Chapter 3

Section 3.4 Factor Theorem and Remainder Theorem

In the last section, we limited ourselves to finding the intercepts, or zeros, of polynomials that factored simply, or we turned to technology. In this section, we will look at algebraic techniques for finding the zeros of polynomials like \( h(t) = t^3 + 4t^2 + t - 6 \).

Long Division

In the last section we saw that we could write a polynomial as a product of factors, each corresponding to a horizontal intercept. If we knew that \( x = 2 \) was an intercept of the polynomial \( x^3 + 4x^2 - 5x - 14 \), we might guess that the polynomial could be factored as \( x^3 + 4x^2 - 5x - 14 = (x - 2)(\text{something}) \). To find that "something," we can use polynomial division.

Example 1

Divide \( x^3 + 4x^2 - 5x - 14 \) by \( x - 2 \)

Start by writing the problem out in long division form

\[
\begin{array}{rrrr}
  & x^2 & + 2x & + 6 \\
\hline
x - 2) & x^3 & + 4x^2 & + 6x - 14 \\
        & -x^3 & + 2x^2 &   \\
        &        & x^2 & + 6x - 14 \\
        &        & -x^2 & - 6x \\
        &        &        & 6x - 14 \\
        &        &        & -6x + 12 \\
        &        &        & 26 \\
\end{array}
\]

Again, divide the leading term of the remainder by the leading term of the divisor. \( 6x^2 \div x = 6x \). We add this to the result, multiply 6x by \( x - 2 \), and subtract.
Repeat the process one last time.

This tells us \( x^3 + 4x^2 - 5x - 14 \) divided by \( x - 2 \) is \( x^2 + 6x + 7 \), with a remainder of zero. This also means that we can factor \( x^3 + 4x^2 - 5x - 14 \) as \( (x - 2)(x^2 + 6x + 7) \).

This gives us a way to find the intercepts of this polynomial.

**Example 2**

Find the horizontal intercepts of \( h(x) = x^3 + 4x^2 - 5x - 14 \).

To find the horizontal intercepts, we need to solve \( h(x) = 0 \). From the previous example, we know the function can be factored as \( h(x) = (x - 2)(x^2 + 6x + 7) \).

\[
h(x) = (x - 2)(x^2 + 6x + 7) = 0 \quad \text{when} \quad x = 2 \quad \text{or} \quad x^2 + 6x + 7 = 0.
\]

This doesn't factor nicely, but we could use the quadratic formula to find the remaining two zeros.

\[
x = \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} = -3 \pm \sqrt{2}.
\]

The horizontal intercepts will be at \((2,0), (-3 - \sqrt{2}, 0)\), and \((-3 + \sqrt{2}, 0)\).
Try it Now
1. Divide $2x^3 - 7x + 3$ by $x + 3$ using long division.

The Factor and Remainder Theorems

When we divide a polynomial, $p(x)$ by some divisor polynomial $d(x)$, we will get a quotient polynomial $q(x)$ and possibly a remainder $r(x)$. In other words, $p(x) = d(x)q(x) + r(x)$.

Because of the division, the remainder will either be zero, or a polynomial of lower degree than $d(x)$. Because of this, if we divide a polynomial by a term of the form $x - c$, then the remainder will be zero or a constant.

If $p(x) = (x - c)q(x) + r$, then $p(c) = (c - c)q(c) + r = 0 + r = r$, which establishes the Remainder Theorem.

<table>
<thead>
<tr>
<th>The Remainder Theorem</th>
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<tbody>
<tr>
<td>If $p(x)$ is a polynomial of degree 1 or greater and $c$ is a real number, then when $p(x)$ is divided by $x - c$, the remainder is $p(c)$.</td>
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</table>

If $x - c$ is a factor of the polynomial $p$, then $p(x) = (x - c)q(x)$ for some polynomial $q$. Then $p(c) = (c - c)q(c) = 0$, showing $c$ is a zero of the polynomial. This shouldn't surprise us - we already knew that if the polynomial factors it reveals the roots.

If $p(c) = 0$, then the remainder theorem tells us that if $p$ is divided by $x - c$, then the remainder will be zero, which means $x - c$ is a factor of $p$.

<table>
<thead>
<tr>
<th>The Factor Theorem</th>
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<tbody>
<tr>
<td>If $p(x)$ is a nonzero polynomial, then the real number $c$ is a zero of $p(x)$ if and only if $x - c$ is a factor of $p(x)$.</td>
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Synthetic Division

Since dividing by $x - c$ is a way to check if a number is a zero of the polynomial, it would be nice to have a faster way to divide by $x - c$ than having to use long division every time. Happily, quicker ways have been discovered.
Let's look back at the long division we did in Example 1 and try to streamline it. First, let's change all the subtractions into additions by distributing through the negatives.

\[
x - 2 \overline{\begin{array}{c} x^2 + 6x + 7 \\ \hline x^3 + 4x^2 - 5x - 14 \\ - x^3 + 2x^2 \\ \hline 6x^2 - 5x - 14 \\ - 6x^2 + 12x \\ \hline 7x - 14 \\ - 7x + 14 \\ \hline 0 \\
\end{array}}
\]

Next, observe that the terms \(-x^3\), \(-6x^2\), and \(-7x\) are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we ‘bring down’ (namely the \(-5x\) and \(-14\)) aren’t really necessary to recopy, so we omit them, too.

\[
x - 2 \overline{\begin{array}{c} x^2 + 6x + 7 \\ \hline x^3 + 4x^2 - 5x - 14 \\ \hline 2x^2 \\ \hline 6x^2 \\ \hline 12x \\ \hline 7x \\ \hline 14 \\ \hline 0 \\
\end{array}}
\]

Now, let’s move things up a bit and, for reasons which will become clear in a moment, copy the \(x^3\) into the last row.

\[
x - 2 \overline{\begin{array}{c} x^2 + 6x + 7 \\ \hline x^3 + 4x^2 - 5x - 14 \\ \hline 2x^2 \\ \hline 12x \\ \hline 7x \\ \hline 14 \\ \hline 0 \\
\end{array}}
\]

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by \(x\) and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial.
This means that we no longer need to write the quotient polynomial down, nor the $x$ in the divisor, to determine our answer.

\[
x - 2 \overline{\begin{array}{c} x^3 + 4x^2 - 5x - 14 \\ \hline 2x^2 \quad 12x \quad 14 \\ x^3 \quad 6x^2 \quad 7x \quad 0 \end{array}}
\]

We’ve streamlined things quite a bit so far, but we can still do more. Let’s take a moment to remind ourselves where the $2x^2$, $12x$ and $14$ came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, $x^2$, $6x$ and $7$, respectively, by the $-2$ in $x - 2$, then by $-1$ when we changed the subtraction to addition. Multiplying by $-2$ then by $-1$ is the same as multiplying by $2$, so we replace the $-2$ in the divisor by $2$. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

\[
\begin{array}{cccc}
2 & | & 1 & 4 & -5 & -14 \\
 & & 2 & 12 & 14 \\
 & & 1 & 6 & 7 & 0
\end{array}
\]

We have constructed a **synthetic division** tableau for this polynomial division problem. Let’s re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we write $2$ in the place of the divisor and the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend. Then "bring down" the first coefficient of the dividend.

\[
\begin{array}{cccc}
2 & | & 1 & 4 & -5 & -14 \\
 & & & 2 & 12 & 14 \\
 & & & 1 & 6 & 7 & 0
\end{array}
\]

Next, take the $2$ from the divisor and multiply by the $1$ that was "brought down" to get $2$. Write this underneath the $4$, then add to get $6$.

\[
\begin{array}{cccc}
2 & | & 1 & 4 & -5 & -14 \\
 & & & 2 & 12 & 14 \\
 & & & 1 & 6 & 7 & 0
\end{array}
\]

Now take the $2$ from the divisor times the $6$ to get $12$, and add it to the $-5$ to get $7$. 

\[
\begin{array}{cccc}
2 & | & 1 & 4 & -5 & -14 \\
 & & & 2 & 12 & 14 \\
 & & & 1 & 6 & 7 & 0
\end{array}
\]
Finally, take the 2 in the divisor times the 7 to get 14, and add it to the $-14$ to get 0.

$$
\begin{array}{c|cccc}
2 & 1 & 4 & -5 & -14 \\
\hline & 2 & 12 & 14 \\
& 1 & 6 & 7 \\
\end{array}
$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works only for these kinds of divisors. Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division.

**Example 3**

Use synthetic division to divide $5x^3 - 2x^2 + 1$ by $x - 3$.

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of $x$ in the dividend. Doing so gives

$$
\begin{array}{c|cccc}
3 & 5 & -2 & 0 & 1 \\
\hline & 15 & 39 & 117 \\
& 5 & 13 & 39 & 118 \\
\end{array}
$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. This means $5x^3 - 2x^2 + 1 = (x - 3)(5x^2 + 13x + 39) + 118$.

It also means that $x - 3$ is *not* a factor of $5x^3 - 2x^2 + 1$.

**Example 4**

Divide $x^3 + 8$ by $x + 2$

For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before.

$$
\begin{array}{c|cccc}
-2 & 1 & 0 & 0 & 8 \\
\hline & -2 & 4 & -8 \\
& 1 & -2 & 4 & 0 \\
\end{array}
$$
The quotient is \( x^2 - 2x + 4 \) and the remainder is zero. Since the remainder is zero, \( x + 2 \) is a factor of \( x^3 + 8 \).

\[
x^3 + 8 = (x + 2)(x^2 - 2x + 4)
\]

**Try it Now**

2. Divide \( 4x^4 - 8x^3 - 5x \) by \( x - 3 \) using synthetic division.

Using this process allows us to find the real zeros of polynomials, presuming we can figure out at least one root. We'll explore how to do that in the next section.

**Example 5**

The polynomial \( p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3 \) has a horizontal intercept at \( x = \frac{1}{2} \) with multiplicity 2. Find the other intercepts of \( p(x) \).

Since \( x = \frac{1}{2} \) is an intercept with multiplicity 2, then \( x - \frac{1}{2} \) is a factor twice. Use synthetic division to divide by \( x - \frac{1}{2} \) twice.

\[
\begin{array}{c|cccccc}
1/2 & 4 & -4 & -11 & 12 & -3 \\
\hline
 & 2 & -1 & -6 & 3 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
1/2 & 4 & -2 & -12 & 6 \\
\hline
 & 2 & 0 & -6 \\
\end{array}
\]

From the first division, we get \( 4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x - 6\right) \). The second division tells us

\[
4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(4x^2 - 12\right).
\]

To find the remaining intercepts, we set \( 4x^2 - 12 = 0 \) and get \( x = \pm \sqrt{3} \).

Note this also means \( 4x^4 - 4x^3 - 11x^2 + 12x - 3 = 4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) \).
### Important Topics of this Section

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</table>

### Try it Now Answers

1. 

\[
\begin{array}{c|ccccc}
    & 2x^2 & -6x & 11 \\
\hline
  x + 3 & 2x^3 & +0x^2 & -7x & +3 \\
  \hline
    & -(2x^3+6x^2) & & & & \\
    & \hline
    & -6x^2 & -7x & +3 \\
  \hline
    & -(6x^2+18x) & & & & \\
    & \hline
    & 11x & +3 \\
  \hline
    & -(11x+33) & & & & \\
    & \hline
    & 30 & & & & \\
\end{array}
\]

The quotient is \(2x^2-6x+11\) with remainder -30.

2. 

\[
\begin{array}{c|cccc}
    3 & 4 & 0 & -8 & -5 & 0 \\
\hline
    \downarrow & 12 & 36 & 84 & 237 \\
    4 & 12 & 28 & 79 & 237 \\
\end{array}
\]

\(4x^4-8x^2-5x\) divided by \(x-3\) is \(4x^3+12x^2+28x+79\) with remainder 237
Section 3.4 Exercises

Use polynomial long division to perform the indicated division.

1. \((4x^2 + 3x - 1) \div (x - 3)\)
2. \((2x^3 - x + 1) \div (x^2 + x + 1)\)
3. \((5x^4 - 3x^3 + 2x^2 - 1) \div (x^2 + 4)\)
4. \((-x^5 + 7x^3 - x) \div (x^3 - x^2 + 1)\)
5. \((9x^3 + 5) \div (2x - 3)\)
6. \((4x^2 - x - 23) \div (x^2 - 1)\)

Use synthetic division to perform the indicated division.

7. \((3x^2 - 2x + 1) \div (x - 1)\)
8. \((x^3 - 5) \div (x - 5)\)
9. \((3 - 4x - 2x^2) \div (x + 1)\)
10. \((4x^2 - 5x + 3) \div (x + 3)\)
11. \((x^3 + 8) \div (x + 2)\)
12. \((4x^3 + 2x - 3) \div (x - 3)\)
13. \((18x^2 - 15x - 25) \div \left(x - \frac{5}{3}\right)\)
14. \((4x^2 - 1) \div \left(x - \frac{1}{2}\right)\)
15. \((2x^3 + x^2 + 2x + 1) \div \left(x + \frac{1}{2}\right)\)
16. \((3x^3 - x + 4) \div \left(x - \frac{2}{3}\right)\)
17. \((2x^3 - 3x + 1) \div \left(x - \frac{1}{2}\right)\)
18. \((4x^4 - 12x^3 + 13x^2 - 12x + 9) \div \left(x - \frac{3}{2}\right)\)
19. \((x^4 - 6x^2 + 9) \div \left(x - \sqrt{3}\right)\)
20. \((x^6 - 6x^4 + 12x^2 - 8) \div \left(x + \sqrt{2}\right)\)

Below you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

21. \(x^3 - 6x^2 + 11x - 6, \ c = 1\)
22. \(x^3 - 24x^2 + 192x - 512, \ c = 8\)
23. \(3x^3 + 4x^2 - x - 2, \ c = \frac{2}{3}\)
24. \(2x^3 - 3x^2 - 11x + 6, \ c = \frac{1}{2}\)
25. \(x^3 + 2x^2 - 3x - 6, \ c = -2\)
26. \(2x^3 - x^2 - 10x + 5, \ c = \frac{1}{2}\)
27. \(4x^4 - 28x^3 + 61x^2 - 42x + 9, \ c = \frac{1}{2}\) is a zero of multiplicity 2
28. \(x^5 + 2x^4 - 12x^3 - 38x^2 - 37x - 12, \ c = -1\) is a zero of multiplicity 3