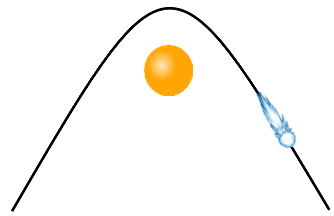
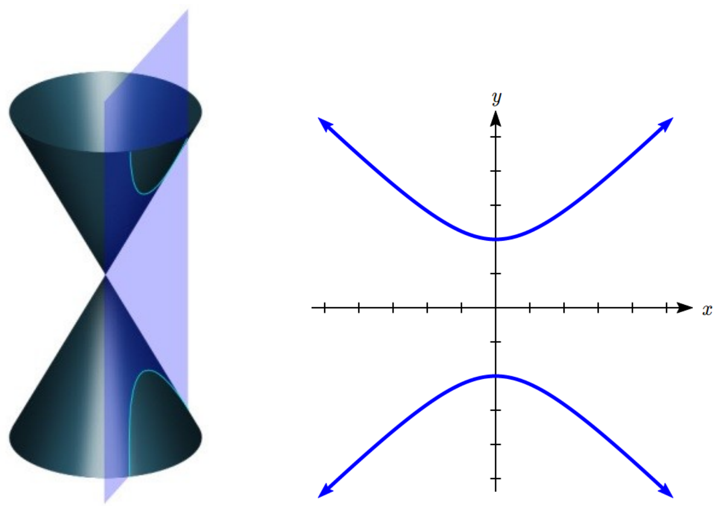
Section 9.2 Hyperbolas



In the last section, we learned that planets have approximately elliptical orbits around the sun. When an object like a comet is moving quickly, it is able to escape the gravitational pull of the sun and follows a path with the shape of a **hyperbola**. Hyperbolas are curves that can help us find the location of a ship, describe the shape of cooling towers, or calibrate seismological equipment.

The hyperbola is another type of conic section created by intersecting a plane with a double cone, as shown below[5].



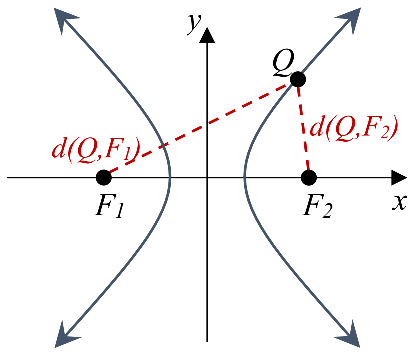
The word "hyperbola" derives from a Greek word meaning "excess." The English word "hyperbole" means exaggeration. We can think of a hyperbola as an excessive or exaggerated ellipse, one turned inside out.

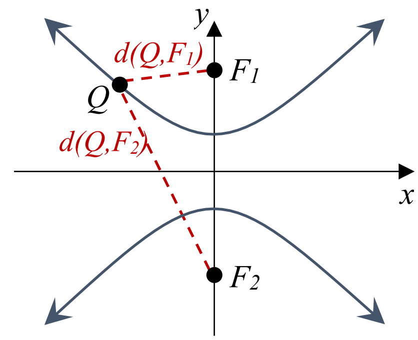
We defined an ellipse as the set of all points where the sum of the distances from that point to two fixed points is a constant. A hyperbola is the set of all points where the absolute value of the difference of the distances from the point to two fixed points is a constant.

[5] Pbroks13 (https://commons.wikimedia.org/wiki/File:Conic_sections_with_plane.svg), "Conic sections with plane", cropped to show only a hyperbola by L Michaels, CC BY 3.0

Hyperbola Definition

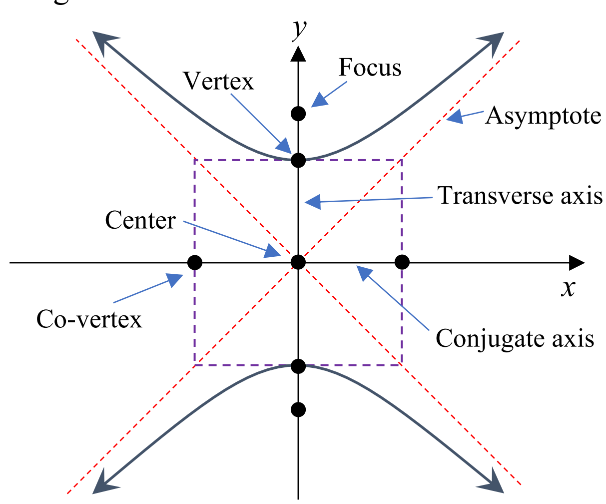
A **hyperbola** is the set of all points $Q(x, y)$ for which the absolute value of the difference of the distances to two fixed points $F_1(x_1, y_1)$ and $F_2(x_2, y_2)$ called the **foci** (plural for focus) is a constant *k*: $\left|d(Q, F_1) - d(Q, F_2)\right| = k$.





The **transverse axis** is the line passing through the foci.
Vertices are the points on the hyperbola which intersect the transverse axis.
The **transverse axis length** is the length of the line segment between the vertices.
The **center** is the midpoint between the vertices (or the midpoint between the foci).
The other axis of symmetry through the center is the **conjugate axis**.
The two disjoint pieces of the curve are called **branches**.
A hyperbola has two **asymptotes**.

Which axis is the transverse axis will depend on the orientation of the hyperbola. As a helpful tool for graphing hyperbolas, it is common to draw a **central rectangle** as a guide. This is a rectangle drawn around the center with sides parallel to the coordinate axes that pass through each vertex and co-vertex. The asymptotes will follow the diagonals of this rectangle.



Hyperbolas Centered at the Origin

From the definition above we can find an equation of a hyperbola. We will find it for a hyperbola centered at the origin $C(0,0)$ opening horizontally with foci at $F_1(c,0)$ and $F_2(-c,0)$ where $c > 0$.

Suppose $Q(x, y)$ is a point on the hyperbola. The distances from Q to F_1 and Q to F_2 are:

$$d(Q,F_1) = \sqrt{(x-c)^2 + (y-0)^2} = \sqrt{(x-c)^2 + y^2}$$

$$d(Q,F_2) = \sqrt{(x-(-c))^2 + (y-0)^2} = \sqrt{(x+c)^2 + y^2}\ .$$

From the definition, the absolute value of the difference should be constant:

$$|d(Q,F_1) - d(Q,F_2)| = \left|\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2}\right| = k$$

Substituting in one of the vertices $(a,0)$, we can determine k in terms of a:

$$\left|\sqrt{(a-c)^2 + 0^2} - \sqrt{(a+c)^2 + 0^2}\right| = k$$

$$\big||a-c| - |a+c|\big| = k \qquad \text{Since } c > a,\ |a-c| = c-a$$

$$|(c-a) - (a+c)| = k$$

$$k = |-2a| = |2a|$$

Using $k = 2a$ and removing the absolute values,

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a \qquad \text{Move one radical}$$

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2} \qquad \text{Square both sides}$$

$$(x-c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \quad \text{Expand}$$

$$x^2 - 2xc + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2$$

Combining like terms leaves

$$-4xc = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} \qquad \text{Divide by 4}$$

$$-xc = a^2 \pm a\sqrt{(x+c)^2 + y^2} \qquad \text{Isolate the radical}$$

$$\pm a\sqrt{(x+c)^2 + y^2} = -a^2 - xc \qquad \text{Square both sides again}$$

$$a^2\left((x+c)^2 + y^2\right) = a^4 + 2a^2xc + x^2c^2 \qquad \text{Expand and distribute}$$

$$a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2 \qquad \text{Combine like terms}$$

$$a^2y^2 + a^2c^2 - a^4 = x^2c^2 - a^2x^2 \qquad \text{Factor common terms}$$

$a^2y^2 + a^2(c^2 - a^2) = (c^2 - a^2)x^2$

Let $b^2 = c^2 - a^2$. Since $c > a$, $b > 0$. Substituting b^2 for $c^2 - a^2$ leaves

$a^2y^2 + a^2b^2 = b^2x^2$ Divide both sides by a^2b^2

$\frac{y^2}{b^2} + 1 = \frac{x^2}{a^2}$ Rewrite

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We can see from the graphs of the hyperbolas that the branches appear to approach asymptotes as x gets large in the negative or positive direction. The equations of the horizontal hyperbola asymptotes can be derived from its standard equation.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Solve for y

$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$ Rewrite 1 as $\frac{x^2}{a^2}\frac{a^2}{x^2}$

$y^2 = b^2\left(\frac{x^2}{a^2} - \frac{x^2}{a^2}\frac{a^2}{x^2}\right)$ Factor out $\frac{x^2}{a^2}$

$y^2 = b^2\frac{x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$ Take the square root

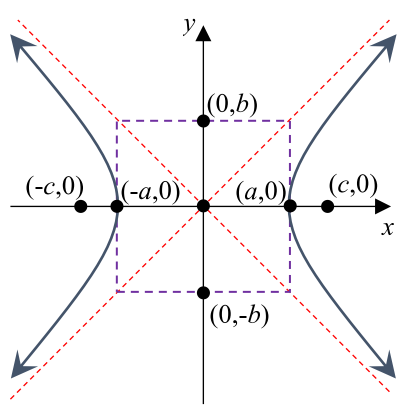
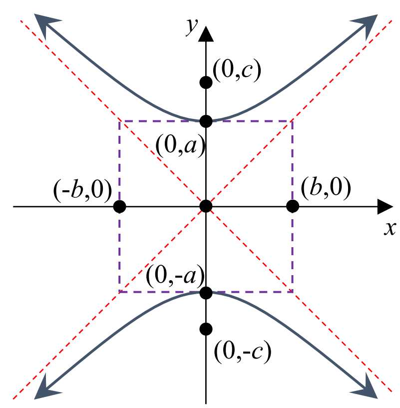
$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}$

As $x \to \pm\infty$ the quantity $\frac{a^2}{x^2} \to 0$ and $\sqrt{1 - \frac{a^2}{x^2}} \to 1$, so the asymptotes are $y = \pm\frac{b}{a}x$.

Similarly, for vertical hyperbolas the asymptotes are $y = \pm\frac{a}{b}x$.

The standard form of an equation of a hyperbola centered at the origin $C(0,0)$ depends on whether it opens horizontally or vertically. The following table gives the standard equation, vertices, foci, asymptotes, construction rectangle vertices, and graph for each.

Equation of a Hyperbola Centered at the Origin in Standard Form

Opens	Horizontally	Vertically
Standard Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Vertices	(-a, 0) and (a, 0)	(0, -a) and (0, a)
Foci	(-c, 0) and (c, 0) where $b^2=c^2-a^2$	(0, -c) and (0, c) Where $b^2=c^2-a^2$
Asymptotes	$y=\pm\frac{b}{a}x$	$y=\pm\frac{a}{b}x$
Construction Rectangle Vertices	(a, b), (-a, b), (a,-b), (-a, -b)	(b, a), (-b, a), (b, -a), (-b, -a)
Graph		

Example 1

Put the equation of the hyperbola $y^2-4x^2=4$ in standard form. Find the vertices, length of the transverse axis, and the equations of the asymptotes. Sketch the graph. Check using a graphing utility.

The equation can be put in standard form $\frac{y^2}{4}-\frac{x^2}{1}=1$ by dividing by 4.

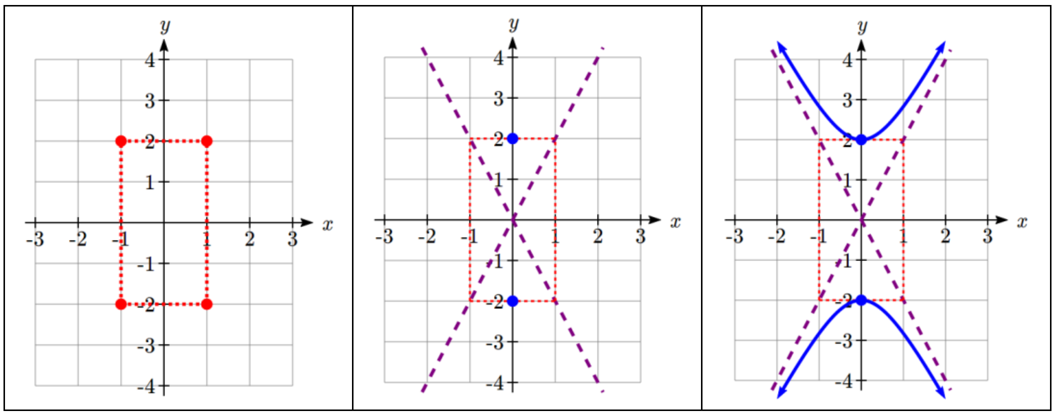
Comparing to the general standard equation $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ we see that $a=\sqrt{4}=2$ and $b=\sqrt{1}=1$.

Since the x term is subtracted, the hyperbola opens vertically and the vertices lie on the y-axis at $(0,\pm a) = (0, \pm 2)$.

The length of the transverse axis is $2(a) = 2(2) = 4$.

Equations of the asymptotes are $y = \pm\frac{a}{b}x$ or $= \pm 2x$.

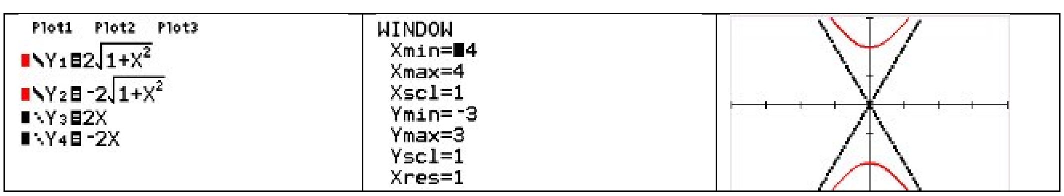
To sketch the graph we plot the vertices of the construction rectangle at $(\pm b,\pm a)$ or (-1,-2), (-1,2), (1,-2), and (1,2). The asymptotes are drawn through the diagonals of the rectangle and the vertices plotted. Then we sketch in the hyperbola, rounded at the vertices and approaching the asymptotes.



To check on a graphing utility, we must solve the equation for y. Isolating y^2 gives us $y^2 = 4\left(1 + x^2\right)$.

Taking the square root of both sides we find $y = \pm 2\sqrt{1+x^2}$.

Under **Y=** enter the two halves of the hyperbola and the two asymptotes as $y = 2\sqrt{1+x^2}$, $y = -2\sqrt{1+x^2}$, $y = 2x$, and $y = -2x$. Set the window to a comparable scale to the sketch with xmin = -4, xmax = 4, ymin= -3, and ymax = 3.



Sometimes we are given the equation. Sometimes we need to find the equation from a graph or other information.

Example 2

Find the standard form of the equation for a hyperbola with vertices at (-6,0) and (6,0) and asymptote $y = \frac{4}{3}x$.

Since the vertices lie on the *x*-axis with a midpoint at the origin, the hyperbola is horizontal with an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The value of a is the distance from the center to a vertex. The distance from (6,0) to (0,0) is 6, so $a = 6$.

The asymptotes follow the form $y = \pm\frac{b}{a}x$. From $y = \frac{4}{3}x$ we see $\frac{4}{3} = \frac{b}{a}$ and substituting $a = 6$ give us $\frac{4}{3} = \frac{b}{6}$. Solving yields $b = 8$.

The equation of the hyperbola in standard form is $\frac{x^2}{6^2} - \frac{y^2}{8^2} = 1$ or $\frac{x^2}{36} - \frac{y^2}{64} = 1$.

Try it Now

1. Find the standard form of the equation for a hyperbola with vertices at (0,-8) and (0,8) and asymptote $y = 2x$

Example 3

Find the standard form of the equation for a hyperbola with vertices at (0, 9) and (0,-9) and passing through the point (8,15).

Since the vertices lie on the *y*-axis with a midpoint at the origin, the hyperbola is vertical with an equation of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. The value of a is the distance from the center to a vertex. The distance from (0,9) to (0,0) is 9, so $a = 9$.

Substituting $a = 9$ and the point (8,15) gives $\frac{15^2}{9^2} - \frac{8^2}{b^2} = 1$. Solving for b yields

$$b = \sqrt{\frac{9^2\left(8^2\right)}{15^2 - 9^2}} = 6.$$

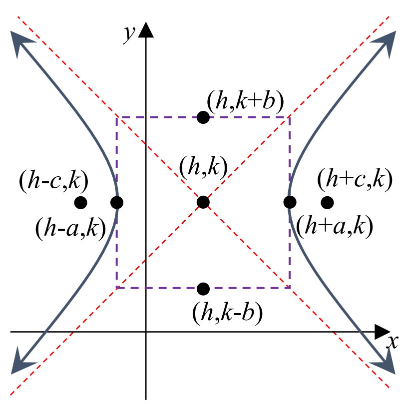
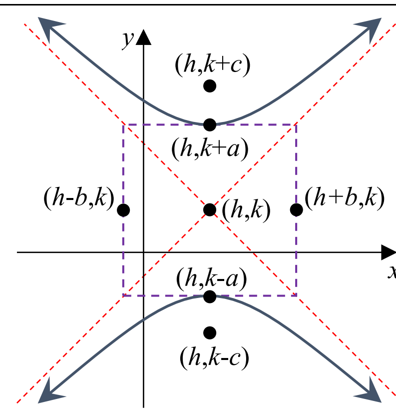
The standard equation for the hyperbola is $\frac{y^2}{9^2} - \frac{x^2}{6^2} = 1$ or $\frac{y^2}{81} - \frac{x^2}{36} = 1$.

Hyperbolas Not Centered at the Origin

Not all hyperbolas are centered at the origin. The standard equation for one centered at (h, k) is slightly different.

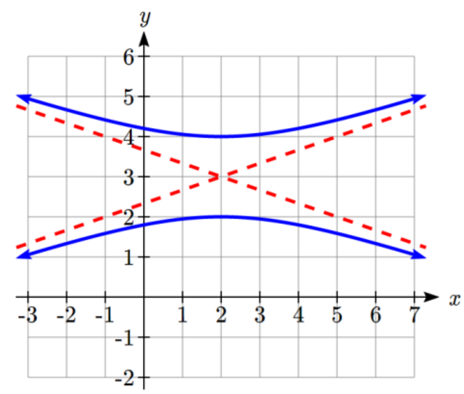
Equation of a Hyperbola Centered at (h, k) in Standard Form

The standard form of an equation of a hyperbola centered at $C(h,k)$ depends on whether it opens horizontally or vertically. The table below gives the standard equation, vertices, foci, asymptotes, construction rectangle vertices, and graph for each.

Opens	Horizontally	Vertically
Standard Equation	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Foci	$(h \pm c, k)$ where $b^2 = c^2 - a^2$	$(h, k \pm c)$ where $b^2 = c^2 - a^2$
Asymptotes	$y-k=\pm\frac{b}{a}(x-h)$	$y-k=\pm\frac{a}{b}(x-h)$
Construction Rectangle Vertices	$(h \pm a, k \pm b)$	$(h \pm b, k \pm a)$
Graph		

Example 4

Write an equation for the hyperbola in the graph shown.



The center is at (2,3), where the asymptotes cross. It opens vertically, so the equation will look like
$\frac{(y-3)^2}{a^2}-\frac{(x-2)^2}{b^2}=1$.

The vertices are at (2,2) and (2,4). The distance from the center to a vertex is $a=4-3=1$.

If we were to draw in the construction rectangle, it would extend from x = -1 to x = 5. The distance from the center to the right side of the rectangle gives $b=5-2=3$.

The standard equation of this hyperbola is $\frac{(y-3)^2}{1^2}-\frac{(x-2)^2}{3^2}=1$, or

$(y-3)^2-\frac{(x-2)^2}{9}=1$.

Example 5

Put the equation of the hyperbola $9x^2+18x-4y^2+16y=43$ in standard form. Find the center, vertices, length of the transverse axis, and the equations of the asymptotes. Sketch the graph, then check on a graphing utility.

To rewrite the equation, we complete the square for both variables to get
$9\left(x^2+2x+1\right)-4\left(y^2-4y+4\right)=43+9-16$
$9(x+1)^2-4(y-2)^2=36$

Dividing by 36 gives the standard form of the equation, $\frac{(x+1)^2}{4}-\frac{(y-2)^2}{9}=1$

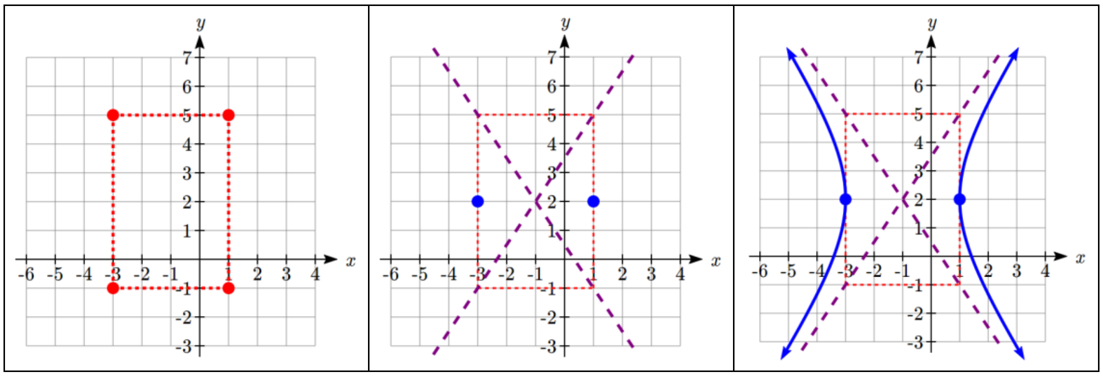
Comparing to the general standard equation $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ we see that

$a=\sqrt{4}=2$ and $b=\sqrt{9}=3$.

Since the y term is subtracted, the hyperbola opens horizontally.
The center is at (h, k) = (-1, 2).
The vertices are at ($h\pm a$, k) or (-3, 2) and (1,2).
The length of the transverse axis is $2(a)=2(2)=4$.

Equations of the asymptotes are $y-k=\pm\frac{b}{a}(x-h)$ or $y-2=\pm\frac{3}{2}(x+1)$.

To sketch the graph we plot the corners of the construction rectangle at (h±a, k±b) or (1, 5), (1, -1), (-3,5), and (-3,-1). The asymptotes are drawn through the diagonals of the rectangle and the vertices plotted. Then we sketch in the hyperbola rounded at the vertices and approaching the asymptotes.

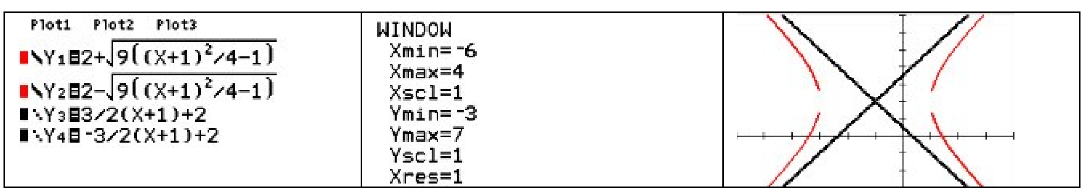


To check on a graphing utility, we must solve the equation for y.

$$y = 2 \pm \sqrt{9\left(\frac{(x+1)^2}{4} - 1\right)}.$$

Under **Y=** enter the two halves of the hyperbola and the two asymptotes as $y = 2 + \sqrt{9\left(\frac{(x+1)^2}{4} - 1\right)}$, $y = 2 - \sqrt{9\left(\frac{(x+1)^2}{4} - 1\right)}$, $y = \frac{3}{2}(x+1)+2$, and $y = -\frac{3}{2}(x+1)+2$. Set the window to a comparable scale to the sketch, then graph.

Note that the gaps you see on the calculator are not really there; they're a limitation of the technology.



Example 6

Find the standard form of the equation for a hyperbola with vertices at $(-2,-5)$ and $(-2,7)$, and asymptote $y = \frac{3}{2}x + 4$.

Since the vertices differ in the y -coordinates, the hyperbola opens vertically with an equation of the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ and asymptote equations of the form $y-k=\pm\frac{a}{b}(x-h)$.

The center will be halfway between the vertices, at $\left(-2, \frac{-5+7}{2}\right)=(-2,1)$.

The value of a is the distance from the center to a vertex. The distance from $(-2,1)$ to $(-2,-5)$ is 6, so $a = 6$.

While our asymptote is not given in the form $y-k=\pm\frac{a}{b}(x-h)$, notice this equation would have slope $\frac{a}{b}$. We can compare that to the slope of the given asymptote equation to find b. Setting $\frac{3}{2}=\frac{a}{b}$ and substituting $a = 6$ gives us $b = 4$.

The equation of the hyperbola in standard form is $\frac{(y-1)^2}{6^2}-\frac{(x+2)^2}{4^2}=1$ or $\frac{(y-1)^2}{36}-\frac{(x+2)^2}{16}=1$.

Try it Now

2. Find the center, vertices, length of the transverse axis, and equations of the asymptotes for the hyperbola $\frac{(x+5)^2}{9}-\frac{(y-2)^2}{36}=1$.

Hyperbola Foci

The location of the foci can play a key role in hyperbola application problems. To find them, we need to find the length from the center to the foci, c, using the equation $b^2 = c^2 - a^2$. It looks similar to, but is not the same as, the Pythagorean Theorem.

Compare this with the equation to find length c for ellipses, which is $b^2 = a^2 - c^2$. If you remember that for the foci to be inside the ellipse they have to come before the vertices $(c < a)$, it's clear why we would calculate a^2 minus c^2. To be inside a hyperbola, the foci have to go beyond the vertices $(c > a)$, so we can see for hyperbolas we need c^2 minus a^2, the opposite.

Example 7

Find the foci of the hyperbola $\dfrac{(y+1)^2}{4} - \dfrac{(x-3)^2}{5} = 1$.

The hyperbola is vertical with an equation of the form $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$.

The center is at $(h, k) = (3, -1)$. The foci are at $(h, k \pm c)$.

To find length c we use $b^2 = c^2 - a^2$. Substituting gives $5 = c^2 - 4$ or $c = \sqrt{9} = 3$.

The hyperbola has foci (3, -4) and (3, 2).

Example 8

Find the standard form of the equation for a hyperbola with foci (5, -8) and (-3, -8) and vertices (4, -8) and (-2, -8).

Since the vertices differ in the x -coordinates, the hyperbola opens horizontally with an equation of the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

The center is at the midpoint of the vertices

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{4+(-2)}{2}, \frac{-8+(-8)}{2}\right) = (1, -8).$$

The value of a is the horizontal length from the center to a vertex, or $a = 4 - 1 = 3$.
The value of c is the horizontal length from the center to a focus, or $= 5 - 1 = 4$.
To find length b we use $b^2 = c^2 - a^2$. Substituting gives $b^2 = 16 - 9 = 7$.

The equation of the hyperbola in standard form is $\dfrac{(x-1)^2}{3^2} - \dfrac{(y-(-8))^2}{7} = 1$ or

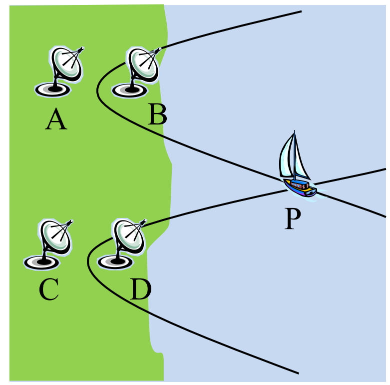
$$\frac{(x-1)^2}{9} - \frac{(y+8)^2}{7} = 1.$$

Try it Now

3. Find the standard form of the equation for a hyperbola with focus (1,9), vertex (1,8), center (1,4).

LORAN

Before GPS, the Long Range Navigation (LORAN) system was used to determine a ship's location. Two radio stations A and B simultaneously sent out a signal to a ship. The difference in time it took to receive the signal was computed as a distance locating the ship on the hyperbola with the A and B radio stations as the foci. A second pair of radio stations C and D sent simultaneous signals to the ship and computed its location on the hyperbola with C and D as the foci. The point P where the two hyperbolas intersected gave the location of the ship.



Example 9

Stations A and B are 150 kilometers apart and send a simultaneous radio signal to the ship. The signal from B arrives 0.0003 seconds before the signal from A. If the signal travels 300,000 kilometers per second, find the equation of the hyperbola on which the ship is positioned.

Stations A and B are at the foci, so the distance from the center to one focus is half the distance between them, giving $c = \frac{1}{2}(150) = 75$ km.

By letting the center of the hyperbola be at (0,0) and placing the foci at (±75,0), the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for a hyperbola centered at the origin can be used.

The difference of the distances of the ship from the two stations is $k = 300{,}000 \frac{\text{km}}{\text{s}} \cdot (0.0003 \text{ s}) = 90 \text{ km}$. From our derivation of the hyperbola equation we determined $k = 2a$, so $a = \frac{1}{2}(90) = 45$.

Substituting a and c into $b^2 = c^2 - a^2$ yields $b^2 = 75^2 - 45^2 = 3600$.

The equation of the hyperbola in standard form is $\frac{x^2}{45^2} - \frac{y^2}{3600} = 1$ or $\frac{x^2}{2025} - \frac{y^2}{3600} = 1$.

To determine the position of a ship using LORAN, we would need an equation for the second hyperbola and would solve for the intersection. We will explore how to do that in the next section.

Important Topics of This Section
Hyperbola Definition
Hyperbola Equations in Standard Form
Hyperbola Foci
Applications of Hyperbolas
Intersections of Hyperbolas and Other Curves

Try it Now Answers

1. The vertices are on the y axis so this is a vertical hyperbola.
   The center is at the origin.
   $a = 8$
   Using the asymptote slope, $\frac{8}{b} = 2$, so $b = 4$.
   $$\frac{y^2}{64} - \frac{x^2}{16} = 1$$

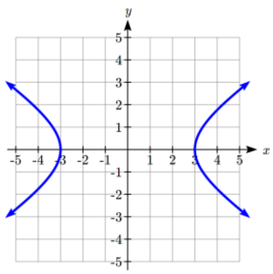
2. Center (-5, 2). This is a horizontal hyperbola. $a = 3$. $b = 6$.
   transverse axis length 6,
   Vertices will be at $(-5\pm3,2) = (-2,2)$ and $(-8,2)$,
   Asymptote slope will be $\frac{6}{3} = 2$. Asymptotes: $y - 2 = \pm 2(x+5)$

3. Focus, vertex, and center have the same x value so this is a vertical hyperbola.
   Using the vertex and center, $a = 9 - 4 = 5$
   Using the focus and center, $c = 8 - 4 = 4$
   $b^2 = 5^2 - 4^2$. $b = 3$.
   $$\frac{(y-4)^2}{16} - \frac{(x-1)^2}{9} = 1$$

Section 9.2 Exercises

In problems 1–4, match each graph to equations A–D.

A. $\frac{x^2}{4}-\frac{y^2}{9}=1$ B. $\frac{x^2}{9}-\frac{y^2}{4}=1$ C. $y^2-\frac{x^2}{9}=1$ D. $\frac{y^2}{9}-x^2=1$

1.


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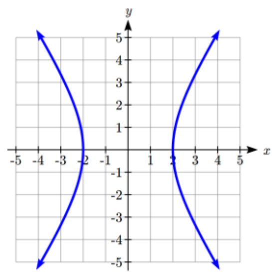

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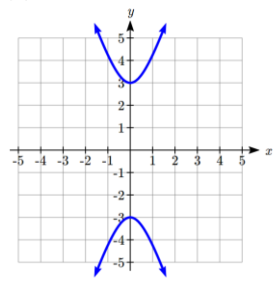

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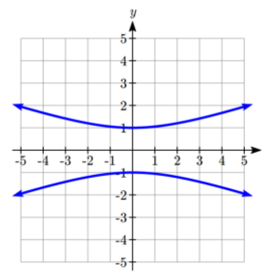


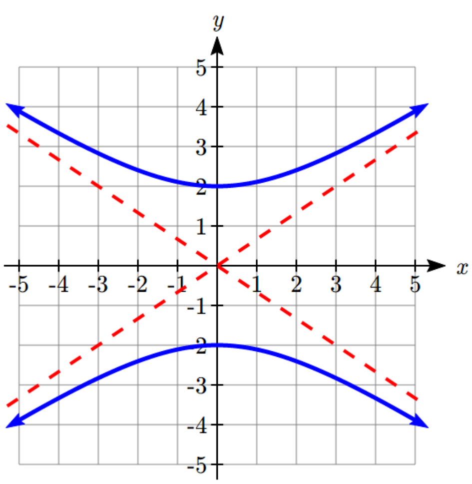
In problems 5–14, find the vertices, length of the transverse axis, and equations of the asymptotes. Sketch the graph. Check using a graphing utility.

5. $\frac{x^2}{4}-\frac{y^2}{25}=1$ 6. $\frac{y^2}{16}-\frac{x^2}{9}=1$ 7. $y^2-\frac{x^2}{4}=1$ 8. $x^2-\frac{y^2}{25}=1$

9. $x^2-9y^2=9$ 10. $y^2-4x^2=4$ 11. $9y^2-16x^2=144$

12. $16x^2-25y^2=400$ 13. $9x^2-y^2=18$ 14. $4y^2-x^2=12$

In problems 15–16, write an equation for the graph.

15.


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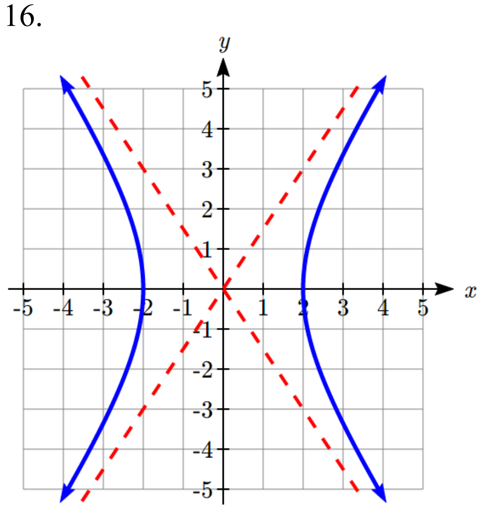


In problems 17–22, find the standard form of the equation for a hyperbola satisfying the given conditions.

17. Vertices at (0,4) and (0, -4); asymptote $y = \frac{1}{2}x$

18. Vertices at (-6,0) and (6,0); asymptote $y = 3x$

19. Vertices at (-3,0) and (3,0); passes through (5,8)

20. Vertices at (0, 4) and (0, -4); passes through (6, 5)

21. Asymptote y = x; passes through (5, 3)

22. Asymptote y = x; passes through (12, 13)

In problems 23–30, match each graph to equations A–H.

A. $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{4} = 1$

B. $\frac{(x+1)^2}{9} - \frac{(y+2)^2}{4} = 1$

C. $\frac{(x+1)^2}{9} - \frac{(y+2)^2}{16} = 1$

D. $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$

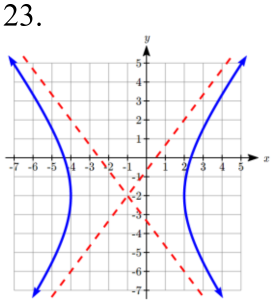
E. $\frac{(y-2)^2}{4} - \frac{(x-1)^2}{9} = 1$

F. $\frac{(y+2)^2}{4} - \frac{(x+1)^2}{9} = 1$

G. $\frac{(y+2)^2}{4} - \frac{(x+1)^2}{16} = 1$

H. $\frac{(y-2)^2}{4} - \frac{(x-1)^2}{16} = 1$

23.



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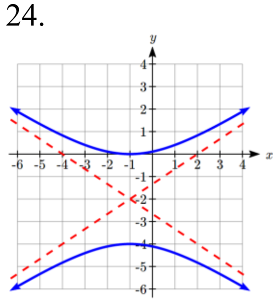

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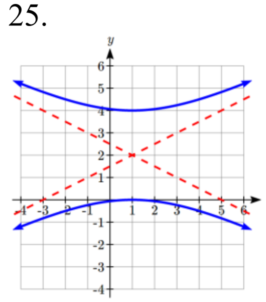

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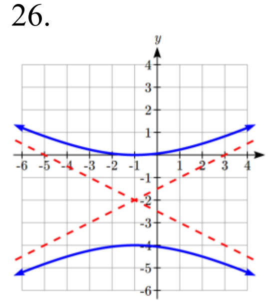

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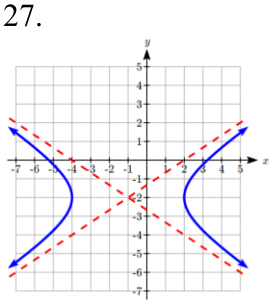

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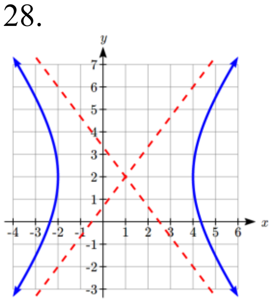

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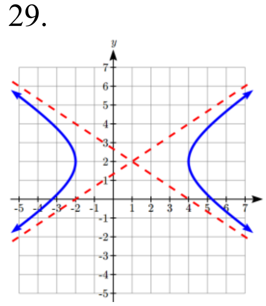

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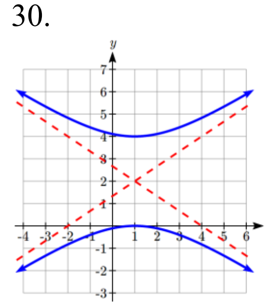


In problems 31–40, find the center, vertices, length of the transverse axis, and equations of the asymptotes. Sketch the graph. Check using a graphing utility.

31. $\dfrac{(x-1)^2}{25}-\dfrac{(y+2)^2}{4}=1$

32. $\dfrac{(y-3)^2}{16}-\dfrac{(x+5)^2}{36}=1$

33. $\dfrac{(y-1)^2}{9}-(x+2)^2=1$

34. $\dfrac{(x-1)^2}{25}-(y-6)^2=1$

35. $4x^2-8x-y^2=12$

36. $4y^2+16y-9x^2=20$

37. $4y^2-16y-x^2-2x=1$

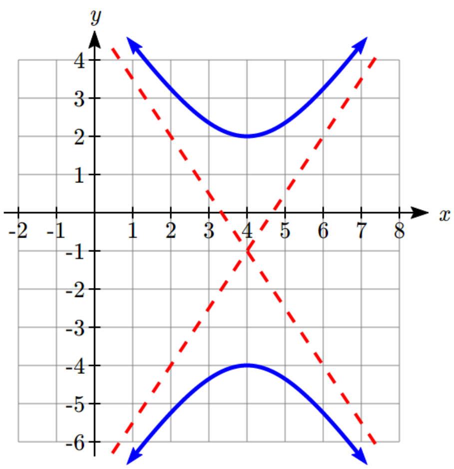
38. $4x^2-16x-y^2+6y=29$

39. $9x^2+36x-4y^2+8y=4$

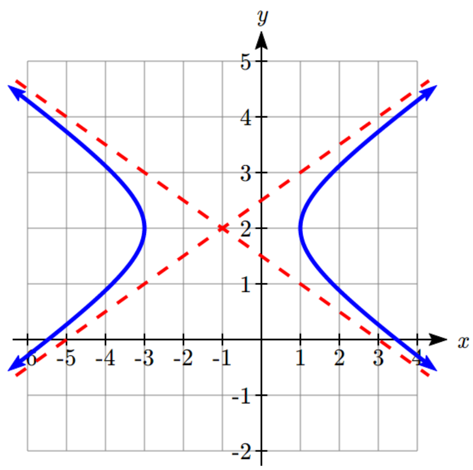
40. $9y^2+36y-16x^2-96x=-36$

In problems 41–42, write an equation for the graph.

41.



42.



In problems 43–44, find the standard form of the equation for a hyperbola satisfying the given conditions.

43. Vertices (-1,-2) and (-1,6); asymptote $y-2=2(x+1)$

44. Vertices (-3,-3) and (5,-3); asymptote $y+3=\dfrac{1}{2}(x-1)$

In problems 45–48, find the center, vertices, length of the transverse axis, and equations of the asymptotes. Sketch the graph. Check using a graphing utility.

45. $y=\pm 4\sqrt{9x^2-1}$

46. $y=\pm\dfrac{1}{4}\sqrt{9x^2+1}$

47. $y=1\pm\dfrac{1}{2}\sqrt{9x^2+18x+10}$

48. $=-1\pm 2\sqrt{9x^2-18x+8}$

In problems 49–54, find the foci.

49. $\dfrac{y^2}{6}-\dfrac{x^2}{19}=1$

50. $x^2-\dfrac{y^2}{35}=1$

51. $\dfrac{(x-1)^2}{15}-(y-6)^2=1$

52. $\dfrac{(y-3)^2}{47}-\dfrac{(x+5)^2}{2}=1$

53. $y=1\pm\dfrac{4}{3}\sqrt{x^2+8x+25}$

54. $y=-3\pm\dfrac{12}{5}\sqrt{x^2-4x-21}$

In problems 55–66, find the standard form of the equation for a hyperbola satisfying the given conditions.

55. Foci (5,0) and (-5,0), vertices (4,0) and (4,0)

56. Foci (0,26) and (0, -26), vertices (0,10) and (0,-10)

57. Focus (0, 13), vertex (0,12), center (0,0)

58. Focus (15, 0), vertex (12, 0), center (0,0)

59. Focus (17, 0) and (-17,0), asymptotes $y=\dfrac{8}{15}x$ and $y=-\dfrac{8}{15}x$

60. Focus (0, 25) and (0, 25), asymptotes $y=\dfrac{24}{7}x$ and $y=-\dfrac{24}{7}x$

61. Focus (10, 0) and (-10, 0), transverse axis length 16

62. Focus (0, 34) and (0, -34), transverse axis length 32

63. Foci (1, 7) and (1, -3), vertices (1, 6) and (1,-2)

64. Foci (4, -2) and (-6, -2), vertices (2, -2) and (-4, -2)

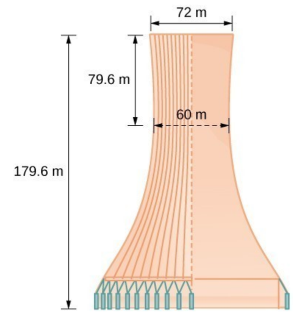
65. Focus (12, 3), vertex (4, 3), center (-1, 3)

66. Focus (-3, 15), vertex (-3, 13), center (-3, -2)

67. **LORAN** Stations A and B are 100 kilometers apart and send a simultaneous radio signal to a ship. The signal from A arrives 0.0002 seconds before the signal from B. If the signal travels 300,000 kilometers per second, find an equation of the hyperbola on which the ship is positioned if the foci are located at A and B.

68. **Thunder and Lightning** Anita and Samir are standing 3050 feet apart when they see a bolt of light strike the ground. Anita hears the thunder 0.5 seconds before Samir does. Sound travels at 1100 feet per second. Find an equation of the hyperbola on which the lighting strike is positioned if Anita and Samir are located at the foci.

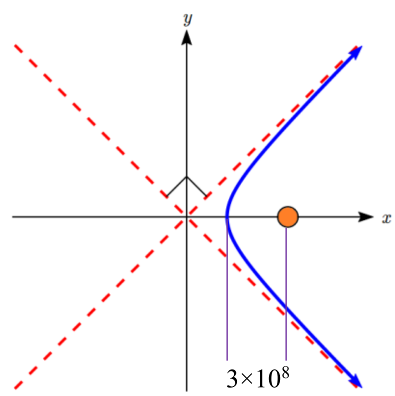
69. **Cooling Tower** The cooling tower for a power plant has sides in the shape of a hyperbola. The tower stands 179.6 meters tall. The diameter at the top is 72 meters. At their closest, the sides of the tower are 60 meters apart. Find an equation that models the sides of the cooling tower.



70. **Calibration** A seismologist positions two recording devices 340 feet apart at points A and B. To check the calibration, an explosive is detonated between the devices 90 feet from point A. The time the explosions register on the devices is noted and the difference calculated. A second explosion will be detonated east of point A. How far east should the second explosion be positioned so that the measured time difference is the same as for the first explosion?

71. **Target Practice** A gun at point A and a target at point B are 200 feet apart. A person at point C hears the gun fire and hit the target at exactly the same time. Find an equation of the hyperbola on which the person is standing if the foci are located at A and B. A fired bullet has a velocity of 2000 feet per second. The speed of sound is 1100 feet per second.

72. **Comet Trajectories** A comet passes through the solar system following a hyperbolic trajectory with the sun as a focus. The closest it gets to the sun is 3×10^8 miles. The figure shows the trajectory of the comet, whose path of entry is at a right angle to its path of departure. Find an equation for the comet's trajectory. Round to two decimal places.



73. The **conjugate** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. Show that $5y^2 - x^2 + 25 = 0$ is the conjugate of $x^2 - 5y^2 + 25 = 0$.

74. The **eccentricity** *e* of a hyperbola is the ratio $\frac{c}{a}$, where *c* is the distance of a focus from the center and *a* is the distance of a vertex from the center. Find the eccentricity of $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

75. An **equilateral hyperbola** is one for which $a = b$. Find the eccentricity of an equilateral hyperbola.

76. The **latus rectum** of a hyperbola is a line segment with endpoints on the hyperbola that passes through a focus and is perpendicular to the transverse axis. Show that $\frac{2b^2}{a}$ is the length of the latus rectum of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

77. **Confocal** hyperbolas have the same foci. Show that, for $0 < k < 6$, all hyperbolas of the form $\frac{x^2}{k} - \frac{y^2}{6-k} = 1$ are confocal.