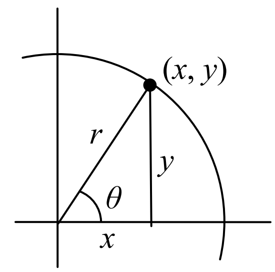
## *Section 5.4 The Other Trigonometric Functions*

In the previous section, we defined the sine and cosine functions as ratios of the sides of a right triangle in a circle. Since the triangle has 3 sides there are 6 possible combinations of ratios. While the sine and cosine are the two prominent ratios that can be formed, there are four others, and together they define the 6 trigonometric functions.

**Tangent, Secant, Cosecant, and Cotangent Functions**

For the point $(x, y)$ on a circle of radius $r$ at an angle of $\theta$, we can define four additional important functions as the ratios of the sides of the corresponding triangle:




The **tangent** function: $\tan(\theta) = \dfrac{y}{x}$

The **secant** function: $\sec(\theta) = \dfrac{r}{x}$

The **cosecant** function: $\csc(\theta) = \dfrac{r}{y}$

The **cotangent** function: $\cot(\theta) = \dfrac{x}{y}$

Geometrically, notice that the definition of tangent corresponds with the slope of the line segment between the origin (0, 0) and the point $(x, y)$. This relationship can be very helpful in thinking about tangent values.

You may also notice that the ratios defining the secant, cosecant, and cotangent are the reciprocals of the ratios defining the cosine, sine, and tangent functions, respectively. Additionally, notice that using our results from the last section,

$$\tan(\theta) = \frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)}$$

Applying this concept to the other trig functions we can state the other reciprocal identities.

**Identities**

The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

These relationships are called **identities**. Identities are statements that are true for all values of the input on which they are defined. Identities are usually something that can be derived from definitions and relationships we already know, similar to how the identities above were derived from the circle relationships of the six trig functions. The Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine. We will discuss the role of identities more after an example.

### Example 1

Evaluate $\tan(45°)$ and $\sec\left(\frac{5\pi}{6}\right)$.

Since we know the sine and cosine values for these angles, it makes sense to relate the tangent and secant values back to the sine and cosine values.

$$\tan(45°) = \frac{\sin(45°)}{\cos(45°)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

Notice this result is consistent with our interpretation of the tangent value as the slope of the line passing through the origin at the given angle: a line at 45 degrees would indeed have a slope of 1.

$$\sec\left(\frac{5\pi}{6}\right) = \frac{1}{\cos\left(\frac{5\pi}{6}\right)} = \frac{1}{-\sqrt{3}/2} = \frac{-2}{\sqrt{3}}, \text{ which could also be written as } \frac{-2\sqrt{3}}{3}.$$

### Try it Now

1. Evaluate $\csc\left(\frac{7\pi}{6}\right)$.

Just as we often need to simplify algebraic expressions, it is often also necessary or helpful to simplify trigonometric expressions. To do so, we utilize the definitions and identities we have established.

**Example 2**

Simplify $\dfrac{\sec(\theta)}{\tan(\theta)}$.

We can simplify this by rewriting both functions in terms of sine and cosine

$\dfrac{\sec(\theta)}{\tan(\theta)} = \dfrac{1/\cos(\theta)}{\sin(\theta)/\cos(\theta)}$ To divide the fractions we could invert and multiply

$= \dfrac{1}{\cos(\theta)}\dfrac{\cos(\theta)}{\sin(\theta)}$ cancelling the cosines,

$= \dfrac{1}{\sin(\theta)} = \csc(\theta)$ simplifying and using the identity

By showing that $\dfrac{\sec(\theta)}{\tan(\theta)}$ can be simplified to $\csc(\theta)$, we have, in fact, established a new identity: that $\dfrac{\sec(\theta)}{\tan(\theta)} = \csc(\theta)$.

Occasionally a question may ask you to "prove the identity" or "establish the identity." This is the same idea as when an algebra book asks a question like "show that $(x-1)^2 = x^2 - 2x + 1$." In this type of question we must show the algebraic manipulations that demonstrate that the left and right side of the equation are in fact equal. You can think of a "prove the identity" problem as a simplification problem where you *know the answer*: you know what the end goal of the simplification should be, and just need to show the steps to get there.

To prove an identity, in most cases you will start with the expression on one side of the identity and manipulate it using algebra and trigonometric identities until you have simplified it to the expression on the other side of the equation. **Do not** treat the identity like an equation to solve – it isn't! The proof is establishing *if* the two expressions are equal, so we must take care to work with one side at a time rather than applying an operation simultaneously to both sides of the equation.

**Example 3**

Prove the identity $\dfrac{1+\cot(\alpha)}{\csc(\alpha)} = \sin(\alpha) + \cos(\alpha)$.

Since the left side seems a bit more complicated, we will start there and simplify the expression until we obtain the right side. We can use the right side as a guide for what might be good steps to make. In this case, the left side involves a fraction while the right side doesn't, which suggests we should look to see if the fraction can be reduced.

Additionally, since the right side involves sine and cosine and the left does not, it suggests that rewriting the cotangent and cosecant using sine and cosine might be a good idea.

$\dfrac{1+\cot(\alpha)}{\csc(\alpha)}$ Rewriting the cotangent and cosecant

$= \dfrac{1+\dfrac{\cos(\alpha)}{\sin(\alpha)}}{\dfrac{1}{\sin(\alpha)}}$ To divide the fractions, we invert and multiply

$= \left(1+\dfrac{\cos(\alpha)}{\sin(\alpha)}\right)\dfrac{\sin(\alpha)}{1}$ Distributing,

$= 1\cdot\dfrac{\sin(\alpha)}{1}+\dfrac{\cos(\alpha)}{\sin(\alpha)}\cdot\dfrac{\sin(\alpha)}{1}$ Simplifying the fractions,

$= \sin(\alpha)+\cos(\alpha)$ Establishing the identity.

Notice that in the second step, we could have combined the 1 and $\dfrac{\cos(\alpha)}{\sin(\alpha)}$ before inverting and multiplying. It is very common when proving or simplifying identities for there to be more than one way to obtain the same result.

We can also utilize identities we have previously learned, like the Pythagorean Identity, while simplifying or proving identities.

### Example 4

Establish the identity $\dfrac{\cos^2(\theta)}{1+\sin(\theta)} = 1-\sin(\theta)$.

Since the left side of the identity is more complicated, it makes sense to start there. To simplify this, we will have to reduce the fraction, which would require the numerator to have a factor in common with the denominator. Additionally, we notice that the right side only involves sine. Both of these suggest that we need to convert the cosine into something involving sine.

Recall the Pythagorean Identity told us $\cos^2(\theta)+\sin^2(\theta)=1$. By moving one of the trig functions to the other side, we can establish:

$\sin^2(\theta)=1-\cos^2(\theta)$ and $\cos^2(\theta)=1-\sin^2(\theta)$

Utilizing this, we now can establish the identity. We start on one side and manipulate:

$\frac{\cos^2(\theta)}{1+\sin(\theta)}$ Utilizing the Pythagorean Identity

$= \frac{1-\sin^2(\theta)}{1+\sin(\theta)}$ Factoring the numerator

$= \frac{(1-\sin(\theta))(1+\sin(\theta))}{1+\sin(\theta)}$ Cancelling the like factors

$= 1-\sin(\theta)$ Establishing the identity

We can also build new identities from previously established identities. For example, if we divide both sides of the Pythagorean Identity by cosine squared (which is allowed since we've already shown the identity is true),

$\frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$ Splitting the fraction on the left,

$\frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$ Simplifying and using the definitions of tan and sec

$1+\tan^2(\theta) = \sec^2(\theta)$.

## Try it Now

2. Use a similar approach to establish that $\cot^2(\theta)+1 = \csc^2(\theta)$.

## Identities

**Alternate forms of the Pythagorean Identity**

$1+\tan^2(\theta) = \sec^2(\theta)$

$\cot^2(\theta)+1 = \csc^2(\theta)$

## Example 5

If $\tan(\theta) = \frac{2}{7}$ and $\theta$ is in the 3rd quadrant, find $\cos(\theta)$.

There are two approaches to this problem, both of which work equally well.

Approach 1

Since $\tan(\theta) = \frac{y}{x}$ and the angle is in the third quadrant, we can imagine a triangle in a circle of some radius so that the point on the circle is (-7, -2). Using the Pythagorean Theorem, we can find the radius of the circle: $(-7)^2 + (-2)^2 = r^2$, so $r = \sqrt{53}$.

Now we can find the cosine value:

$$\cos(\theta) = \frac{x}{r} = \frac{-7}{\sqrt{53}}$$

Approach 2

Using the $1 + \tan^2(\theta) = \sec^2(\theta)$ form of the Pythagorean Identity with the known tangent value,

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + \left(\frac{2}{7}\right)^2 = \sec^2(\theta)$$

$$\frac{53}{49} = \sec^2(\theta)$$

$$\sec(\theta) = \pm\sqrt{\frac{53}{49}} = \pm\frac{\sqrt{53}}{7}$$

Since the angle is in the third quadrant, the cosine value will be negative so the secant value will also be negative. Keeping the negative result, and using definition of secant,

$$\sec(\theta) = -\frac{\sqrt{53}}{7}$$

$$\frac{1}{\cos(\theta)} = -\frac{\sqrt{53}}{7} \qquad \text{Inverting both sides}$$

$$\cos(\theta) = -\frac{7}{\sqrt{53}} = -\frac{7\sqrt{53}}{53}$$

## Try it Now

3. If $\sec(\phi) = -\frac{7}{3}$ and $\frac{\pi}{2} < \phi < \pi$, find $\tan(\phi)$ and $\sin(\phi)$.

## Important Topics of This Section

6 Trigonometric Functions:
- Sine
- Cosine
- Tangent
- Cosecant
- Secant
- Cotangent

Trig identities

## Try it Now Answers

1. -2

2.

$$\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2\theta}=1$$

$$\frac{\cos^2(\theta)}{\sin^2(\theta)}+\frac{\sin^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)}$$

$$\cot^2(\theta)+1=\csc^2(\theta)$$

3. $\sin(\phi)=\dfrac{\sqrt{40}}{7}$ $\quad\tan(\phi)=\dfrac{\sqrt{40}}{-3}$

## Section 5.4 Exercises

1. If $\theta = \frac{\pi}{4}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

2. If $\theta = \frac{7\pi}{4}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

3. If $\theta = \frac{5\pi}{6}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

4. If $\theta = \frac{\pi}{6}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

5. If $\theta = \frac{2\pi}{3}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

6. If $\theta = \frac{4\pi}{3}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

7. Evaluate: a. $\sec(135°)$ b. $\csc(210°)$ c. $\tan(60°)$ d. $\cot(225°)$

8. Evaluate: a. $\sec(30°)$ b. $\csc(315°)$ c. $\tan(135°)$ d. $\cot(150°)$

9. If $\sin(\theta) = \frac{3}{4}$, and $\theta$ is in quadrant II, find $\cos(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

10. If $\sin(\theta) = \frac{2}{7}$, and $\theta$ is in quadrant II, find $\cos(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

11. If $\cos(\theta) = -\frac{1}{3}$, and $\theta$ is in quadrant III, find $\sin(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

12. If $\cos(\theta) = \frac{1}{5}$, and $\theta$ is in quadrant I, find $\sin(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

13. If $\tan(\theta) = \frac{12}{5}$, and $0 \le \theta < \frac{\pi}{2}$, find $\sin(\theta), \cos(\theta), \sec(\theta), \csc(\theta), \cot(\theta)$.

14. If $\tan(\theta) = 4$, and $0 \le \theta < \frac{\pi}{2}$, find $\sin(\theta), \cos(\theta), \sec(\theta), \csc(\theta), \cot(\theta)$.

15. Use a calculator to find sine, cosine, and tangent of the following values:

a. 0.15 b. 4 c. 70° d. 283°

16. Use a calculator to find sine, cosine, and tangent of the following values:

a. 0.5 b. 5.2 c. 10° d. 195°

Simplify each of the following to an expression involving a single trig function with no fractions.

17. $\csc(t)\tan(t)$

18. $\cos(t)\csc(t)$

19. $\dfrac{\sec(t)}{\csc(t)}$

20. $\dfrac{\cot(t)}{\csc(t)}$

21. $\dfrac{\sec(t)-\cos(t)}{\sin(t)}$

22. $\dfrac{\tan(t)}{\sec(t)-\cos(t)}$

23. $\dfrac{1+\cot(t)}{1+\tan(t)}$

24. $\dfrac{1+\sin(t)}{1+\csc(t)}$

25. $\dfrac{\sin^2(t)+\cos^2(t)}{\cos^2(t)}$

26. $\dfrac{1-\sin^2(t)}{\sin^2(t)}$

Prove the identities.

27. $\dfrac{\sin^2(\theta)}{1+\cos(\theta)} = 1-\cos(\theta)$

28. $\tan^2(t) = \dfrac{1}{\cos^2(t)} - 1$

29. $\sec(a) - \cos(a) = \sin(a)\tan(a)$

30. $\dfrac{1+\tan^2(b)}{\tan^2(b)} = \csc^2(b)$

31. $\dfrac{\csc^2(x)-\sin^2(x)}{\csc(x)+\sin(x)} = \cos(x)\cot(x)$

32. $\dfrac{\sin(\theta)-\cos(\theta)}{\sec(\theta)-\csc(\theta)} = \sin(\theta)\cos(\theta)$

33. $\dfrac{\csc^2(\alpha)-1}{\csc^2(\alpha)-\csc(\alpha)} = 1+\sin(\alpha)$

34. $1+\cot(x) = \cos(x)\left(\sec(x)+\csc(x)\right)$

35. $\dfrac{1+\cos(u)}{\sin(u)} = \dfrac{\sin(u)}{1-\cos(u)}$

36. $2\sec^2(t) = \dfrac{1-\sin(t)}{\cos^2(t)} + \dfrac{1}{1-\sin(t)}$

37. $\dfrac{\sin^4(\gamma)-\cos^4(\gamma)}{\sin(\gamma)-\cos(\gamma)} = \sin(\gamma)+\cos(\gamma)$

38. $\dfrac{\left(1+\cos(A)\right)\left(1-\cos(A)\right)}{\sin(A)} = \sin(A)$