7.1 Solutions to Exercises

1. Dividing both sides by 2, we have $\sin\theta = -\frac{1}{2}$. Since $\sin\theta$ is negative only in quadrants III and IV, using our knowledge of special angles, $\theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$.

3. Dividing both sides by 2, $\cos\theta = \frac{1}{2}$. Using our knowledge of quadrants, this occurs in quadrants I and IV. In quadrant I, $\theta = \frac{\pi}{3}$; in quadrant IV, $\theta = \frac{5\pi}{3}$.

5. Start by dividing both sides by 2 to get $\sin\left(\frac{\pi}{4}x\right) = \frac{1}{2}$. We know that $\sin\theta = \frac{1}{2}$ for $\theta = \frac{\pi}{6} + 2k\pi$ and $\theta = \frac{5\pi}{6} + k\pi$ for any integer k. Therefore, $\frac{\pi}{4}x = \frac{\pi}{6} + 2k\pi$ and $\frac{\pi}{4}x = \frac{5\pi}{6} + 2k\pi$. Solving the first equation by multiplying both sides by $\frac{4}{\pi}$ (the reciprocal of $\frac{\pi}{4}$) and distributing, we get $x = \frac{4}{6} + 8k$, or $x = \frac{2}{3} + 8k$. The second equation is solved in exactly the same way to arrive at $x = \frac{10}{3} + 8k$.

7. Divide both sides by 2 to arrive at $\cos 2t = -\frac{\sqrt{3}}{2}$. Since $\cos\theta = -\frac{\sqrt{3}}{2}$ when $\theta = \frac{5\pi}{6} + 2k\pi$ and when $\theta = \frac{7\pi}{6} + 2k\pi$. Thus, $2t = \frac{5\pi}{6} + 2k\pi$ and $2t = \frac{7\pi}{6} + 2k\pi$. Solving these equations for t results in $t = \frac{5\pi}{12} + k\pi$ and $t = \frac{7\pi}{12} + k\pi$.

9. Divide both sides by 3; then, $\cos\left(\frac{\pi}{5}x\right) = \frac{2}{3}$. Since $\frac{2}{3}$ is not the cosine of any special angle we know, we must first determine the angles in the interval $[0, 2\pi)$ that have a cosine of $\frac{2}{3}$. Your calculator will calculate $\cos^{-1}\left(\frac{2}{3}\right)$ as approximately 0.8411. But remember that, by definition, $\cos^{-1}\theta$ will always have a value in the interval $[0, \pi]$ -- and that there will be another angle in $(\pi, 2\pi)$ that has the same cosine value. In this case, 0.8411 is in quadrant I, so the other angle must be in quadrant IV: $2\pi - 0.8411 \approx 5.4421$. Therefore, $\frac{\pi}{5}x = 0.8411 + 2k\pi$ and $\frac{\pi}{5}x = 5.442 + 2k\pi$. Multiplying both sides of both equations by $\frac{5}{\pi}$ gives us $x = 1.3387 + 10k$ and $x = 8.6612 + 10k$.

11. Divide both sides by 7: $\sin 3t = -\frac{2}{7}$. We need to know the values of θ that give us $\sin\theta = -\frac{2}{7}$. Your calculator provides one answer: $\sin^{-1}\left(-\frac{2}{7}\right) \approx -0.2898$. However, $\sin^{-1}\theta$ has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which only covers quadrants I and IV. There is another angle in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ with the same sine value; in this case, in quadrant III: $\pi + 0.2898 \approx 3.4314$. Therefore, $3t = -0.2898 + 2k\pi$ and $3t = 3.4314 + 2k\pi$. Dividing both sides of both equations by 3 gives us $t = 1.1438 + \frac{2\pi}{3}k$ and $t = -0.0966 + \frac{2\pi}{3}k$.

13. Resist the urge to divide both sides by $\cos x$ -- although you can do this, you then have to separately consider the case where $\cos x = 0$. Instead, regroup all expressions onto one side of the equation:

$$10\sin x\cos x - 6\cos x = 0$$

Now factor $\cos x$:

$$\cos x\,(10\sin x - 6) = 0$$

So either $\cos x = 0$ or $10\sin x - 6 = 0$. On the interval $[0, 2\pi)$, $\cos x = 0$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, which provides us with two solutions. If $10\sin x - 6 = 0$, then $10\sin x = 6$ and $\sin x = \frac{6}{10}$. Using a calculator or computer to calculate $\sin^{-1}\frac{6}{10}$ gives us approximately 0.644, which is in quadrant I. We know there is another value for x in the interval $[0, 2\pi)$: in quadrant II at $\pi - 0.644 \approx 2.498$. Our solutions are $\frac{\pi}{2}$, $\frac{3\pi}{2}$, 0.644 and 2.498.

15. Add 9 to both sides to get $\csc 2x = 9$. If we rewrite this as $\frac{1}{\sin 2x} = 9$, we have $9\sin 2x = 1$ and $\sin 2x = \frac{1}{9}$. $\sin\theta = \frac{1}{9}$ at $\theta \approx 0.1113$ (the value from a calculator) and $\theta \approx 3.0303$ (using the reference angle in quadrant II). Therefore, $2x = 0.1113 + 2k\pi$ and $2x = 3.0303 + 2k\pi$. Solving these equations gives us $x = 0.056 + k\pi$ and $x = 1.515 + k\pi$ for integral k. We choose $k = 0$ and $k = 1$ for both equations to get four values: 0.056, 1.515, 3.198 and 4.657; these are the only values that lie in the interval $[0, 2\pi)$.

17. Factoring $\sin x$, we get $\sin x\,(\sec x - 2) = 0$. Therefore, either $\sin x = 0$ or $\sec x - 2 = 0$. On the interval $[0, 2\pi)$, $\sin x = 0$ at $x = 0$ and $x = \pi$, so these are our first two answers.

If $\sec x - 2 = 0$, then $\sec x = 2$ and $\frac{1}{\cos x} = 2$. This leads us to $2\cos x = 1$ and $\cos x = \frac{1}{2}$. Recognizing this as a well-known angle, we conclude that (again, on the interval $[0, 2\pi)$), $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

19. If $\sin^2(x) = \frac{1}{4}$, then $\sin x = \pm\frac{1}{2}$. On the interval $[0, 2\pi)$, this occurs at $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$.

21. If $\sec^2 x = 7$, then $\sec x = \pm\sqrt{7}$, $\frac{1}{\cos x} = \pm\sqrt{7}$, and $\cos x = \pm\frac{1}{\sqrt{7}} = \pm\frac{\sqrt{7}}{7}$.

Using a calculator for $\cos^{-1}\left(\frac{\sqrt{7}}{7}\right)$, we get $x \approx 1.183$. There is another angle on the interval $[0, 2\pi)$ whose cosine is $\frac{1}{7}$, in quadrant IV: $x = 2\pi - 1.183 \approx 5.1$. The two angles where $\cos x = -\frac{\sqrt{7}}{7}$ must lie in quadrants III and IV at $x = \pi - 1.183 \approx 1.959$ and $x = \pi + 1.183 \approx 4.325$.

23. This is quadratic in $\sin w$: think of it as $2x^2 + 3x + 1 = 0$, where $x = \sin w$. This is simple enough to factor:

$$2x^2 + 3x + 1 = (2x + 1)(x + 1) = 0$$

This means that either $2x + 1 = 0$ and $x = -\frac{1}{2}$, or $x + 1 = 0$ and $x = -1$. Therefore, either $\sin w = -\frac{1}{2}$ or $\sin w = -1$. We know these special angles: these occur on the interval $[0, 2\pi)$ when $w = \frac{7\pi}{6}$ or $w = \frac{11\pi}{6}$ (for $\sin w = -\frac{1}{2}$) or when $w = \frac{3\pi}{2}$ (for $\sin w = -1$).

25. If we subtract 1 from both sides, we can see that this is quadratic in $\cos t$:

$$2(\cos^2 t + \cos t - 1 = 0$$

If we let $x = \cos t$, we have:

$$2x^2 + x - 1 = (2x - 1)(x + 1) = 0$$

Either $2x - 1 = 0$ and $x = \frac{1}{2}$, or $x + 1 = 0$ and $x = -1$. Therefore, $\cos t = \frac{1}{2}$ or $\cos t = -1$. On the interval $[0, 2\pi)$, these are true when $t = \frac{\pi}{3}$ or $t = \frac{5\pi}{3}$ (for $\cos t = \frac{1}{2}$) or when $t = \pi$ (for $\cos t = -1$).

27. If we rearrange the equation, it is quadratic in $\cos x$:

$$4\cos^2 x - 15\cos x - 4 = 0$$

If we let $u = \cos x$, we can write this as:

$$4u^2 - 15u - 4 = 0$$

This factors as:

$$(4u + 1)(u - 4) = 0$$

Therefore, either $4u + 1 = 0$ and $u = -\frac{1}{4}$, or $u - 4 = 0$ and $u = 4$. Substituting back, we have:

$$\cos x = -\frac{1}{4}$$

We reject the other possibility that $\cos x = 4$ since $\cos x$ is always in the interval $[-1, 1]$.

Your calculator will tell you that $\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.823$. This is in quadrant II, and the cosine is negative, so the other value must lie in quadrant III. The reference angle is $\pi - 1.823$, so the other angle is at $\pi + (\pi - 1.823) = 2\pi - 1.823 \approx 4.460$.

29. If we substitute $1 - \cos^2 t$ for $\sin^2 t$, we can see that this is quadratic in $\cos t$:

$$12\sin^2 t + \cos t - 6 = 12(1 - \cos^2 t) + \cos t - 6 = -12\cos^2 t + \cos t + 6 = 0$$

Setting $u = \cos t$:

$$-12u^2 + u + 6 = (-4u + 3)(3u + 2) = 0$$

This leads us to $-4u + 3 = 0$ or $3u + 2 = 0$, so either $u = \frac{3}{4}$ or $u = -\frac{2}{3}$.

Substituting back, $\cos t = \frac{3}{4}$ gives us (via a calculator) $t \approx 0.7227$. This is in quadrant I, so the corresponding angle must lie in quadrant IV at $t = 2\pi - 0.7227 \approx 5.5605$.

Similarly, $\cos t = -\frac{2}{3}$ gives us $t \approx 2.3005$. This is in quadrant II; the corresponding angle with the same cosine value must be in quadrant III at $t = 2\pi - 2.3005 \approx 3.9827$.

31. Substitute $1 - \sin^2 \phi$ for $\cos^2 \phi$:

$$1 - \sin^2 \phi = -6 \sin \phi$$

$$-\sin^2 \phi + 6 \sin \phi + 1 = 0$$

This is quadratic in $\sin \phi$, so set $u = \sin \phi$ and we have:

$$-u^2 + 6u + 1 = 0$$

This does not factor easily, but the quadratic equation gives us:

$$u = \frac{-6 \pm \sqrt{36 - 4(-1)(1)}}{2(-1)} = \frac{-6 \pm \sqrt{40}}{-2} = \frac{-6 \pm 2\sqrt{10}}{-2} = 3 \pm \sqrt{10}$$

Thus, $u \approx 6.1623$ and $u \approx -0.1623$. Substituting back, we have $\sin \phi = -0.1623$. We reject $\sin \phi = 6.1623$ since $\sin \phi$ is always between -1 and 1. Using a calculator to calculate $\sin^{-1}(-0.1623)$, we get $\phi \approx -.1630$. Unfortunately, this is not in the required interval $[0, 2\pi)$, so we add 2π to get $\phi \approx 6.1202$. This is in quadrant IV; the corresponding angle with the same sine value must be in quadrant III at $\pi + 0.1630 \approx 3.3046$.

33. If we immediately substitute $v = \tan x$, we can write:

$$v^3 = 3v$$

$$v^3 - 3v = 0$$

$$v(v^2 - 3) = 0$$

Thus, either $v = 0$ or $v^2 - 3 = 0$, meaning $v^2 = 3$ and $v = \pm\sqrt{3}$.

Substituting back, $\tan x = 0$ at $x = 0$ and $x = \pi$. Similarly, $\tan x = \sqrt{3}$ at $x = \frac{\pi}{3}$ and $x = \frac{4\pi}{3}$ and $\tan x = -\sqrt{3}$ at $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

35. Substitute $v = \tan x$ so that:

$$v^5 = v$$

$$v^5 - v = 0$$

$$v(v^4 - 1) = 0$$

Either $v = 0$ or $v^4 - 1 = 0$ and $v^4 = 1$ and $v = \pm 1$. Substituting back, $\tan x = 0$ at $x = 0$ and $x = \pi$. Similarly, $\tan x = 1$ at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. Finally, $\tan x = -1$ for $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

37. The structure of the equation is not immediately apparent.

Substitute $u = \sin x$ and $v = \cos x$, and we have:

$$4uv + 2u - 2v - 1 = 0$$

The structure is now reminiscent of the result of multiplying two binomials in different variables. For example, $(x + 1)(y + 1) = xy + x + y + 1$. In fact, our equation factors as:

$$(2u - 1)(2v + 1) = 0$$

Therefore, either $2u - 1 = 0$ (and $u = \frac{1}{2}$) or $2v + 1 = 0$ (and $v = -\frac{1}{2}$). Substituting back, $\sin x = \frac{1}{2}$ or $\cos x = -\frac{1}{2}$. This leads to $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$ (for $\sin x = \frac{1}{2}$) and $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$ (for $\cos x = -\frac{1}{2}$

39. Rewrite $\tan x$ as $\frac{\sin x}{\cos x}$ to give:

$$\frac{\sin x}{\cos x} - 3\sin x = 0$$

Using a common denominator of $\cos x$, we have:

$$\frac{\sin x}{\cos x} - \frac{3\sin x \cos x}{\cos x} = 0$$

and

$$\frac{\sin x - 3\sin x \cos x}{\cos x} = 0$$

$$\sin x - 3\sin x \cos x = 0$$

$$\sin x\,(1 - 3\cos x) = 0$$

Therefore, either $\sin x = 0$ or $1 - 3\cos x = 0$, which means $\cos x = \frac{1}{3}$.

For $\sin x = 0$, we have $x = 0$ and $x = \pi$ on the interval $[0, 2\pi)$. For $\cos x = \frac{1}{3}$, we need $\cos^{-1}\left(\frac{1}{3}\right)$, which a calculator will indicate is approximately 1.231. This is in quadrant I, so the corresponding angle with a cosine of $\frac{1}{3}$ is in quadrant IV at $2\pi - 1.231 \approx 5.052$.

41. Rewrite both $\tan t$ and $\sec t$ in terms of $\sin t$ and $\cos t$:

$$2\frac{\sin^2 t}{\cos^2 t} = 3\frac{1}{\cos t}$$

We can multiply both sides by $\cos^2 t$:

$$2\sin^2 t = 3\cos t$$

Now, substitute $1-\cos^2 t$ for $\sin^2 t$ to yield:

$$2(1-\cos^2 t) - 3\cos t = 0$$

This is beginning to look quadratic in $\cos t$. Distributing and rearranging, we get:

$$-2\cos^2 t - 3\cos t + 2 = 0$$

Substitute $u = \cos t$:

$$-2u^2 - 3u + 2 = 0$$

$$(-2u+1)(u+2) = 0$$

Therefore, either $-2u+1=0$ (and $u=\frac{1}{2}$) or $u+2=0$ (and $u=-2$). Since $u=\cos t$ will never have a value of -2, we reject the second solution.

$\cos t = \frac{1}{2}$ at $t=\frac{\pi}{3}$ and $t=\frac{5\pi}{3}$ on the interval $[0, 2\pi)$.

7.2 Solutions to Exercises

1. sin(75°) = sin(45°+30°) = sin(45°)cos(30°)+cos(45°)sin(30°) = $\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$

3. cos(165°) = cos(120° + 45°) = cos(120°)cos(45°)-sin(120°)sin(45°) = $\frac{-(\sqrt{6}+\sqrt{2})}{4}$

5. $\cos\left(\frac{7\pi}{12}\right)=\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{2}-\sqrt{6}}{4}$

7. $\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

9. $\sin\left(x + \frac{11\pi}{6}\right) = \sin(x)\cos\left(\frac{11\pi}{6}\right) + \cos(x)\sin\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}\sin(x) - \frac{1}{2}\cos(x)$

11. $\cos\left(x - \frac{5\pi}{6}\right) = \cos(x)\cos\left(\frac{5\pi}{6}\right) + \sin(x)\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}\cos(x) + \frac{1}{2}\sin(x)$

13. $\csc\left(\frac{\pi}{2} - t\right) = \frac{1}{\sin\left(\frac{\pi}{2}-t\right)} = \frac{1}{\sin\left(\frac{\pi}{2}\right)\cos(t)-\cos\left(\frac{\pi}{2}\right)\sin(t)} = \frac{1}{\cos(t)} = \sec(t)$

15. $\cot\left(\frac{\pi}{2} - x\right) = \frac{\cos\left(\frac{\pi}{2}\right)\cos(x)+\sin\left(\frac{\pi}{2}\right)\sin(x)}{\sin\left(\frac{\pi}{2}\right)\cos(t)-\cos\left(\frac{\pi}{2}\right)\sin(t)} = \frac{\sin(x)}{\cos(x)} = \tan(x)$

17. $16\sin(16x)\sin(11x) = 16 \cdot \frac{1}{2}(\cos(16x - 11x) - \cos(16x + 11x)) = 8\cos(5x) - 8\cos(27x)$

19. $2\sin(5x)\cos(3x) = \sin(5x + 3x) + \sin(5x - 3x) = \sin(8x) + \sin(2x)$

21. $\cos(6t) + \cos(4t) = 2\cos\left(\frac{6t+4t}{2}\right)\cos\left(\frac{6t-4t}{2}\right) = 2\cos(5t)\cos(t)$

23. $\sin(3x) + \sin(7x) = 2\sin\left(\frac{3x+7x}{2}\right)\cos\left(\frac{3x-7x}{2}\right) = 2\sin(5x)\cos(-2x)$

25. We know that $\sin(a) = \frac{2}{3}$ and $\cos(b) = -\frac{1}{4}$ and that the angles are in quadrant II. We can find $\cos(a)$ and $\sin(b)$ using the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, or by using the known values of $\sin(a)$ and $\cos(b)$ to draw right triangles. Using the latter method: we know two sides of both a right triangle including angle *a* and a right triangle including angle *b*. The triangle including angle *a* has a hypotenuse of 3 and an opposite side of 2. We may use the pythagorean theorem to find the side adjacent to angle *a*. Using the same method we may find the side opposite to *b*.

For the triangle containing angle *a*:

$$\text{Adjacent} = \sqrt{3^2 - 2^2} = \sqrt{5}$$

However, this side lies in quadrant II, so it will be $-\sqrt{5}$.

For the triangle containing angle *b*:

$$\text{Opposite} = \sqrt{4^2 - 1^2} = \sqrt{15}$$

In quadrant II, *y* is positive, so we do not need to change the sign.

From this, we know: $\sin(a) = \frac{2}{3}, \cos(a) = -\frac{\sqrt{5}}{3}, \cos(b) = -\frac{1}{4}, \sin(b) = \frac{\sqrt{15}}{4}$.

a. $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) = \frac{-2-5\sqrt{3}}{12}$

b. $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) = \frac{\sqrt{5}+2\sqrt{15}}{12}$

27. $\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

    $\sin(3x - 6x) = -0.9$

    $\sin(-3x) = -0.9$

    $-\sin(3x) = -0.9$

    $3x = \sin^{-1}(0.9) + 2\pi k$ or $3x = \pi - \sin^{-1}(0.9) + 2\pi k$, where *k* is an integer

    $x = \frac{\sin^{-1}(0.9)+2\pi k}{3}$ or $x = \frac{\pi-\sin^{-1}(0.9)+2\pi k}{3}$

    $x \approx 0.373 + \frac{2\pi}{3}k$ or $x \approx 0.674 + \frac{2\pi}{3}k$, where *k* is an integer

29. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

    $\cos(2x - x) = 1$

    $x = 0 + 2\pi k$, where *k* is an integer

31. $\cos(5x) = -\cos(2x)$

    $\cos(5x) + \cos(2x) = 0$

    $2\cos\left(\frac{5x+2x}{2}\right)\cos\left(\frac{5x-2x}{2}\right) = 0$

    $\cos\left(\frac{7x}{2}\right) = 0$ or $\cos\left(\frac{3x}{2}\right) = 0$

    $\frac{7x}{2} = \frac{\pi}{2} + \pi k$ or $\frac{3x}{2} = \frac{\pi}{2} + \pi k$, where *k* is an integer

    $x = \frac{\pi+2\pi k}{7}$ or $x = \frac{\pi+2\pi k}{3}$

33. $\cos(6\theta) - \cos(2\theta) = \sin(4\theta)$

$-2\sin\left(\frac{6\theta+2\theta}{2}\right)\sin\left(\frac{6\theta-2\theta}{2}\right) = \sin(4\theta)$

$-2\sin(4\theta)\sin(2\theta) - \sin(4\theta) = 0$

$\sin(4\theta)\,(-2\sin(2\theta) - 1) = 0$

$\sin(4\theta) = 0$ or $-2\sin(2\theta) - 1 = 0$

$\sin(4\theta) = 0$ or $\sin(2\theta) = -\frac{1}{2}$

$4\theta = \pi k$ or $2\theta = \frac{7\pi}{6} + 2\pi k$ or $\frac{11\pi}{6} + 2\pi k$

$\theta = \frac{\pi k}{4}$ or $\frac{7\pi}{12} + \pi k$ or $\frac{11\pi}{12} + \pi k$

35. $A = \sqrt{4^2 + 6^2} = 2\sqrt{13}$

$\cos(c) = \frac{2}{\sqrt{13}}, \sin(c) = -\frac{3}{\sqrt{13}}$

Since $\sin(C)$ is negative but $\cos(C)$ is positive, we know that C is in quadrant IV.

$C = \sin^{-1}\left(-\frac{3}{\sqrt{13}}\right)$

Therefore the expression can be written as $2\sqrt{13}\sin\left(x + \sin^{-1}\left(-\frac{3}{\sqrt{13}}\right)\right)$ or approximately $2\sqrt{13}\sin(x - 0.9828)$.

37. $A = \sqrt{5^2 + 2^2} = \sqrt{29}$

$\cos(C) = \frac{5}{\sqrt{29}}, \sin(C) = \frac{2}{\sqrt{29}}$

Since both $\sin(C)$ and $\cos(C)$ are positive, we know that C is in quadrant I.

$C = \sin^{-1}\left(\frac{2}{\sqrt{29}}\right)$

Therefore the expression can be written as $\sqrt{29}\sin\left(3x + \sin^{-1}\left(\frac{2}{\sqrt{29}}\right)\right)$ or approximately $\sqrt{29}\sin(3x + 0.3805)$.

39. This will be easier to solve if we combine the 2 trig terms into one sinusoidal function of the form $A\sin(Bx + C)$.

$A=\sqrt{5^2+3^2}=\sqrt{34}$, $\cos(C)=-\frac{5}{\sqrt{34}}$, $\sin(C)=\frac{3}{\sqrt{34}}$

C is in quadrant II, so $C=\pi-\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)$

Then:

$\sqrt{34}\sin\left(x+\pi-\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)\right)=1$

$-\sin\left(x-\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)\right)=\frac{1}{\sqrt{34}}$ (Since $\sin(x+\pi)=-\sin(x)$)

$x-\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)=\sin^{-1}\left(-\frac{1}{\sqrt{34}}\right)$ or, to get the second solution $x-\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)=\pi-\sin^{-1}\left(-\frac{1}{\sqrt{34}}\right)$

$x\approx 0.3681$ or $x\approx 3.8544$ are the first two solutions.

41. This will be easier to solve if we combine the 2 trig terms into one sinusoidal function of the form $A\sin(Bx+C)$.

$A=\sqrt{5^2+3^2}=\sqrt{34}$, $\cos(C)=\frac{3}{\sqrt{34}}$, $\sin(C)=-\frac{5}{\sqrt{34}}$

C is in quadrant IV, so $C=\sin^{-1}\left(-\frac{5}{\sqrt{34}}\right)$.

Then:

$\sqrt{34}\sin\left(2x+\sin^{-1}\left(-\frac{5}{\sqrt{34}}\right)\right)=3$

$\sin\left(2x+\sin^{-1}\left(-\frac{5}{\sqrt{34}}\right)\right)=\frac{3}{\sqrt{34}}$

$2x+\sin^{-1}\left(-\frac{5}{\sqrt{34}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)$ and $2x+\sin^{-1}\left(-\frac{5}{\sqrt{34}}\right)=\pi-\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)$

$$x=\frac{\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)-\sin^{-1}\left(-\frac{5}{\sqrt{34}}\right)}{2} \text{ or } \frac{\pi-\sin^{-1}\left(\frac{3}{\sqrt{34}}\right)-\sin^{-1}\left(-\frac{5}{\sqrt{34}}\right)}{2}.$$

43. $\frac{\sin(7t)+\sin(5t)}{\cos(7t)+\cos(5t)}=\frac{2\sin\left(\frac{7t+5t}{2}\right)\cos\left(\frac{7t-5t}{2}\right)}{2\cos\left(\frac{7t+5t}{2}\right)\cos\left(\frac{7t-5t}{2}\right)}=\frac{2\sin(6t)\cos(t)}{2\cos(6t)\cos(t)}=\tan(6t)$

45. $\tan\left(\frac{\pi}{4}-t\right)=\dfrac{\sin\left(\frac{\pi}{4}-t\right)}{\cos\left(\frac{\pi}{4}-t\right)}$

$$=\frac{\sin\left(\frac{\pi}{4}\right)\cos(t)-\cos\left(\frac{\pi}{4}\right)\sin(t)}{\cos\left(\frac{\pi}{4}\right)\cos(t)+\sin\left(\frac{\pi}{4}\right)\sin(t)}$$

$$=\frac{\frac{\sqrt{2}}{2}(\cos(t)-\sin(t))}{\frac{\sqrt{2}}{2}(\cos(t)+\sin(t))}$$

$$=\frac{(\cos(t))\left(1-\frac{\sin(t)}{\cos(t)}\right)}{(\cos(t))\left(1+\frac{\sin(t)}{\cos(t)}\right)}$$

$$=\frac{1-\tan(t)}{1+\tan(t)}$$

47. $\dfrac{\cos(a+b)}{\cos(a-b)}=\dfrac{\cos(a)\cos(b)-\sin(a)\sin(b)}{\cos(a)\cos(b)+\sin(a)\sin(b)}$

$$=\frac{\cos(a)\cos(b)\left(1-\frac{\sin(a)\sin(b)}{\cos(a)\cos(b)}\right)}{\cos(a)\cos(b)\left(1+\frac{\sin(a)\sin(b)}{\cos(a)\cos(b)}\right)}$$

$$=\frac{1-\tan(a)\tan(b)}{1+\tan(a)\tan(b)}$$

49. Using the Product-to-Sum identity:

$$2\sin(a+b)\sin(a-b)=2\left(\frac{1}{2}\right)\cos\big((a+b)-(a-b)\big)-\cos\big((a+b)+(a-b)\big)$$

$$=\cos(2b)-\cos(2a)$$

51. $\dfrac{\cos(a+b)}{\cos(a)\cos(b)}=\dfrac{\cos(a)\cos(b)-\sin(a)\sin(b)}{\cos(a)\cos(b)}$

$$=\frac{\cos(a)\cos(b)}{\cos(a)\cos(b)}-\frac{\sin(a)\sin(b)}{\cos(a)\cos(b)}$$

$$=1-\tan(a)\tan(b)$$

7.3 Solutions to Exercises

1. a. $\sin(2x) = 2\sin x \cos x$

To find $\cos x$:

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \frac{1}{64} = \frac{63}{64}$

$\cos x = \pm\sqrt{\frac{63}{64}}$ Note that we need the positive root since we are told x is in quadrant 1.

$\cos x = \frac{3\sqrt{7}}{8}$

So: $\sin(2x) = 2\left(\frac{1}{8}\right)\left(\frac{3\sqrt{7}}{8}\right) = \frac{3\sqrt{7}}{32}$

b. $\cos(2x) = 2\cos^2 x - 1$

$= (2\left(\frac{63}{64}\right)) - 1 = \frac{63-32}{32} = \frac{31}{32}$

c. $\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{3\sqrt{7}}{32}\Big/\frac{31}{32} = \frac{3\sqrt{7}}{31}$

3. $\cos^2 x - \sin^2 x = \cos(2x)$, so $\cos^2(28°) - \sin^2(28°) = \cos(56°)$

5. $1 - 2\sin^2(x) = \cos(2x)$, so $1 - 2\sin^2(17°) = \cos(2\,(17°)) = \cos(34°)$

7. $\cos^2(9x) - \sin^2(9x) = \cos\big(2(9x)\big) = \cos(18x)$

9. $4\sin(8x)\cos(8x) = 2\,(2\sin(8x)\cos(8x)) = 2\sin(16x)$

11. $6\sin(2t) + 9\sin t = 6 \cdot 2\sin t\cos t + 9\sin t = 3\sin t\,(4\cos t + 3)$, so we can solve $3\sin t\,(4\cos t + 3) = 0$:

$\sin t = 0$ or $\cos t = -3/4$

$t = 0, \pi$ or $t \approx 2.4186, 3.8643$.

13. $9\cos(2\theta) = 9\cos^2\theta - 4$

$9(\cos^2\theta - \sin^2\theta) = 9\cos^2\theta - 4$

$9\sin^2\theta - 4 = 0$

$(3\sin\theta - 2)(3\sin\theta + 2) = 0$

$\sin\theta = \frac{2}{3}, -\frac{2}{3}$

$\theta = \sin^{-1}\frac{2}{3},\ \sin^{-1}\frac{-2}{3}$

$\theta \approx 0.7297, 2.4119, 3.8713, 5.5535$

15. $\sin(2t) = \cos t$

$2\sin t\cos t = \cos t$

$2\sin t\cos t - \cos t = 0$

$\cos t\,(2\sin t - 1) = 0$

$\cos t = 0$ or $2\sin t - 1 = 0$

If $\cos t = 0$, then $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $2\sin t - 1 = 0$, then $\sin t = \frac{1}{2}$, so $t = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. So $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

17. $\cos(6x) - \cos(3x) = 0$

$2\cos^2(3x) - 1 - \cos(3x) = 0$

$2\cos^2(3x) - \cos(3x) - 1 = 0$

$(2\cos(3x) + 1)(\cos(3x) - 1) = 0$

$\cos(3x) = -\frac{1}{2}$ or 1

Since we need solutions for x in the interval $[0, 2\pi)$, we will look for all solutions for $3x$ in the interval $[0, 6\pi)$. If $\cos(3x) = -\frac{1}{2}$, then there are two possible sets of solutions. First, $3x = \frac{2\pi}{3} + 2\pi k$ where $k = 0, 1,$ or 2, so $x = \frac{2\pi}{9} + \frac{2\pi k}{3}$ where $k = 0, 1,$ or 2. Second, $3x = \frac{4\pi}{3} + 2\pi k$ where $k = 0, 1,$ or 2, so $x = \frac{4\pi}{9} + \frac{2\pi k}{3}$ where $k = 0, 1,$ or 2 If $\cos(3x) = 1$, then $3x = 2\pi k$ where $k = 0, 1,$ or 2, so $x = \frac{2\pi k}{3}$ where $k = 0, 1,$ or 2.

19. $\cos^2(5x) = \frac{\cos(10x)+1}{2}$ because $\cos^2 x = \frac{\cos(2x)+1}{2}$ (power reduction identity)

21. $\sin^4(8x) = \sin^2(8x) \cdot \sin^2(8x)$

$= \frac{(1-\cos(16x))}{2} \cdot \frac{(1-\cos(16x))}{2}$ (because power reduction identity $\sin^2 x = \frac{(1-\cos(2x))}{2}$

$= \frac{1-2\cos(16x)}{4} + \frac{\cos^2(16x)}{4}$

$= \frac{1-2\cos(16x)}{4} + \frac{\cos(32x)+1}{8}$ because $\cos^2 x = \frac{\cos(2x)+1}{2}$ (power reduction identity)

$= \frac{1}{4} - \frac{\cos(16x)}{2} + \frac{\cos(32x)}{8} + \frac{1}{8}$

23. $\cos^2 x \sin^4 x$

$= \cos^2 x \cdot \sin^2 x \cdot \sin^2 x$

$= \frac{1+\cos(2x)}{2} \cdot \frac{1-\cos(2x)}{2} \cdot \frac{1-\cos(2x))}{2}$

$= \frac{1-\cos^2(2x)}{4} \cdot \frac{1-\cos(2x)}{2}$

$= \frac{1-\frac{\cos(4x)+1}{2}}{4} \cdot \frac{1-\cos(2x)}{2}$

$= \frac{1-\cos(4x)}{8} \cdot \frac{1-\cos(2x)}{2}$

$= \frac{1-\cos(2x)-\cos(4x)+\cos(2x)\cos(4x)}{16}$

25. Since $\csc x = 7$ and x is in quadrant 2, $\sin x = \frac{1}{7}$ (reciprocal of cosecant) and $\cos x = -\frac{4\sqrt{3}}{7}$ (Pythagorean identity).

a. $\sin\frac{x}{2} = \sqrt{\frac{(1-\cos(x))}{2}} = \sqrt{\frac{7+4\sqrt{3}}{14}}$ (Note that the answer is positive because x is in quadrant 2, so $\frac{x}{2}$ is in quadrant 1.)

b. $\cos\left(\frac{x}{2}\right) = \sqrt{\frac{(\cos x+1)}{2}} = \sqrt{\frac{7-4\sqrt{3}}{14}}$ (Note that the answer is positive because x is in quadrant 2, so $\frac{x}{2}$ is in quadrant 1.)

c. $\tan\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \sqrt{\frac{7+4\sqrt{3}}{7-4\sqrt{3}}} = \sqrt{\frac{(7+4\sqrt{3})^2}{(7-4\sqrt{3})(7+4\sqrt{3})}} = \sqrt{\frac{(7+4\sqrt{3})^2}{1}} = 7 + 4\sqrt{3}$

27. $(\sin t - \cos t)^2 = 1 - \sin(2t)$

Left side: $(\sin^2 t - 2\sin t\cos t + \cos^2 t)$

$= 1 - 2\sin t\cos t$ (because $(\sin^2 t + \cos^2 t = 1)$

$= 1 - \sin(2t)$, the right side (because $\sin(2t) = 2\sin t\cos t$)

29. $\sin(2x) = \frac{2\tan(x)}{1+\tan^2(x)}$

The right side: $\frac{\frac{2\sin(x)}{\cos(x)}}{1+\frac{\sin^2 x}{\cos^2 x}} \cdot \frac{\cos^2(x)}{\cos^2(x)} = \frac{2\sin(x)\cos(x)}{\cos^2(x)+\sin^2(x)} = 2\sin x\cos x = \sin(2x)$, the left side.

31. $\cot x - \tan x = 2\cot(2x)$

The left side: $\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}$

$= \frac{\cos^2 x - \sin^2 x}{\sin(x)\cos(x)} = \frac{\cos(2x)}{\frac{\sin(2x)}{2}} = 2\cot(2x)$

33. $\cos(2\alpha) = \frac{1-\tan^2(\alpha)}{1+\tan^2(\alpha)}$

The left side: $\frac{1-\tan^2(\alpha)}{1+\tan^2(\alpha)} = \frac{\frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}} = \frac{\frac{\cos(2\alpha)}{\cos^2\alpha}}{\frac{1}{\cos^2\alpha}} = \cos(2\alpha)$

35. $\sin(3x) = 3\sin(\mathrm{x})\cos^2 x - \sin^3 x$

Left side: $\sin(x+2x) = \sin(x)\cos(2x) + \cos(x)\sin(2x)$ addition rule.

$= \sin(x)\left(\cos^2(x) - \sin^2(x)\right) + \cos(x)\left(2\sin(x)\cos(x)\right)$

$= \cos^2(x)\sin(\mathrm{x}) - \sin^3(x) + 2\cos^2(x)\sin(x)$

$= 3\cos^2(x)\sin(\mathrm{x}) - \sin^3(x)$

7.4 Solutions to Exercises

1. By analysis, the function has a period of 12 units. The frequency is 1/12 Hz. The average of the *y*-values from $0 \le x < 12$ is -1, and since the terms repeat identically there is no change in the midline over time. Therefore the midline is $y = f(x) = -1$. The high point ($y = 2$) and low point ($y = -4$) are both 3 units away from the midline. Therefore, amplitude = 3 units. The function also starts at a minimum, which means that its phase must be shifted by one quarter of a cycle, or 3

units, to the right. Therefore, phase shift = 3.
Now insert known values into the function:

$$y = A\sin\left(\left(\frac{2\pi}{\text{period}}\right)\left(x - (\text{phase shift})\right)\right) + \text{midline}$$

$$y = 3\sin\left(\frac{2\pi}{12}(x-3)\right) - 1$$

This can be reduced to $y = 3\sin\left(\frac{\pi}{6}(x-3)\right) - 1$

Alternatively, had we chosen to use the cosine function: $y = -3\cos\left(\frac{\pi}{6}x\right) - 1$.

3. By analysis of the function, we determine:

A = amplitude = 8 units

Solving $\frac{2\pi}{\text{period}} = 6\pi$, we get: period = 1/3 seconds

Frequency = 3 Hz

5. In this problem, it is assumed that population increases linearly. Using the starting average as well as the given rate, the average population is then $y(x) = 650 + (160/12)x = 650 + (40/3)x$, where x is measured as the number of months since January.

Based on the problem statement, we know that the period of the function must be twelve months with an amplitude of 19. Since the function starts at a low-point, we can model it with a cosine function since -[cos(0)] = -1

Since the period is twelve months, the factor inside the cosine operator is equal to $\frac{2\pi}{12} = \frac{\pi}{6}$. Thus, the cosine function is $-19\cos\left(\frac{\pi}{6}x\right)$.

Therefore, our equation is: $y = f(x) = 650 + \frac{40}{3}x - 19\cos\left(\frac{\pi}{6}x\right)$

7. By analysis of the problem statement, the amplitude of the sinusoidal component is 33 units with a period of 12 months. Since the sinusoidal component starts at a minimum, its phase must be shifted by one quarter of a cycle, or 3 months, to the right.

$y = g(x) = 33\sin\left(\frac{\pi}{6}(x-3)\right)$

Using the starting average as well as the given rate, the average population is then:

$y = f(x) = 900(1.07)^x$

$g(x) + f(x) = 33\sin\left(\frac{\pi}{6}(x-3)\right) + 900(1.07)^x$

Alternatively, if we had used the cosine function, we'd get:

$h(x) = -33\cos\left(\frac{\pi}{6}x\right) + 900(1.07)^x$

9. The frequency is 18Hz, therefore period is 1/18 seconds. Starting amplitude is 10 cm. Since the amplitude decreases with time, the sinusoidal component must be multiplied by an exponential function. In this case, the amplitude decreases by 15% every second, so each new amplitude is 85% of the prior amplitude. Therefore, our equation is $y = f(x) = 10\cos(36\pi x) \cdot (0.85)^x$.

11. The initial amplitude is 17 cm. Frequency is 14 Hz, therefore period is 1/14 seconds.

For this spring system, we will assume an exponential model with a sinusoidal factor.
The general equation looks something like this: $D(t) = A(R)^t \cdot \cos(Bt)$ where *A* is amplitude, *R* determines how quickly the oscillation decays, and *B* determines how quickly the system oscillates. Since $D(0) = 17$, we know $A = 17$. Also, $B = \frac{2\pi}{\text{period}} = 28$.

We know $D(3) = 13$, so, plugging in:

$13 = 17(R)^3\cos(28\pi \cdot 3)$

$13 = 17(R)^3 \cdot 1$

$R^3 = \frac{13}{17}$

$R = \sqrt[3]{\frac{13}{17}} \approx 0.9145$

Thus, the solution is $D(t) = 17(0.9145)^t\cos(28\pi t)$.

13. By analysis:

(a) must have constant amplitude with exponential growth, therefore the correct graph is IV.

(b) must have constant amplitude with linear growth, therefore the correct graph is III.

15. Since the period of this function is 4, and values of a sine function are on its midline at the endpoints and center of the period, $f(0)$ and $f(2)$ are both points on the midline. We'll start by looking at our function at these points:

At $f(0)$, plugging into the general form of the equation, $6 = ab^0 + c\sin(0)$, so $a = 6$.

At $f(2)$: $96 = 6b^2 + c\sin(\pi)$, so $b = 4$.

At $f(1)$: $29 = 6(4)^1 + c\sin\left(\frac{\pi}{2}\right)$. Solving gives $c = 5$.

This gives a solution of $y = 6 \cdot 4^x + 5\sin\left(\frac{\pi}{2}x\right)$.

17. Since the period of this function is 4, and values of a sine function are on its midline at the endpoints and center of the period, $f(0)$ and $f(2)$ are both points on the midline.

At $f(0)$, plugging in gives $7 = a\sin(0) + m + b \cdot 0$, so $m = 7$.

At $f(2)$: $11 = a\sin(\pi) + 7 + 2b$. Since $\sin(\pi) = 0$, we get $b = 2$.

At $f(1)$: $6 = a\sin\left(\frac{\pi}{2}\right) + 7 + 2 \cdot 1$. Simplifying, $a = -3$.

This gives an equation of $y = -3\sin\left(\frac{\pi}{2}x\right) + 2x + 7$.

19. Since the first two places $\cos(\theta) = 0$ are when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, which for $\cos\left(\frac{\pi}{2}x\right)$ occur when $x = 1$ or $x = 3$, we'll start by looking at the function at these points:

At $f(1)$, plugging in gives $3 = ab^1\cos\left(\frac{\pi}{2}\right) + c$. Since $\cos\left(\frac{\pi}{2}\right) = 0$, $c = 3$. (Note that looking at $f(3)$ would give the same result.)

At $f(0)$: $11 = ab^0\cos(0) + 3$. Simplifying, we see $a = 8$.

At $f(2)$: $1 = 8b^2\cos(\pi) + 3$. Since $\cos(\pi) = -1$, it follows that $b = \frac{1}{2}$:

$1 = -8b^2 + 3$

$-8b^2 = -2$

$b^2 = \frac{1}{4}$

$b = \pm\frac{1}{2}$, but since we require exponential expressions to have a positive number as the base,

$b = \frac{1}{2}$. Therefore, the final equation is: $y = 8\left(\frac{1}{2}\right)^x \cos\left(\frac{\pi}{2}x\right) + 3$.