Logic

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like "and" and "or" – to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

Boolean Logic

We can often classify items as belonging to sets. If you went the library to search for a book and they asked you to express your search using unions, intersections, and complements of sets, that would feel a little strange. Instead, we typically using words like "and", "or", and "not" to connect our keywords together to form a search. These words, which form the basis of **Boolean logic**, are directly related to our set operations. (Boolean logic was developed by the 19th-century English mathematician George Boole.)

> **Boolean Logic**
> Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.
>
> In connection to sets, a search is true if the element is part of the set.

Suppose M is the set of all mystery books, and C is the set of all comedy books. If we search for "mystery", we are looking for all the books that are an element of the set M; the search is true for books that are in the set.

When we search for "mystery *and* comedy", we are looking for a book that is an element of both sets, in the intersection. If we were to search for "mystery *or* comedy", we are looking for a book that is a mystery, a comedy, or both, which is the union of the sets. If we searched for "*not* comedy", we are looking for any book in the library that is not a comedy, the complement of the set C.

> **Connection to Set Operations**
> A and B elements in the intersection $A \cap B$
> A or B elements in the union $A \cup B$
> not A elements in the complement A^c

Notice here that *or* is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks "do you want to go to the park or the movies?" they usually are proposing an exclusive choice – one option or the other, but not both. In Boolean logic, the *or* is not exclusive – more like being asked at a restaurant "would you like fries or a drink with that?" Answering "both, please" is an acceptable answer.

Example 1

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.

We could start with the search "Mexico *and* university", but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read: Mexico *and* university *not* "New Mexico"

In most internet search engines, it is not necessary to include the word *and*; the search engine assumes that if you provide two keywords you are looking for both. In Google's search, the keyword *or* has be capitalized as OR, and a negative sign in front of a word is used to indicate *not*. Quotes around a phrase indicate that the entire phrase should be looked for. The search from the previous example on Google could be written:
Mexico university -"New Mexico"

Example 2

Describe the numbers that meet the condition:
    even and less than 10 and greater than 0

The numbers that satisfy all three requirements are {2, 4, 6, 8}

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedent, just like in algebraic order of operations.

Example 3

Describe the numbers that meet the condition:
    odd number and less than 20 and greater than 0 and (multiple of 3 or multiple of 5)

The first three conditions limit us to the set {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5. This leaves us with the set {3, 5, 9, 15}

Notice that we would have gotten a very different result if we had written
    (odd number and less than 20 and greater than 0 and multiple of 3) or multiple of 5
The first grouped set of conditions would give {3, 9, 15}. When combined with the last condition, though, this set expands without limits:
{3, 5, 9, 15, 20, 25, 30, 35, 40, 45, …}

Example 4

The English phrase "Go to the store and buy me eggs and bagels or cereal" is ambiguous; it is not clear whether the requestors is asking for eggs always along with either bagels or cereal, or whether they're asking for either the combination of eggs and bagels, or just cereal.

For this reason, using parentheses clarifies the intent:
Eggs and (bagels or cereal) means   Option 1: Eggs and bagels, Option 2: Eggs and cereal
(Eggs and bagels) or cereal means   Option 1: Eggs and bagels, Option 2: Cereal

Be aware that when a string of conditions is written without grouping symbols, it is often interpreted from the left to right, resulting in the latter interpretation.

Conditional Statements

Beyond searching, Boolean logic is commonly used in spreadsheet applications like Excel to do conditional calculations. A **statement** is something that is either true or false. A statement like $3 < 5$ is true; a statement like "a rat is a fish" is false. A statement like "$x < 5$" is true for some values of x and false for others. When an action is taken or not depending on the value of a statement, it forms a **conditional**.

> **Statements and Conditionals**
> A **statement** is either true or false.
> A **conditional** is a compound statement of the form
>   "if p then q" or "if p then q, else s".

Example 5

In common language, an example of a conditional statement would be "If it is raining, then we'll go to the mall. Otherwise we'll go for a hike."

The statement "If it is raining" is the condition – this may be true or false for any given day. If the condition is true, then we will follow the first course of action, and go to the mall. If the condition is false, then we will use the alternative, and go for a hike.

Example 6

As mentioned earlier, conditional statements are commonly used in spreadsheet applications like Excel or Google Sheets. In Excel, you can enter an expression like
 =IF(A1<2000, A1+1, A1*2)

Notice that after the IF, there are three parts. The first part is the condition, and the second two are calculations. Excel will look at the value in cell A1 and compare it to 2000. If that condition is true, then the first calculation is used, and 1 is added to the value of A1 and the result is stored. If the condition is false, then the second calculation is used, and A1 is multiplied by 2 and the result is stored.

In other words, this statement is equivalent to saying "If the value of A1 is less than 2000, then add 1 to the value in A1. Otherwise, multiply A1 by 2".

Example 7

The expression =IF(A1>5, 2*A1, 3*A1) is used. Find the result if A1 is 3, and the result if A1 is 8.

This is equivalent to saying
If A1 > 5, then calculate 2*A1. Otherwise, calculate 3*A1

If A1 is 3, then the condition is false, since 3 > 5 is not true, so we do the alternate action, and multiply by 3, giving 3*3 = 9

If A1 is 8, then the condition is true, since 8 > 5, so we multiply the value by 2, giving 2*8=16

Example 8

An accountant needs to withhold 15% of income for taxes if the income is below $30,000, and 20% of income if the income is $30,000 or more. Write an expression that would calculate the amount to withhold.

Our conditional needs to compare the value to 30,000. If the income is less than 30,000, we need to calculate 15% of the income: 0.15*income. If the income is more than 30,000, we need to calculate 20% of the income: 0.20*income.

In words we could write "If income < 30,000, then multiply by 0.15, otherwise multiply by 0.20". In Excel, we would write:

=IF(A1<30000, 0.15*A1, 0.20*A1)

As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example 9

A parent might say to their child "if you clean your room and take out the garbage, then you can have ice cream."

Here, there are two simpler conditions:
1) The child cleaning her room
2) The child taking out the garbage

Since these conditions were joined with *and*, the combined conditional will be true only if both simpler conditions are true; if either chore is not completed, then the parent's condition is not met.

Notice that if the parent had said "if you clean your room *or* take out the garbage, then you can have ice cream", then the child would need to complete only one chore to meet the condition.

Suppose you wanted to have something happen when a certain value is between 100 and 300. To create the condition "A1 < 300 and A1 > 100" in Excel, you would need to enter "AND(A1<300, A1>100)". Likewise, for the condition "A1=4 or A1=6" you would enter "OR(A1=4, A1=6)"

Example 10

In a spreadsheet, cell A1 contains annual income, and A2 contains number of dependents. A certain tax credit applies if someone with no dependents earns less than $10,000, or if someone with dependents earns less than $20,000. Write a rule that describes this.

There are two ways the rule is met:
income is less than 10,000 *and* dependents is 0, *or*
income is less than 20,000 *and* dependents is not 0.

Informally, we could write these as
(A1 < 10000 *and* A2 = 0) *or* (A1 < 20000 *and* A2 > 0)

In Excel's format, we'd write
IF(OR(AND(A1<10000, A2=0), AND(A1<20000, A2>0)), "you qualify", "you don't qualify")

Quantified Statements

Words that describe an entire set, such as "all", "every", or "none", are called **universal quantifiers** because that set could be considered a universal set. In contrast, words or phrases such as "some", "one", or "at least one" are called **existential quantifiers** because they describe the existence of at least one element in a set.

> **Quantifiers**
> A **universal quantifier** states that an entire set of things share a characteristic.
>
> An **existential quantifier** states that a set contains at least one element.

Something interesting happens when we **negate** – or state the opposite of – a quantified statement.

Example 11

Suppose your friend says "Everybody cheats on their taxes." What is the minimum amount of evidence you would need to prove your friend wrong?

To show that it is not true that everybody cheats on their taxes, all you need is <u>one</u> person who does not cheat on their taxes. It would be perfectly fine to produce more people who do not cheat, but one counterexample is all you need.

It is important to note that you do <u>not</u> need to show that absolutely nobody cheats on their taxes.

Example 12

Suppose your friend says "One of these six cartons of milk is leaking." What is the minimum amount of evidence you would need to prove your friend wrong?

In this case, you would need to check <u>all</u> six cartons and show that none of them is leaking. You cannot disprove your friend's statement by checking only one of the cartons.

When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

Negating a quantified statement
The negation of "all A are B" is "at least one A is not B".
The negation of "no A are B" is "at least one A is B".
The negation of "at least one A is B" is "no A are B".
The negation of "at least one A is not B" is "all A are B".

Example 13

"Somebody brought a flashlight." Write the negation of this statement.

The negation is "Nobody brought a flashlight."

Example 14

"There are no prime numbers that are even." Write the negation of this statement.

The negation is "At least one prime number is even."

Try it Now 1
Write the negation of "All Icelandic children learn English in school."

Truth Tables: Conjunction (and), Disjunction (or), Negation (not)

Before we focus on truth tables, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol ∧ is used for *and*:   *A* and *B* is notated *A* ∧ *B*

The symbol ∨ is used for *or*:   *A* or *B* is notated *A* ∨ *B*

The symbol ~ is used for *not*:   not *A* is notated ~*A*

You can remember the first two symbols by relating them to the shapes for the union and intersection. *A* ∧ *B* would be the elements that exist in both sets, in A ∩ B. Likewise, *A* ∨ *B* would be the elements that exist in either set, in A ∪ B. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 15

Translate each statement into symbolic notation. Let *P* represent "I like Pepsi" and let *C* represent "I like Coke".

a. I like Pepsi or I like Coke.
b. I like Pepsi and I like Coke.
c. I do not like Pepsi.
d. It is not the case that I like Pepsi or Coke.
e. I like Pepsi and I do not like Coke.

a. *P* ∨ *C*

b. *P* ∧ *C*

c. ~*P*

d. ~(*P* ∨ *C*)

e. *P* ∧ ~*C*

As you can see, we can use parentheses to organize more complicated statements.

Try it Now 2

Translate "We have carrots or we will not make soup" into symbols. Let *C* represent "we have carrots" and let *S* represent "we will make soup".

Because complex Boolean statements can get tricky to think about, we can create a **truth table** to keep track of what truth values for the simple statements make the complex statement true and false.

> **Truth table**
> A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 16

Suppose you're picking out a new couch, and your significant other says "get a sectional *or* something with a chaise".

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use S to designate "is a sectional", and C to designate "has a chaise".
A truth table for this situation would look like this:

S	C	S or C
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if S is true and C is also true, then the complex statement "S or C" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

In the previous example about the couch, the truth table was really just summarizing what we already know about how the *or* statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

> **Basic truth tables**
>
> **Conjunction**
>
> | A | B | $A \wedge B$ |
> |-----|-----|------------|
> | T | T | T |
> | T | F | F |
> | F | T | F |
> | F | F | F |
>
> **Disjunction**
>
> | A | B | $A \vee B$ |
> |-----|-----|-----------|
> | T | T | T |
> | T | F | T |
> | F | T | T |
> | F | F | F |
>
> **Negation**
>
> | A | $\sim A$ |
> |-----|---------|
> | T | F |
> | F | T |

Truth tables really become useful when we analyze more complex Boolean statements.

Example 17

Create a truth table for the statement $A \lor \sim B$

When we create the truth table, we need to list all the possible truth value combinations for A and B. Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

A	B
T	T
T	F
F	T
F	F

After creating columns with those initial values, we create a third column for the expression $\sim B$. Now we will temporarily ignore the column for A and write the truth values for $\sim B$.

A	B	$\sim B$
T	T	F
T	F	T
F	T	F
F	F	T

Next we can find the truth values of $A \lor \sim B$, using the first and third columns.

A	B	$\sim B$	$A \lor \sim B$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

The truth table shows that $A \lor \sim B$ is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game <u>or</u> if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If A = Anaheim wins its game and B = Boston wins its game, then $A \lor \sim B$ represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is <u>false</u> that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Try it Now 3

Create a truth table for this statement: $\sim A \land B$

Example 18

Create a truth table for the statement $A \land \sim(B \lor C)$

It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for A, B, and C. Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates T, F, T, F... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, $B \lor C$. Now we will temporarily ignore the column for A and focus on B and C, writing the truth values for $B \lor C$.

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

A	B	C	$B \lor C$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Next we can find the negation of $B \lor C$, working off the $B \lor C$ column we just created. (Ignore the first three columns and simply negate the values in the $B \lor C$ column.)

A	B	C	$B \lor C$	$\sim(B \lor C)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of A and $\sim(B \lor C)$. (Ignore the second, third, and fourth columns.)

A	B	C	$B \lor C$	$\sim(B \lor C)$	$A \land \sim(B \lor C)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is true in only one case: when *A* is true, *B* is false, and *C* is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. TFF is the only scenario in which Anaheim will make the playoffs.

Try it Now 4
Create a truth table for this statement: (~*A* ∧ *B*) ∨ ~*B*

Truth Tables: Conditional, Biconditional

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

> **Conditional**
> A conditional is a logical compound statement in which a statement *p*, called the antecedent, implies a statement *q*, called the consequent.
>
> A conditional is written as *p* → *q* and is translated as "if *p*, then *q*".

Example 19

The English statement "If it is raining, then there are clouds is the sky" is a conditional statement. It makes sense because if the antecedent "it is raining" is true, then the consequent "there are clouds in the sky" must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consquent becomes irrelevant.

Example 20

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday's game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

There are four possible outcomes:
1) You pay for expedited shipping and receive the jersey by Friday
2) You pay for expedited shipping and don't receive the jersey by Friday
3) You don't pay for expedited shipping and receive the jersey by Friday
4) You don't pay for expedited shipping and don't receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the *only* way to receive the jersey by Friday.

Example 21

A friend tells you "If you upload that picture to Facebook, you'll lose your job." Under what conditions can you say that your friend was wrong?

There are four possible outcomes:
1) You upload the picture and lose your job
2) You upload the picture and don't lose your job
3) You don't upload the picture and lose your job
4) You don't upload the picture and don't lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't say that their statement was wrong. Even if you didn't upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn't upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

Truth table for the conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 22

Construct a truth table for the statement $(m \land \sim p) \to r$

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, $m \land \sim p$.

m	p	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

m	p	r	$\sim p$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

m	p	r	$\sim p$	$m \land \sim p$
T	T	T	F	F
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

Now we can create a column for the conditional. Because it can be confusing to keep track of all the Ts and Fs, why don't we copy the column for r to the right of the column for $m \land \sim p$? This makes it a lot easier to read the conditional from left to right.

m	p	r	$\sim p$	$m \land \sim p$	r	$(m \land \sim p) \to r$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	F	T

When m is true, p is false, and r is false—the fourth row of the table—then the antecedent $m \land \sim p$ will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by $(m \land \sim p) \to r$, consider this: let $m =$ we order meatballs, $p =$ we order pasta, and $r =$ Rob is happy. The statement $(m \land \sim p) \to r$ is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

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Related Statements

The original conditional is	"if p, then q"	$p \rightarrow q$
The converse is	"if q, then p"	$q \rightarrow p$
The inverse is	"if not p, then not q"	$\sim p \rightarrow \sim q$
The contrapositive is	"if not q, then not p"	$\sim q \rightarrow \sim p$

</div>

Example 23

Consider again the conditional "If it is raining, then there are clouds in the sky." It seems reasonable to assume that this is true.

The converse would be "If there are clouds in the sky, then it is raining." This is not always true.

The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true.

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Equivalent

<div style="border: 1px solid purple; padding: 10px;">
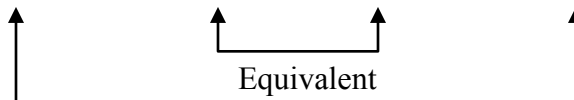
Equivalence

A conditional statement and its contrapositive are logically equivalent.
The converse and inverse of a conditional statement are logically equivalent.

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In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 24

Suppose this statement is true: "If I eat this giant cookie, then I will feel sick." Which of the following statements must also be true?

a. If I feel sick, then I ate that giant cookie.
b. If I don't eat this giant cookie, then I won't feel sick.
c. If I don't feel sick, then I didn't eat that giant cookie.

a. This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.

b. This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn't guarantee that I won't feel sick.

c. This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don't feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

Try it Now 5
"If you microwave salmon in the staff kitchen, then I will be mad at you." If this statement is true, which of the following statements must also be true?
a. If you don't microwave salmon in the staff kitchen, then I won't be mad at you.
b. If I am not mad at you, then you didn't microwave salmon in the staff kitchen.
c. If I am mad at you, then you microwaved salmon in the staff kitchen.

Consider the statement "If you park here, then you will get a ticket." What set of conditions would prove this statement false?

a. You don't park here and you get a ticket.
b. You don't park here and you don't get a ticket.
c. You park here and you don't get a ticket.

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here *and* you do not get a ticket."

> **The Negation of a Conditional**
> The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.
>
> $$\sim(p \rightarrow q) \text{ is equivalent to } p \wedge \sim q$$

Example 25

> Which of the following statements is equivalent to the negation of "If you don't grease the pan, then the food will stick to it" ?
>
> a. I didn't grease the pan and the food didn't stick to it.
> b. I didn't grease the pan and the food stuck to it.
> c. I greased the pan and the food didn't stick to it.
>
> a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
>
> b. This is essentially the original statement with no negation; the "if…then" has been replaced by "and".
>
> c. This essentially agrees with the original statement and cannot disprove it.

Try it Now 6

"If you go swimming less than an hour after eating lunch, then you will get cramps." Which of the following statements is equivalent to the negation of this statement?
a. I went swimming more than an hour after eating lunch and I got cramps.
b. I went swimming less than an hour after eating lunch and I didn't get cramps.
c. I went swimming more than an hour after eating lunch and I didn't get cramps.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if t, then p also includes the inverse of the statement: if not t, then not p. A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional**.

> **Biconditional**
> A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.
>
> A biconditional is written as $p \leftrightarrow q$ and is translated as "p if and only if q".

Because a biconditional statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$, we may think of it as a conditional statement combined with its converse: if p, then q and if q, then p. The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth table for the biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 26

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

a. It is noon on Thursday and the garbage truck did not come down my street this morning.
b. It is Monday and the garbage truck is coming down my street.
c. It is Wednesday at 11:59PM and the garbage truck did not come down my street today.

a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.

b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.

c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try it Now 7

Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.
a. I am exercising and I am not wearing my running shoes.
b. I am wearing my running shoes and I am not exercising.
c. I am not exercising and I am not wearing my running shoes.

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Create a truth table for the statement $(A \lor B) \leftrightarrow \sim C$

Whenever we have three component statements, we start by listing all the possible truth value combinations for A, B, and C. After creating those three columns, we can create a fourth column for the antecedent, $A \lor B$. Now we will temporarily ignore the column for C and focus on A and B, writing the truth values for $A \lor B$.

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

A	B	C	A ∨ B
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Next we can create a column for the negation of C. (Ignore the $A \lor B$ column and simply negate the values in the C column.)

A	B	C	A ∨ B	~C
T	T	T	T	F
T	T	F	T	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

Finally, we find the truth values of $(A \lor B) \leftrightarrow \sim C$. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

A	B	C	A ∨ B	~C	(A ∨ B) ↔ ~C
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	F

To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So $(A \lor B) \leftrightarrow \sim C$ means "You will not get a crummy review if and only if you do project A or project B." Looking at a few of the rows of the truth table, we can see

how this works out. In the first row, *A*, *B*, and *C* are all true: you did both projects and got a crummy review, which is not what your boss told you would happen! That is why the final result of the first row is false. In the fourth row, *A* is true, *B* is false, and *C* is false: you did project *A* and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, *A*, *B*, and *C* are all false: you didn't do either project and did not get a crummy review. This is <u>not</u> what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

De Morgan's Laws

A contemporary of Boole's, Augustus De Morgan, formalized two rules of logic that had previously been known informally. They allow us to rewrite the negation of a conjunction as a disjunction, and vice-versa.

For example, suppose you want to schedule a meeting with two colleagues at 4:30PM on Friday, and you need both of them to be available at that time. What situation would make it impossible to have the meeting? It is NOT the case that colleague *a* is available AND colleague *b* is available: $\sim(a \wedge b)$. This situation is equivalent to either colleague *a* NOT being available OR colleague *b* NOT being available: $\sim a \vee \sim b$.

> **De Morgan's Laws**
> The negation of a conjunction is equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an "and" statement, negate each part and change the "and" to "or".
>
> $$\sim(p \wedge q) \text{ is equivalent to } \sim p \vee \sim q$$
>
> The negation of a disjunction is equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an "or" statement, negate each part and change the "or" to "and".
>
> $$\sim(p \vee q) \text{ is equivalent to } \sim p \wedge \sim q$$

Example 28

For Valentine's Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

a. You did not get them flowers or did not get them candy.
b. You did not get them flowers and did not get them candy.
c. You got them flowers or got them candy.

a. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
b. This statement is equivalent to the original; $\sim(f \vee c)$ is equivalent to $\sim f \wedge \sim c$.
c. This statement says that you got them something, but we know that you did not.

Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

> **Argument types**
> An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.
>
> A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 29

The argument "when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store" is an inductive argument.

The premises are:
I forgot my purse last week
I forgot my purse today

The conclusion is:
I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 30

The argument "every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M." is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

Evaluating inductive arguments
An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. Common scientific theories, like Newton's theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

Evaluating deductive arguments
A deductive argument is considered valid if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion *must* be true.

Evaluating Deductive Arguments with Euler Diagrams

We can interpret a deductive argument visually with an Euler diagram, which is essentially the same thing as a Venn diagram. This can make it easier to determine whether the argument is valid or invalid.

Example 31

Consider the deductive argument "All cats are mammals and a tiger is a cat, so a tiger is a mammal." Is this argument valid?
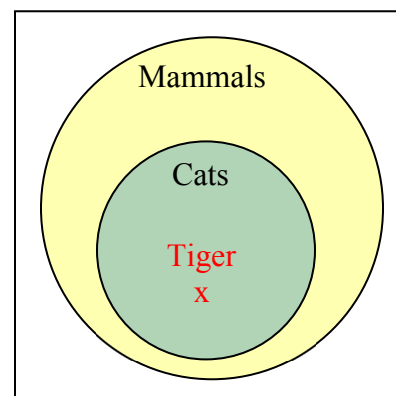
The premises are:
All cats are mammals.
A tiger is a cat.

The conclusion is:
A tiger is a mammal.

Both the premises are true. To see that the premises must logically lead to the conclusion, we can use a Venn diagram. From the first premise, we draw the set of cats as a subset of the set of mammals. From the second premise, we are told that a tiger is contained within the set of cats. From that, we can see in the Venn diagram that the tiger must also be inside the set of mammals, so the conclusion is valid.

Analyzing arguments with Euler diagrams
To analyze an argument with an Euler diagram:
1) Draw an Euler diagram based on the premises of the argument
2) The argument is invalid if there is a way to draw the diagram that makes the conclusion false
3) The argument is valid if the diagram cannot be drawn to make the conclusion false
4) If the premises are insufficient to determine the location of an element or a set mentioned in the conclusion, then the argument is invalid.

Try it Now 9
Determine the validity of this argument:
Premise: All cats are scared of vacuum cleaners.
Premise: Max is a cat.
Conclusion: Max is scared of vacuum cleaners.

Example 32

Premise: All firefighters know CPR.
Premise: Jill knows CPR.
Conclusion: Jill is a firefighter.

From the first premise, we know that firefighters all lie inside the set of those who know CPR. (Firefighters are a subset of people who know CPR.) From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know whether she also is a member of the smaller subset that is firefighters.

Since the conclusion does not necessarily follow from the premises, this is an invalid argument. It's possible that Jill is a firefighter, but the structure of the argument doesn't allow us to conclude that she definitely is.
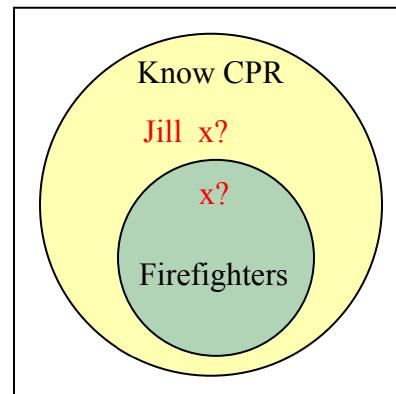
It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are concerned with whether the premises are enough to prove the conclusion.

Try it Now 10
Determine the validity of this argument:
Premise: All bicycles have two wheels.
Premise: This Harley-Davidson has two wheels.
Conclusion: This Harley-Davidson is a bicycle.

Try it Now 11
Determine the validity of this argument:
Premise: No cows are purple.
Premise: Fido is not a cow.
Conclusion: Fido is purple.

In addition to these categorical style premises of the form "all ___", "some ____", and "no ____", it is also common to see premises that are conditionals.

Premise: If you live in Seattle, you live in Washington.
Premise: Marcus does not live in Seattle.
Conclusion: Marcus does not live in Washington.

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether Marcus lives in Washington or not. This is an invalid argument.
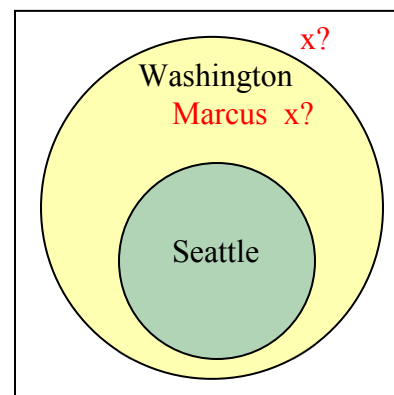
Try it Now 12
Determine the validity of this argument:
Premise: If you have lipstick on your collar, then you are cheating on me.
Premise: If you are cheating on me, then I will divorce you.
Premise: You do not have lipstick on your collar.
Conclusion: I will not divorce you.

Evaluating Deductive Arguments with Truth Tables

Arguments can also be analyzed using truth tables, although this can be a lot of work.

> **Analyzing arguments using truth tables**
> To analyze an argument with a truth table:
> 1. Represent each of the premises symbolically
> 2. Create a conditional statement, joining all the premises to form the antecedent, and using the conclusion as the consequent.
> 3. Create a truth table for the statement. If it is always true, then the argument is valid.

Example 34

Consider the argument
Premise: If you bought bread, then you went to the store.
Premise: You bought bread.
Conclusion: You went to the store.

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll let b represent "you bought bread" and s represent "you went to the store". Then the argument becomes:
Premise: $b \rightarrow s$
Premise: b
Conclusion: s

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(b \rightarrow s) \wedge b] \rightarrow s$?

b	s	$b \rightarrow s$
T	T	T
T	F	F
F	T	T
F	F	T

b	s	$b \rightarrow s$	$(b \rightarrow s) \wedge b$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

b	s	$b \rightarrow s$	$(b \rightarrow s) \wedge b$	$[(b \rightarrow s) \wedge b] \rightarrow s$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth table for $[(b \rightarrow s) \wedge b] \rightarrow s$ is always true, this is a valid argument.

Try it Now 13

Determine whether the argument is valid:
Premise: If I have a shovel, I can dig a hole.
Premise: I dug a hole.
Conclusion: Therefore, I had a shovel.

Example 35

Premise: If I go to the mall, then I'll buy new jeans.
Premise: If I buy new jeans, I'll buy a shirt to go with it.
Conclusion: If I go to the mall, I'll buy a shirt.

Let m = I go to the mall, j = I buy jeans, and s = I buy a shirt.
The premises and conclusion can be stated as:
Premise: $m \to j$
Premise: $j \to s$
Conclusion: $m \to s$

We can construct a truth table for $[(m \to j) \wedge (j \to s)] \to (m \to s)$. Try to recreate each step and see how the truth table was constructed.

m	j	s	$m{\to}j$	$j{\to}s$	$(m{\to}j) \wedge (j{\to}s)$	$m{\to}s$	$[(m{\to}j) \wedge (j{\to}s)] \to (m{\to}s)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the final column of the truth table, we can see this is a valid argument.

Forms of Valid Arguments

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

The Law of Detachment (*Modus Ponens*)
The law of detachment applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:
Premise: $p \to q$
Premise: p
Conclusion: q

The Latin name, *modus ponens*, translates to "mode that affirms".

Example 36

Recall this argument from an earlier example:
Premise: If you bought bread, then you went to the store.
Premise: You bought bread.
Conclusion: You went to the store.

In symbolic form:
Premise: $b \rightarrow s$
Premise: b
Conclusion: s

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

The Law of Contraposition (*Modus Tollens*)
The law of contraposition applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:
Premise: $p \rightarrow q$
Premise: $\sim q$
Conclusion: $\sim p$

The Latin name, *modus tollens*, translates to "mode that denies".

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \rightarrow \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

Example 37

Premise: If I drop my phone into the swimming pool, my phone will be ruined.
Premise: My phone isn't ruined.
Conclusion: I didn't drop my phone into the swimming pool.

If we let d = I drop the phone in the pool and r = the phone is ruined, then we can represent the argument this way:
Premise $d \rightarrow r$
Premise $\sim r$
Conclusion: $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

Try it Now 14
Is this argument valid?
Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
Premise: Your toothbrush is dry.
Conclusion: You didn't brush your teeth before bed.

> **The Transitive Property (Hypothetical Syllogism)**
> The transitive property has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:
>
> Premise: $p \rightarrow q$
> Premise: $q \rightarrow r$
> Conclusion: $p \rightarrow r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don't have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

Example 38

Premise: If a soccer player commits a reckless foul, she will receive a yellow card.
Premise: If Hayley receives a yellow card, she will be suspended for the next match.
Conclusion: If Hayley commits a reckless foul, she will be suspended for the next match.

If we let r = committing a reckless foul, y = receiving a yellow card, and s = being suspended, then our argument looks like this:

Premise $r \rightarrow y$
Premise $y \rightarrow s$
Conclusion: $r \rightarrow s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

Try it Now 15

Is this argument valid?
Premise: If the old lady swallows a fly, she will swallow a spider.
Premise: If the old lady swallows a spider, she will swallow a bird.
Premise: If the old lady swallows a bird, she will swallow a cat.
Premise: If the old lady swallows a cat, she will swallow a dog.
Premise: If the old lady swallows a dog, she will swallow a goat.
Premise: If the old lady swallows a goat, she will swallow a cow.
Premise: If the old lady swallows a cow, she will swallow a horse.
Premise: If the old lady swallows a horse, she will die, of course.
Conclusion: If the old lady swallows a fly, she will die, of course.

> **Disjunctive Syllogism**
> In a disjunctive syllogism, the premises consist of an *or* statement and the negation of one of the options. The conclusion is the other option. The general form is:
> Premise: $p \lor q$
> Premise: $\sim p$
> Conclusion: q

The order of the two parts of the disjunction isn't important. In other words, we could have the premises $p \lor q$ and $\sim q$, and the conclusion p.

Example 39

Premise: I can either drive or take the train.
Premise: I refuse to drive.
Conclusion: I will take the train.

If we let d = I drive and t = I take the train, then the symbolic representation of the argument is:
Premise $d \lor t$
Premise $\sim d$
Conclusion: t

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

Try it Now 16
Is this argument valid?
Premise: Alison was required to write a 10-page paper or give a 5-minute speech.
Premise: Alison did not give a 5-minute speech.
Conclusion: Alison wrote a 10-page paper.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent *invalid* arguments, which are also called *fallacies*.

The Fallacy of the Converse
The fallacy of the converse arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:
Premise: $p \rightarrow q$
Premise: q
Conclusion: p

Notice that the second premise and the conclusion look like the converse of the first premise, $q \rightarrow p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

Example 40

Premise: If I drink coffee after noon, then I have a hard time falling asleep that night.
Premise: I had a hard time falling asleep last night.
Conclusion: I drank coffee after noon yesterday.

If we let c = I drink coffee after noon and h = I have a hard time falling asleep, then our argument looks like this:
Premise $c \rightarrow h$
Premise h
Conclusion: c

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, …

Try it Now 17
Is this argument valid?
Premise: If you pull that fire alarm, you will get in big trouble.
Premise: You got in big trouble.
Conclusion: You must have pulled the fire alarm.

> **The Fallacy of the Inverse**
> The fallacy of the inverse occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:
> Premise: $\qquad p \to q$
> Premise: $\qquad \sim p$
> Conclusion: $\qquad \sim q$

Again, notice that the second premise and the conclusion look like the inverse of the first premise, $\sim p \to \sim q$, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

Example 41

Premise: $\qquad$ If you listen to the Grateful Dead, then you are a hippie.
Premise: $\qquad$ Sky doesn't listen to the Grateful Dead.
Conclusion: $\quad$ Sky is not a hippie.

If we let g = listen to the Grateful Dead and h = is a hippie, then this is the argument:
Premise $\qquad g \to h$
Premise $\qquad \sim g$
Conclusion: $\quad \sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

Try it Now 18
Is this argument valid?
Premise: $\qquad$ If a hockey player trips an opponent, he will be assessed a 2-minute penalty.
Premise: $\qquad$ Alexei did not trip an opponent.
Conclusion: $\quad$ Alexei will not be assessed a 2-minute penalty.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of *Alice's Adventures in Wonderland*, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

Example 42

Solve the puzzle. In other words, find a logical conclusion from these premises.
All babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

Let b = is a baby, d = is despised, i = is illogical, and m = can manage a crocodile. Then we can write the premises as:
$b \rightarrow i$
$m \rightarrow \sim d$
$i \rightarrow d$

Writing the second premise correctly can be a challenge; it can be rephrased as "If you can manage a crocodile, then you are not despised."

Using the transitive property with the first and third premises, we can conclude that $b \rightarrow d$; that all babies are despised. Using the contrapositive of the second premise, $d \rightarrow \sim m$, we can then use the transitive property with $b \rightarrow d$ to conclude that $b \rightarrow \sim m$; that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

Example 43

Premise: If I work hard, I'll get a raise.
Premise: If I get a raise, I'll buy a boat.
Conclusion: If I don't buy a boat, I must not have worked hard.

If we let h = working hard, r = getting a raise, and b = buying a boat, then we can represent our argument symbolically:
Premise $h \rightarrow r$
Premise $r \rightarrow b$
Conclusion: $\sim b \rightarrow \sim h$

Using the transitive property with the two premises, we can conclude that $h \rightarrow b$; if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement $h \rightarrow b$ is equivalent to $\sim b \rightarrow \sim h$. Therefore, the conclusion is indeed a logical syllogism derived from the premises.

Try it Now 19
Is this argument valid?
Premise: If I go to the party, I'll be really tired tomorrow.
Premise: If I go to the party, I'll get to see friends.
Conclusion: If I don't see friends, I won't be tired tomorrow.

Logical Fallacies in Common Language

In the previous discussion, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

> **Ad hominem**
> An ad hominem argument attacks the person making the argument, ignoring the argument itself.

Example 44

"Jane says that whales aren't fish, but she's only in the second grade, so she can't be right."

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

Example 45

"Jane says that whales aren't fish, but everyone knows that they're really mammals. She's so stupid."

This certainly isn't very nice, but it is *not* ad hominem since a valid counterargument is made along with the personal insult.

> **Appeal to ignorance**
> This type of argument assumes something it true because it hasn't been proven false.

Example 46

"Nobody has proven that photo isn't of Bigfoot, so it must be Bigfoot."

> **Appeal to authority**
> These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person's opinion is not shared by other experts, or when the authority is irrelevant to the claim.

Example 47

"A diet high in bacon can be healthy; Doctor Atkins said so."

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

Example 48

> "Jennifer Hudson lost weight with Weight Watchers, so their program must work."
>
> Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person's experience would.

> **Appeal to consequence**
> An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

Example 49

> "Humans will travel faster than light: faster-than-light travel would be beneficial for space travel."

> **False dilemma**
> A false dilemma argument falsely frames an argument as an "either or" choice, without allowing for additional options.

Example 50

> "Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens."
>
> This argument ignores the possibility that the lights could be something other than an airplane or aliens.

> **Circular reasoning**
> Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example 51

> "I shouldn't have gotten a C in that class; I'm an A student!"
>
> In this argument, the student is claiming that because they're an A student, though shouldn't have gotten a C. But because they got a C, they're not an A student.

> **Post hoc (post hoc ergo propter hoc)**
> A post hoc argument claims that because two things happened sequentially, then the first must have caused the second.

Example 52

> "Today I wore a red shirt, and my football team won! I need to wear a red shirt every time they play to make sure they keep winning."

> **Straw man**
> A straw man argument involves misrepresenting the argument in a less favorable way to make it easier to attack.

Example 53

"Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists"

Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack Senator Jones' position.

> **Correlation implies causation**
> Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

Example 54

"Months with high ice cream sales also have a high rate of deaths by drowning. Therefore, ice cream must be causing people to drown."

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

Try it Now 20
Identify the logical fallacy in each of the arguments

a. Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
b. Since the 1950s, both the atmospheric carbon dioxide level and obesity levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
c. The oven was working fine until you started using it, so you must have broken it.
d. You can't give me a D in the class because I can't afford to retake it.
e. The senator wants to increase support for food stamps. He wants to take the taxpayers' hard-earned money and give it away to lazy people. This isn't fair, so we shouldn't do it.

It may be difficult to identify one particular fallacy for an argument. Consider this argument: "Emma Watson says she's a feminist, but she posed for these racy pictures. I'm a feminist, and no self-respecting feminist would do that." Could this be ad hominem, saying that Emma Watson has no self-respect? Could it be appealing to authority because the person making the argument claims to be a feminist? Could it be a false dilemma because the argument assumes that a woman is either a feminist or not, with no gray area in between?

Try it Now Answers

1. At least one Icelandic child did not learn English in school.

2. $C \vee \sim S$

3.

A	B	$\sim A$	$\sim A \wedge B$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

4.

A	B	$\sim A$	$\sim A \wedge B$	$\sim B$	$(\sim A \wedge B) \vee \sim B$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

5. Choice b is correct because it is the contrapositive of the original statement.

6. Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.

7. Choices a & b are false; c is true.

8. Failing to meet just one of the three conditions is all it takes to be disqualified. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word here is "or" instead of "and".

9. Valid. Cats are a subset of creatures that are scared by vacuum cleaners. Max is in the set of cats, so he must also be in the set of creatures that are scared by vacuum cleaners.
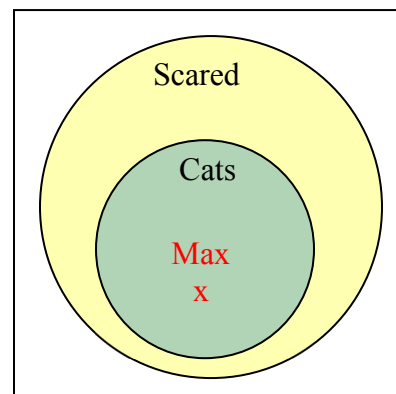
10. Invalid. The set of bicycles is a subset of the set of vehicles with two wheels; the Harley-Davidson is in the set of two-wheeled vehicles but not necessarily in the smaller circle.

Two wheels

Harley x?

Bicycles

x?
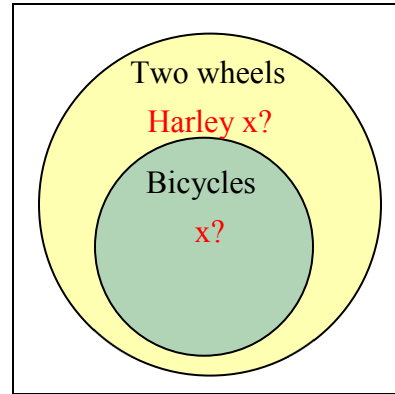
11. Invalid. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.

Fido   x?

Cows

x?Purple things
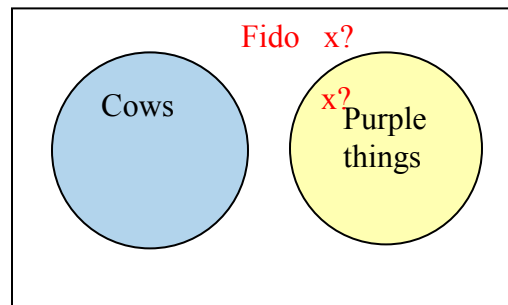
12. Invalid. Lipstick on your collar is a subset of scenarios in which you are cheating, and cheating is a subset of the scenarios in which I will divorce you. Although it is wonderful that you don't have lipstick on your collar, you could still be cheating on me, and I will divorce you. In fact, even if you aren't cheating on me, I might divorce you for another reason. You'd better shape up.

You   x?

Divorce
x?

Cheating
x?

Lipstick
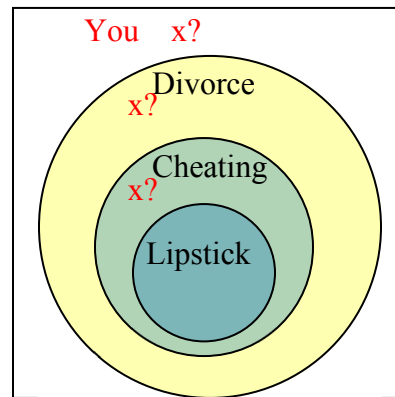
13. Let S = have a shovel, D = dig a hole.
The first premise is equivalent to $S \rightarrow D$. The second premise is D. The conclusion is S.
We are testing $[(S \rightarrow D) \wedge D] \rightarrow S$

S	D	$S{\rightarrow}D$	$(S{\rightarrow}D) \wedge D$	$[(S{\rightarrow}D) \wedge D] \rightarrow S$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

This is not a tautology, so this is an invalid argument.

14. Let b = brushed teeth and w = toothbrush is wet.
    Premise: $b \rightarrow w$
    Premise: $\sim w$
    Conclusion: $\sim b$
    This argument is valid by the Law of Contraposition.

15. This argument is valid by the Transitive Property, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \rightarrow s$, $s \rightarrow b$, $b \rightarrow c$, $c \rightarrow d$, $d \rightarrow g$, $g \rightarrow w$, $w \rightarrow h$, $h \rightarrow x$ can be reduced to $f \rightarrow x$. (Because we had already used c and d, we decided to use w for cow and x for death.) If the old lady swallows the fly, she will eventually eat a horse and die.

16. Let p = wrote a paper and s = gave a speech.
    Premise: $p \lor s$
    Premise: $\sim s$
    Conclusion: p
    This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.

17. Let f = pulled fire alarm and t = got in big trouble.
    Premise: $f \rightarrow t$
    Premise: t
    Conclusion: f

    This argument is invalid because it has the form of the Fallacy of the Converse. The young rascal may have gotten in trouble for any number of reasons besides pulling the fire alarm.

18. Let t = tripped and p = got a penalty.
    Premise: $t \rightarrow p$
    Premise: $\sim t$
    Conclusion: $\sim p$

    This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.

19. Let p = go to party, t = be tired, and f = see friends.
   Premise: $p \rightarrow t$
    Premise: $p \rightarrow f$
    Conclusion: $\sim f \rightarrow \sim t$

    We could try to rewrite the second premise using the contrapositive to state $\sim f \rightarrow \sim p$, but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.

A Venn diagram can help, if we set it up correctly. The "party" circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the "friends" circle, but we cannot determine whether I am in the "tired" circle. All we really know for sure is that I didn't go to the party.
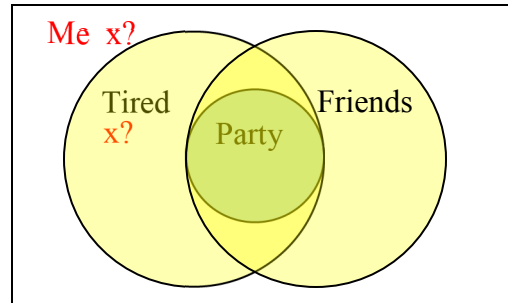
20.a. Circular
   b. Correlation does not imply causation
   c. Post hoc
   d. Appeal to consequence
   e. Straw man

Exercises

Boolean Logic

For questions 1-2, list the set of integers that satisfy the given conditions.

1. A positive multiple of 5 and not a multiple of 2

2. Greater than 12 and less than or equal to 18

Quantified Statements

For questions 3-4, write the negation of each quantified statement.

3. Everyone failed the quiz today.

4. Someone in the car needs to use the restroom.

Truth Tables

5. Translate each statement from symbolic notation into English sentences. Let A represent "Elvis is alive" and let G represent "Elvis gained weight".
    a. $A \lor G$
    b. $\sim(A \land G)$
    c. $G \rightarrow \sim A$
    d. $A \leftrightarrow \sim G$

For questions 6-9, create a truth table for each statement.

6. $A \land \sim B$

7. $\sim(\sim A \lor B)$

8. $(A \land B) \rightarrow C$

9. $(A \lor B) \rightarrow \sim C$

Questions 10-13: In this lesson, we have been studying the *inclusive* or, which allows both A and B to be true. The *exclusive* or does not allow both to be true; it translates to "either A or B, but not both."

10. For each situation, decide whether the "or" is most likely exclusive or inclusive.
    a. An entrée at a restaurant includes soup or a salad.
    b. You should bring an umbrella or a raincoat with you.
    c. We can keep driving on I-5 or get on I-405 at the next exit.
    d. You should save this document on your computer or a flash drive.

11. Complete the truth table for the exclusive or.

A	B	A ⊻ B
T	T	
T	F	
F	T	
F	F	

12. Complete the truth table for (A ∨ B) ∧ ~(A ∧ B).

A	B	A ∨ B	A ∧ B	~(A ∧ B)	(A ∨ B) ∧ ~(A ∧ B)
T	T				
T	F				
F	T				
F	F				

13. Compare your answers for questions 11 and 12. Can you explain the similarities?

Conditional Statements

14. Consider the statement "If you are under age 17, then you cannot attend this movie."
    a. Write the converse.
    b. Write the inverse.
    c. Write the contrapositive.

15. Assume that the statement "If you swear, then you will get your mouth washed out with soap" is true. Which of the following statements must also be true?
    a. If you don't swear, then you won't get your mouth washed out with soap.
    b. If you don't get your mouth washed out with soap, then you didn't swear.
    c. If you get your mouth washed out with soap, then you swore.

For questions 16-18, write the negation of each conditional statement.

16. If you don't look both ways before crossing the street, then you will get hit by a car.

17. If Luke faces Vader, then Obi-Wan cannot interfere.

18. If you weren't talking, then you wouldn't have missed the instructions.

19. Assume that the biconditional statement "You will play in the game if and only if you attend all practices this week" is true. Which of the following situations could happen?
    a. You attended all practices this week and didn't play in the game.
    b. You didn't attend all practices this week and played in the game.
    c. You didn't attend all practices this week and didn't play in the game.

De Morgan's Laws

For questions 20-21, use De Morgan's Laws to rewrite each conjunction as a disjunction, or each disjunction as a conjunction.

20. It is not true that Tina likes Sprite or 7-Up.

21. It is not the case that you need a dated receipt and your credit card to return this item.

22. Go back and look at the truth tables in Exercises 6 & 7. Explain why the results are identical.

Deductive Arguments

For questions 23-28, use a Venn diagram or truth table or common form of an argument to decide whether each argument is valid or invalid.

23. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.

24. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.

25. Jamie must scrub the toilets or hose down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie will hose down the garbage cans.

26. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!

27. Every student brought a pencil or a pen. Marcie brought a pencil. Therefore, Marcie did not bring a pen.

28. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.

Logical Fallacies

For questions 28-30, name the type of logical fallacy being used.

29. If you don't want to drive from Boston to New York, then you will have to take the train.

30. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The "Cheatriots" have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.

31. Whenever our smoke detector beeps, my kids eat cereal for dinner. The loud beeping sound must make them want to eat cereal for some reason.