle on the interval $[0,\pi]$, we know that the sine of that angle must be positive, so $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)=\sin(\theta)=\frac{3}{5}$

Example 6

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

While we could use a similar technique as in the last example, we will demonstrate a different technique here. From the inside, we know there is an angle so $\tan(\theta)=\frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle.



Using the Pythagorean Theorem, we can find the hypotenuse of this triangle:

$4^2+7^2=hypotenuse^2$

$hypotenuse=\sqrt{65}$

Now, we can evaluate the sine of the angle as opposite side divided by hypotenuse

$\sin(\theta)=\frac{7}{\sqrt{65}}$

This gives us our desired composition

$\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)=\sin(\theta)=\frac{7}{\sqrt{65}}$.

Try it Now

4. Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

We can also find compositions involving algebraic expressions.

Example 7

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$, for $-3 \le x \le 3$.

We know there is an angle θ so that $\sin(\theta) = \frac{x}{3}$. Using the Pythagorean Theorem,

$\sin^2(\theta) + \cos^2(\theta) = 1$ Using our known expression for sine

$\left(\frac{x}{3}\right)^2 + \cos^2(\theta) = 1$ Solving for cosine

$\cos^2(\theta) = 1 - \frac{x^2}{9}$

$\cos(\theta) = \pm\sqrt{\frac{9 - x^2}{9}} = \pm\frac{\sqrt{9 - x^2}}{3}$

Since we know that the inverse sine must give an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can deduce that the cosine of that angle must be positive. This gives us

$$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9 - x^2}}{3}$$

Try it Now

5. Find a simplified expression for $\sin\left(\tan^{-1}(4x)\right)$, for $-\frac{1}{4} \le x \le \frac{1}{4}$.

Important Topics of This Section

Inverse trig functions: arcsine, arccosine and arctangent
Domain restrictions
Evaluating inverses using unit circle values and the calculator
Simplifying numerical expressions involving the inverse trig functions
Simplifying algebraic expressions involving the inverse trig functions

Try it Now Answers

1. a) $-\frac{\pi}{2}$ b) $-\frac{\pi}{4}$ c) π d) $\frac{\pi}{3}$

2. 1.9823 or 113.578°

3. $\frac{3\pi}{4}$

4. $\frac{4\sqrt{2}}{9}$

5. $\frac{4x}{\sqrt{16x^2+1}}$

Section 6.3 Exercises

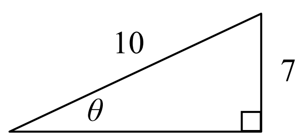
Evaluate the following expressions.

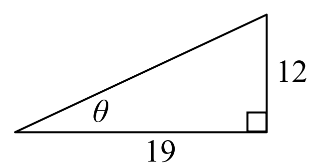
1. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
2. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
3. $\sin^{-1}\left(-\frac{1}{2}\right)$
4. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
5. $\cos^{-1}\left(\frac{1}{2}\right)$
6. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
7. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
8. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
9. $\tan^{-1}(1)$
10. $\tan^{-1}\left(\sqrt{3}\right)$
11. $\tan^{-1}\left(-\sqrt{3}\right)$
12. $\tan^{-1}(-1)$

Use your calculator to evaluate each expression.

13. $\cos^{-1}(-0.4)$
14. $\cos^{-1}(0.8)$
15. $\sin^{-1}(-0.8)$
16. $\tan^{-1}(6)$

Find the angle θ.

17. 

18. 

Evaluate the following expressions.

19. $\sin^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)$
20. $\cos^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$
21. $\sin^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$
22. $\cos^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$
23. $\cos\left(\sin^{-1}\left(\frac{3}{7}\right)\right)$
24. $\sin\left(\cos^{-1}\left(\frac{4}{9}\right)\right)$
25. $\cos\left(\tan^{-1}(4)\right)$
26. $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

Find a simplified expression for each of the following.

27. $\sin\left(\cos^{-1}\left(\frac{x}{5}\right)\right)$, for $-5 \le x \le 5$
28. $\tan\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$, for $-2 \le x \le 2$
29. $\sin\left(\tan^{-1}(3x)\right)$
30. $\cos\left(\tan^{-1}(4x)\right)$