Weighted Voting

In a corporate shareholders meeting, each shareholders' vote counts proportional to the amount of shares they own. An individual with one share gets the equivalent of one vote, while someone with 100 shares gets the equivalent of 100 votes. This is called **weighted voting**, where each vote has some weight attached to it. Weighted voting is sometimes used to vote on candidates, but more commonly to decide "yes" or "no" on a proposal, sometimes called a motion. Weighted voting is applicable in corporate settings, as well as decision making in parliamentary governments and voting in the United Nations Security Council.

In weighted voting, we are most often interested in the power each voter has in influencing the outcome.

Beginnings

We'll begin with some basic vocabulary for weighted voting systems.

Each individual or entity casting a vote is called a **player** in the election. They're often notated as $P_1, P_2, P_3, \ldots, P_N$, where N is the total number of voters.

Each player is given a **weight**, which usually represents how many votes they get.

The **quota** is the minimum weight needed for the votes or weight needed for the proposal to be approved.

A weighted voting system will often be represented in a shorthand form:

$$[q: w_1, w_2, w_3, \ldots, w_n]$$

In this form, q is the quota, w_1 is the weight for player 1, and so on.

Example: In a small company, there are 4 shareholders. Mr. Smith has a 30% ownership stake in the company, Mr. Garcia has a 25% stake, Mrs. Hughes has a 25% stake, and Mrs. Lee has a 20% stake. They are trying to decide whether to open a new location. The company by-laws state that more than 50% of the ownership has to approve any decision like this. This could be represented by the weighted voting system:

[51: 30, 25, 25, 20]

Here we have treated the percentage ownership as votes, so Mr. Smith gets the equivalent of 30 votes, having a 30% ownership stake. Since more than 50% is required to approve the decision, the quota is 51, the smallest whole number over 50.

In order to have a meaningful weighted voting system, it is necessary to put some limits on the quota. The quota must be at more than ½ the total number of votes. Likewise, the quota can't be larger than the total number of votes.

Why? Consider the voting system [q: 3, 2, 1]

Here there are 6 total votes. If the quota was set at only 3, then player 1 could vote yes, players 2 and 3 could vote no, and both would reach quota, which doesn't lead to a decision being made. If the quota was set to 7, then no group of voters could ever reach quota, and no decision can be made.

A Look at Power

Consider the voting system [10: 11, 3, 2]. Notice that in this system, player 1 can reach quota without the support of any other player. When this happens, we say that player 1 is a **dictator**. A player will be a dictator if their weight is equal to or greater than the quota. The dictator can also block any proposal from passing; the other players cannot reach quota without the dictator.

In the voting system [8: 6, 3, 2], no player is a dictator. However, in this system, the quota can only be reached if player 1 is in support of the proposal; player 2 and 3 cannot reach quota without player 1's support. In this case, player 1 is said to have **veto power**. Notice that player 1 is not a dictator, since player 1 would still need player 2 or 3's support to reach quota.

With the system [10: 7, 6, 2], player 3 is said to be a **dummy**, meaning they have no influence in the outcome. The only way the quota can be met is with the support of both players 1 and 2 (both of which would have veto power here); the vote of player 3 cannot affect the outcome.

Example: In the voting system [16: 7, 6, 3, 3, 2], are any players dictators? Do any have veto power? Are any dummies?

No player can reach quota alone, so there are no dictators.

Without player 1, the rest of the players' weights add to 14, which doesn't reach quota, so player 1 has veto power. Likewise, without player 2, the rest of the players' weights add to 15, which doesn't reach quota, so player 2 also has veto power.

Since player 1 and 2 can reach quota with either player 3 or player 4's support, neither player 3 or player 4 have veto power. However they cannot reach quota with player 5's support alone, so player 5 has no influence on the outcome and is a dummy.

To better define power, we need to introduce the idea of a **coalition**. A coalition is a group of players voting the same way. In the example above, $\{P_1, P_2, P_4\}$ would represent the coalition of players 1, 2 and 4. This coalition has a combined weight of 7+6+3 = 16, which meets quota, so this would be a winning coalition.

A player is said to be **critical** in a coalition if them leaving the coalition would change it from a winning coalition to a losing coalition. In the coalition $\{P_1, P_2, P_4\}$, every player is critical. In the coalition $\{P_3, P_4, P_5\}$, no player is critical, since it wasn't a winning coalition to begin with. In the coalition $\{P_1, P_2, P_3, P_4, P_5\}$, only players 1 and 2 are critical; any other player could leave the coalition and it would still meet quota.

Example: In the Scottish Parliament, there were (as of 2009) 5 political parties: 47 representatives for the Scottish National Party, 46 for the Labour Party, 17 for the Conservative Party, 16 for the Liberal Democrats, and 2 for the Scottish Green Party. Typically all representatives from a party vote as a block, so the parliament can be treated like the weighted voting system:

[65: 47, 46, 17, 16, 2]

Consider the coalition $\{P_1, P_3, P_4\}$. No two players alone could meet the quota, so all three players are critical in this coalition.

In the coalition $\{P_1, P_3, P_4, P_5\}$, any player except P_1 could leave the coalition and it would still meet quota, so only P_1 is critical in this coalition.

Notice that a player with veto power will be critical in every winning coalition, since removing their support would prevent a proposal from passing.

Likewise, a dummy will never be critical, since their support will never change a losing coalition to a winning one.

Calculating Power: Banzhaf Power Index

The **Banzhaf power index** was originally created in 1946 by Lionel Penrose, but was reintroduced by John Banzhaf in 1965. The power index is a numerical way of looking at power in a weighted voting situation.

Banzhaf power index is calculated by:

1) List all winning coalitions
2) In each coalition, identify the players who are critical
3) Count up how many times each player is critical
4) Convert these counts to fractions or decimals by dividing by the total times any player is critical

Example: Consider the voting system [16: 7, 6, 3, 3, 2]. The winning coalitions are listed below, with the critical players underlined.

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$
$\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$
$\{\underline{P_1}, \underline{P_2}, P_3, P_4\}$
$\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_5\}$
$\{\underline{P_1}, \underline{P_2}, \underline{P_4}, P_5\}$
$\{\underline{P_1}, \underline{P_2}, P_3, P_4, P_5\}$

Counting up times that each player is critical:

$P_1 = 6$
$P_2 = 6$
$P_3 = 2$
$P_4 = 2$
$P_5 = 0$
Total of all: 16

Divide each player's count by 16 to convert to fractions or percents:

P_1 = 6/16 = 3/8 = 37.5%
P_2 = 6/16 = 3/8 = 37.5%
P_3 = 2/16 = 1/8 = 12.5%
P_4 = 2/16 = 1/8 = 12.5%
P_5 = 0/16 = 0 = 0%

This power index measures a player's ability to influence the outcome of the vote. Notice that player 5 has a power index of 0, indicating that there is no coalition in which they would be critical and could influence the outcome. This means player 5 is a dummy, as we noted earlier.

Example: Revisiting the Scottish Parliament, with voting system [65: 47, 46, 17, 16, 2], the winning coalitions are listed, with the critical players underlined.

$\{\underline{P_1}, \underline{P_2}\}$
$\{\underline{P_1}, \underline{P_2}, P_3\}$
$\{\underline{P_1}, \underline{P_2}, P_4\}$
$\{\underline{P_1}, \underline{P_2}, P_5\}$
$\{P_1, P_2, P_3, P_4\}$
$\{P_1, P_2, P_3, P_5\}$
$\{\underline{P_1}, P_2, P_4, P_5\}$
$\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$
$\{\underline{P_1}, \underline{P_3}, \underline{P_5}\}$
$\{\underline{P_1}, \underline{P_4}, \underline{P_5}\}$
$\{\underline{P_1}, P_3, P_4, P_5\}$
$\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$
$\{\underline{P_2}, \underline{P_3}, \underline{P_5}\}$
$\{\underline{P_2}, \underline{P_3}, P_4, P_5\}$
$\{P_1, P_2, P_3, P_4, P_5\}$

Counting up times that each player is critical:

District	**Times critical**	**Power index**
P_1 (Scottish National Party)	9	9/27 = 33.3%
P_2 (Labour Party)	7	7/27 = 25.9%
P_3 (Conservative Party)	5	5/27 = 18.5%
P_4 (Liberal Democrats Party)	3	3/27 = 11.1%
P_5 (Scottish Green Party)	3	3/27 = 11.1%

Interestingly, even though the Liberal Democrats party has only one less representative than the Conservative Party, and 14 more than the Scottish Green Party, their Banzhaf power index is the same as the Scottish Green Party's. In parliamentary governments, forming coalitions is an essential part of getting results, and a party's ability to help a coalition reach quota defines its influence.

Example: Banzhaf used this index to argue that the weighted voting system used in the Nassau County Board of Supervisors in New York was unfair. The county was divided up into 6 districts, each getting voting weight proportional to the population in the district, as shown below:

District	**Weight**
Hempstead #1	31
Hempstead #2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

Translated into a weighted voting system, assuming a simple majority is needed for a proposal to pass:

[58: 31, 31, 28, 21, 2, 2]

Listing the winning coalitions:

{H1, H2}
{H1, OB}
{H2, OB}
{H1, H2, NH}
{H1, H2, LB}
{H1, H2, GC}
{H1, H2, NH, LB}
{H1, H2, NH, GC}
{H1, H2, LB, GC}
{H1, H2, NH, LB. GC}
{H1, OB, NH}
{H1, OB, LB}
{H1, OB, GC}
{H1, OB, NH, LB}
{H1, OB, NH, GC}
{H1, OB, LB, GC}
{H1, OB, NH, LB. GC}
{H2, OB, NH}
{H2, OB, LB}
{H2, OB, GC}
{H2, OB, NH, LB}
{H2, OB, NH, GC}
{H2, OB, LB, GC}
{H2, OB, NH, LB, GC}
{H1, H2, OB}
{H1, H2, OB, NH}
{H1, H2, OB, LB}
{H1, H2, OB, GC}
{H1, H2, OB, NH, LB}
{H1, H2, OB, NH, GC}
{H1, H2, OB, NH, LB, GC}

There's a lot of them! Counting up how many times each player is critical,

District	**Times critical**	**Power index**
Hempstead #1	16	16/48 = 1/3 = 33%
Hempstead #2	16	16/48 = 1/3 = 33%
Oyster Bay	16	16/48 = 1/3 = 33%
North Hempstead	0	0/48 = 0%
Long Beach	0	0/48 = 0%
Glen Cove	0	0/48 = 0%

It turns out that the three smaller districts are dummies. Any winning coalition requires two of the larger districts.

The weighted voting system that most Americans are most familiar with is the Electoral College system used to elect the President. In the Electoral College, states are given a number of votes equal to the number of their congress representatives (house + senate). Most states give all their electoral votes to the candidate that wins a majority in their state, turning the Electoral College into a weighted voting system, in which the states are the players. As I'm sure you can imagine, there are billions of possible winning coalitions, so the power index for the Electoral College has to be computed by a computer using approximation techniques.

Calculating Power: Shapley-Shubik Power Index

The **Shapley-Shubik** power index was introduced in 1954 by economists Lloyd Shapley and Martin Shubik, and provides a different approach for calculating power.

In situations like political alliances, the order in which players join an alliance could be considered the most important consideration. In particular, if a proposal is introduced, the player that joins the coalition and allows it to reach quota might be considered the most essential. The Shapley-Shubik power index counts how likely a player is to be **pivotal**. What does it mean for a player to be pivotal?

First, we need to change our approach to coalitions. Previously, the coalition {P_1, P_2} and {P_2, P_1} would be considered equivalent, since they contain the same players. We now need to consider the *order* in which players join the coalition. For that, we will consider **sequential coalitions –** coalitions that contain all the players in which the order players are listed reflect the order they joined the coalition. For example, the sequential coalition <P_2, P_1, P_3> would mean that P_2 joined the coalition first, then P_1, and finally P_3. The angle brackets < > are used instead of curly brackets to distinguish sequential coalitions.

A **pivotal player** is the player in a sequential coalition that changes a coalition from a losing coalition to a winning one. Notice there can only be one pivotal player in any sequential coalition.

Example: In the weighted voting system [8: 6, 4, 3, 2], which player is critical in the sequential coalition <P_3, P_2, P_4, P_1>?

The sequential coalition shows the order in which players joined the coalition. Consider the running totals as each player joins:

P_3	Total weight: 3	Not winning
P_3, P_2	Total weight: 3+4 = 7	Not winning
P_3, P_2, P_4	Total weight: 3+4+2 = 9	Winning
P_3, P_2, P_4, P_1	Total weight: 3+4+2+6 = 15	Winning

Since the coalition *becomes* winning when P_4 joins, P_4 is the pivotal player in this coalition.

Shabley-Shubik Power Index is calculated by:
1) List all sequential coalitions
2) In each sequential coalition, determine the pivotal player
3) Count up how many times each player is pivotal
4) Convert these counts to fractions or decimals by dividing by the total number of sequential coalitions

How many sequential coalitions should we expect to have? If there are N players in the voting system, then there are N possibilities for the first player in the coalition, $N-1$ possibilities for the second player in the coalition, and so on. Combining these possibilities, the total number of coalitions would be: $N(N-1)(N-2)(N-3)\cdots(3)(2)(1)$. This calculation is called a **factorial**, and is notated $N!$ The number of sequential coalitions with N players is $N!$

Example: How many sequential coalitions will there be in a voting system with 7 players?

There will be 7! sequential coalitions. $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

As you can see, computing the Shapley-Shubik power index by hand would be very difficult for voting systems that are not very small.

Example: Consider the weighted voting system [6: 4, 3, 2]. We will list all the sequential coalitions and identify the pivotal player. We will have 3! = 6 sequential coalitions. The coalitions are listed, and the pivotal player is underlined.

$< P_1, \underline{P_2}, P_3 >$ $< P_1, \underline{P_3}, P_2 >$ $< P_2, \underline{P_1}, P_3 >$
$< P_2, P_3, \underline{P_1} >$ $< P_3, P_2, \underline{P_1} >$ $< P_3, \underline{P_1}, P_2 >$

P_1 is pivotal 4 times, P_2 is pivotal 1 time, and P_3 is pivotal 1 time.

Player	**Times pivotal**	**Power index**
P_1	4	4/6 = 66.7%
P_2	1	1/6 = 16.7%
P_3	1	1/6 = 16.7%

For comparison, the Banzhaf power index for the same weighted voting system would be P_1: 60%, P_2: 20%, P_3: 20%. While the Banzhaf power index and Shapley-Shubik power index are usually not terribly different, the two different approaches usually produce somewhat different results.

Exercises

Skills

1. Consider the weighted voting system [47: 10,9,9,5,4,4,3,2,2]
    a. How many players are there?
    b. What is the total number (weight) of votes?
    c. What is the quota in this system?

2. Consider the weighted voting system [31: 10,10,8,7,6,4,1,1]
    a. How many players are there?
    b. What is the total number (weight) of votes?
    c. What is the quota in this system?

3. Consider the weighted voting system [*q*: 7,5,3,1,1]
    a. What is the smallest value that the quota *q* can take?
    b. What is the largest value that the quota *q* can take?
    c. What is the value of the quota if at least two-thirds of the votes are required to pass a motion?

4. Consider the weighted voting system [*q*: 10,9,8,8,8,6]
    a. What is the smallest value that the quota *q* can take?
    b. What is the largest value that the quota *q* can take?
    c. What is the value of the quota if at least two-thirds of the votes are required to pass a motion?

5. Consider the weighted voting system [13: 13, 6, 4, 2]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.

6. Consider the weighted voting system [11: 9, 6, 3, 1]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.

7. Consider the weighted voting system [19: 13, 6, 4, 2]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.

8. Consider the weighted voting system [17: 9, 6, 3, 1]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.

9. Consider the weighted voting system [15: 11, 7, 5, 2]
   a. What is the weight of the coalition {P_1,P_2,P_4}
   b. In the coalition {P_1,P_2,P_4} which players are critical?

10. Consider the weighted voting system [17: 13, 9, 5, 2]
   a. What is the weight of the coalition {P_1,P_2,P_3}
   b. In the coalition {P_1,P_2,P_3} which players are critical?

11. Find the Banzhaf power distribution of the weighted voting system [27: 16, 12, 11, 3]

12. Find the Banzhaf power distribution of the weighted voting system [33: 18, 16, 15, 2]

13. Consider the weighted voting system [q: 15, 8, 3, 1] Find the Banzhaf power distribution of this weighted voting system,
   a. When the quota is 15
   b. When the quota is 16
   c. When the quota is 18

14. Consider the weighted voting system [q: 15, 8, 3, 1] Find the Banzhaf power distribution of this weighted voting system,
   a. When the quota is 19
   b. When the quota is 23
   c. When the quota is 26

15. Consider the weighted voting system [17: 13, 9, 5, 2]. In the sequential coalition $\langle P_3,P_2,P_1,P_4 \rangle$ which player is pivotal?

16. Consider the weighted voting system [15: 13, 9, 5, 2]. In the sequential coalition $\langle P_1,P_4,P_2,P_3 \rangle$ which player is pivotal?

17. Find the Shapley-Shubik power distribution for the system [24: 17, 13, 11]

18. Find the Shapley-Shubik power distribution for the system [25: 17, 13, 11]

Concepts

19. Consider the weighted voting system [*q*: 7, 3, 1]
   a. Which values of *q* result in a dictator (list all possible values)
   b. What is the smallest value for *q* that results in exactly one player with veto power but no dictators?
   c. What is the smallest value for *q* that results in exactly two players with veto power?

20. Consider the weighted voting system [*q*: 9, 4, 2]
    a. Which values of *q* result in a dictator (list all possible values)
    b. What is the smallest value for *q* that results in exactly one player with veto power?
    c. What is the smallest value for *q* that results in exactly two players with veto power?

21. Using the Shapley-Shubik method, is it possible for a dummy to be pivotal?

22. If a specific weighted voting system requires a unanimous vote for a motion to pass:
    a. Which player will be pivotal in any sequential coalition?
    b. How many winning coalitions will there be?

23. Consider a weighted voting system with three players. If Player 1 is the only player with veto power, there are no dictators, and there are no dummies:
    a. Find the Banzhof power distribution.
    b. Find the Shapley-Shubik power distribution

24. Consider a weighted voting system with three players. If Players 1 and 2 have veto power but are not dictators, and Player 3 is a dummy:
    a. Find the Banzhof power distribution.
    b. Find the Shapley-Shubik power distribution

25. An executive board consists of a president (P) and three vice-presidents (V_1,V_2,V_3). For a motion to pass it must have three yes votes, one of which must be the president's. Find a weighted voting system to represent this situation.

26. On a college's basketball team, the decision of whether a student is allowed to play is made by four people: the head coach and the three assistant coaches. To be allowed to play, the student needs approval from the head coach and at least one assistant coach. Find a weighted voting system to represent this situation.

27. In a corporation, the shareholders receive 1 vote for each share of stock they hold, which is usually based on the amount of money the invested in the company. Suppose a small corporation has two people who invested $30,000 each, two people who invested $20,000 each, and one person who invested $10,000. If they receive one share of stock for each $1000 invested, and any decisions require a majority vote, set up a weighted voting system to represent this corporation's shareholder votes.

28. A contract negotiations group consists of 4 workers and 3 managers. For a proposal to be accepted, a majority of workers and a majority of managers must approve of it. Calculate the Banzhaf power distribution for this situation. Who has more power: a worker or a manager?

29. The United Nations Security Council consists of 15 members, 10 of which are elected, and 5 of which are permanent members. For a resolution to pass, 9 members must support it, which must include all 5 of the permanent members. Set up a weighted voting system to represent the UN Security Council and calculate the Banzhaf power distribution.

Exploration

30. In the U.S., the Electoral College is used in presidential elections. Each state is awarded a number of electors equal to the number of representatives (based on population) and senators (2 per state) they have in congress. Since most states award the winner of the popular vote in their state all their state's electoral votes, the Electoral College acts as a weighted voting system. To explore how the Electoral College works, we'll look at a mini-country with only 4 states. Here is the outcome of a hypothetical election:

State	**Smalota**	**Medigan**	**Bigonia**	**Hugodo**
Population	50,000	70,000	100,000	240,000
Votes for A	40,000	50,000	80,000	50,000
Votes for B	10,000	20,000	20,000	190,000

a. If this country did not use an Electoral College, which candidate would win the election?
b. Suppose that each state gets 1 electoral vote for every 10,000 people. Set up a weighted voting system for this scenario, calculate the Banzhaf power index for each state, then calculate the winner if each state awards all their electoral votes to the winner of the election in their state.
c. Suppose that each state gets 1 electoral vote for every 10,000 people, plus an additional 2 votes. Set up a weighted voting system for this scenario, calculate the Banzhaf power index for each state, then calculate the winner if each state awards all their electoral votes to the winner of the election in their state.
d. Suppose that each state gets 1 electoral vote for every 10,000 people, and awards them based on the number of people who voted for each candidate. Additionally, they get 2 votes that are awarded to the majority winner in the state. Calculate the winner under these conditions.
e. Does it seem like an individual state has more power in the Electoral College under the vote distribution from part c or from part d?
f. Research the history behind the Electoral College to explore why the system was introduced instead of using a popular vote. Based on your research and experiences, state and defend your opinion on whether the Electoral College system is or is not fair.