

DALE HOFFMAN

CONTEMPORARY
CALCULUS

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Welcome to Calculus

Calculus was first developed more than 300 years ago by Sir Isaac Newton and Gottfried Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to a wide variety of problems besides planetary motion. They have used calculus to help understand physical, biological, economic and social phenomena and to describe and solve related problems.

The discovery, development and application of calculus is a great intellectual achievement—and now you have the opportunity to share in that achievement. You should feel exhilarated. You may also be somewhat concerned (a common reaction among students just beginning to study calculus). You need to be concerned enough to work to master calculus, yet confident enough to keep going when you (at first) don't understand something.

Part of the beauty of calculus is that it relies upon a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe and solve problems in a variety of fields. This book tries to emphasize both the beauty and the power.

In Section 0.1 (Preview) we will look at the main ideas that will continue throughout the book: the problems of finding tangent lines and computing areas. We will also consider a process that underlies both of those problems: the limiting process of approximating a solution and then getting better and better approximations until we finally get an exact solution.

Sections 0.2 (Lines), 0.3 (Functions) and 0.4 (Combinations of Functions) contain review material. These sections emphasize concepts and skills you will need in order to succeed in calculus. You should have worked with most of these concepts in previous courses, but the emphasis and use of the material here may be different than in those earlier classes.

Section 0.5 (Mathematical Language) discusses a few key mathematical phrases. It considers their use and meaning and some of their equivalent forms. It will be difficult to understand the meaning and subtleties of calculus if you don't understand how these phrases are used and what they mean.

0.1 A Preview of Calculus

Calculus can be viewed as an attempt—a historically successful attempt—to solve two fundamental problems. In this section we begin to examine geometric forms of those two problems and some fairly simple attempts to solve them. At first, the problems themselves may not appear very interesting or useful—and the methods for solving them may seem crude—but these simple problems and methods have led to one of the most beautiful, powerful and useful creations in mathematics: Calculus.

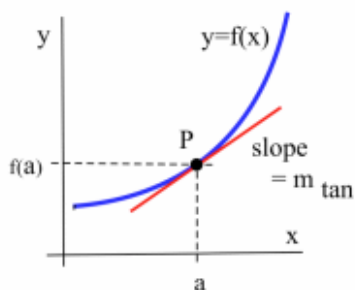


Figure 1: The graph of $y = f(x)$ along with its tangent line at point P .

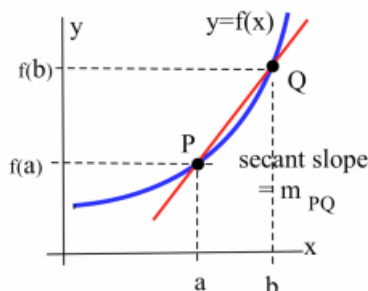


Figure 2: The graph of $y = f(x)$ with a secant line through points P and Q .

Finding the Slope of a Tangent Line

Suppose we have the graph of a function $y = f(x)$ and we want to find the equation of a line that is tangent to the graph at a particular point P on the graph. (We will offer a precise definition of “tangent” in Section 1.0; for now, think of the tangent line as a line that touches the curve at the point P and stays close to the graph of $y = f(x)$ near P .)

We know that P is on the tangent line, so if its x -coordinate is $x = a$, then the y -coordinate of P must be $y = f(a)$: $P = (a, f(a))$. The only other information we need to find the equation of the tangent line is its slope, m_{tan} , but that is where the difficulty arises.

In algebra, we needed two points in order to determine a slope. So far, we only have the point P . Let’s simply pick a second point, call it Q , on the graph of $y = f(x)$. If the x -coordinate of Q is b , then the y -coordinate is $f(b)$: $Q = (b, f(b))$. So the slope of the line through P and Q is

$$m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{f(b) - f(a)}{b - a}$$

If we drew the graph of $y = f(x)$ on a wall, put nails at the points P and Q , and laid a straightedge on the nails, then the straightedge would have slope m_{PQ} . But the slope m_{PQ} can be very different from the value we want (the slope m_{tan} of the tangent line). The key idea is that when the point Q is *close* to the point P , then the slope m_{PQ} should be *close* to the slope we want, m_{tan} . Physically, if we slide the nail at Q along the graph toward the fixed point P , then the slope, $m_{PQ} = \frac{f(b) - f(a)}{b - a}$, of the straightedge gets closer and closer to the slope, m_{tan} , of the tangent line. If the value of b is very close to a , then the point Q is very close to P , and the value of m_{PQ} is very close to the value of m_{tan} .

Rather than defacing walls with graphs and nails, we can instead calculate $m_{PQ} = \frac{f(b) - f(a)}{b - a}$ and examine the values of m_{PQ} as b gets closer and closer to a . We say that m_{tan} is the **limiting value** of m_{PQ} as b gets very close to a , and we write:

$$m_{\text{tan}} = \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}$$

Eventually we will call the slope m_{tan} of the tangent line the **derivative** of the function $f(x)$ at the point P , and call this part of calculus **differential calculus**. Chapters 2 and 3 begin the study of differential calculus.

The slope of the tangent line to the graph of a function will tell us important information about the function and will allow us to solve problems such as:

- The U.S. Postal Service requires that the length plus the girth of a package not exceed 84 inches. What is the largest volume that can be mailed in a rectangular box?

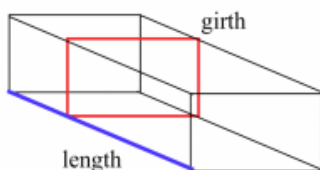


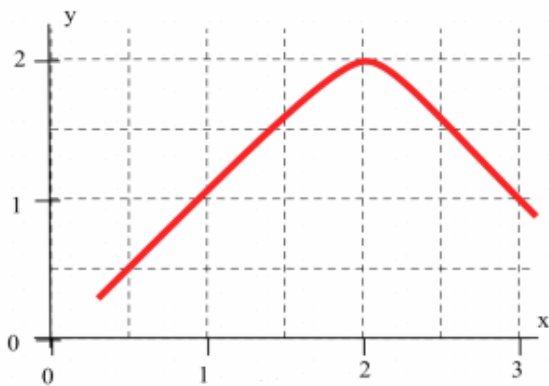
Figure 3: Sketch of a box satisfying USPS regulations.

- An oil tanker was leaking oil and a 4-inch-thick oil slick had formed. When first measured, the slick had a radius of 200 feet, and the radius was increasing at a rate of 3 feet per hour. At that time, how fast was the oil leaking from the tanker?

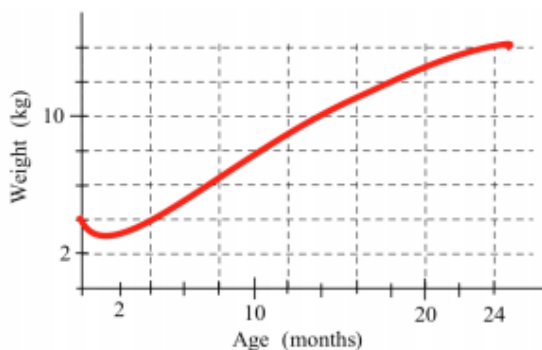
Derivatives will even help us solve such “traditional” mathematical problems as finding solutions of equations like $x^2 = 2 + \sin(x)$ and $x^9 + 5x^5 + x^3 + 3 = 0$.

Problems

- Sketch the lines tangent to the curve shown below at $x = 1, 2$ and 3 . Estimate the slope of each of the tangent lines you drew.

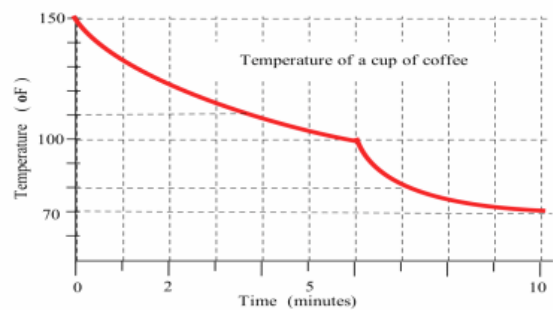


- A graph of the weight of a “typical” child from age 0 to age 24 months appears below. (Each of your answers should have the units “kg per month.”)
 - What was the average weight gain from month 0 to month 24?
 - What was the average weight gain from month 9 to month 12? From month 12 to month 15?
 - Approximately how fast was the child gaining weight at age 12 months? At age 3 months?



- The graph below shows the temperature of a cup of coffee during a 10-minute period. (Each of your answers in (a)–(c) should have the units “degrees per minute.”)

- What was the average rate of cooling from minute 0 to minute 10?
- What was the average rate of cooling from minute 7 to minute 8? From minute 8 to minute 9?
- What was the rate of cooling at minute 8? At minute 2?
- When was cold milk added to the coffee?



- Describe a method for determining the slope of a steep hill at a point midway up the hill
 - using a ruler, a long piece of string, a glass of water and a loaf of bread.
 - using a protractor, a piece of string and a helium-filled balloon.



Figure 4: A leaf.

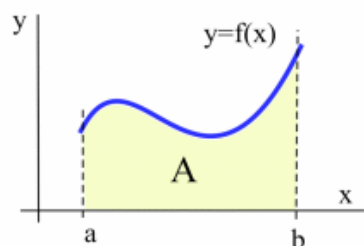
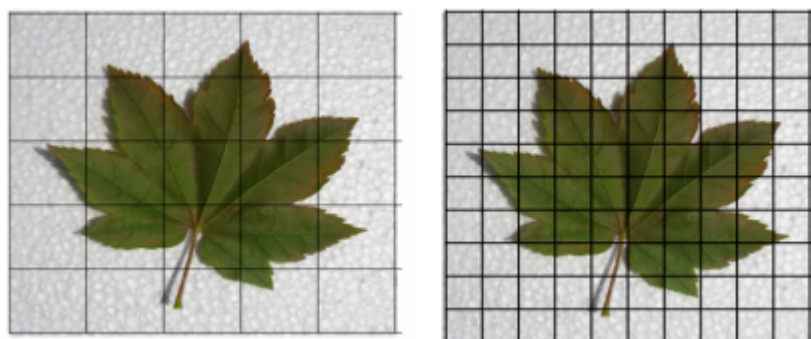
Figure 5: Squares in the left image each have area 1 cm^2 , while squares in the right image each have area $\frac{1}{4} \text{ cm}^2$.

Figure 6: What is the area of the shaded region?

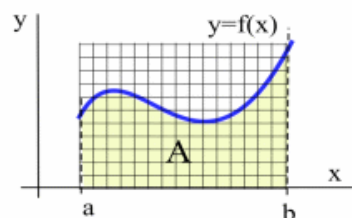


Figure 7: Using graph paper to approximate the shaded area.

Finding the Area of a Shape

Suppose we need to find the area of a leaf as part of a study of how much energy a plant gets from sunlight. One method for finding the area would be to trace the shape of the leaf onto a piece of paper and then divide the region into “easy” shapes such as rectangles and triangles (whose areas we can easily calculate). We could add all of these “easy” areas together to approximate the area of the leaf.

A modification of this method would be to trace the shape onto a piece of graph paper and then count the number of squares completely inside the edge of the leaf to get a lower estimate of the area, and count the number of squares that touch the leaf to get an upper estimate of the area. If we repeat this process with smaller and smaller squares, we will have to do more counting and adding, but our estimates should be closer together—and closer to the actual area of the leaf.

(We could also approximate the area of the leaf using a sheet of paper, scissors and an accurate scale. How?)

We can calculate the area A between the graph of a function $y = f(x)$ and the x -axis by using similar methods. We can divide the area into strips of width w and determine the lower and upper values of $y = f(x)$ on each strip. Then we can approximate the area of each rectangle and add all of the little areas together to get A_w , an approximation of the exact area. The key idea is that if w is small, then the rectangles are narrow, and the approximate area A_w should be very close to the actual area A . If we take narrower and narrower rectangles, the approximate areas get closer and closer to the actual area:

$$A = \lim_{w \rightarrow 0} A_w$$

The process described above is the basis for a technique called *integration*, and this part of calculus is called *integral calculus*. Integral calculus and integration will begin in Chapter 4.

The process of taking the limit of a sum of “little” quantities will give us important information about a function and will also allow us to solve problems such as:

- Find the length of the graph of $y = \sin(x)$ over one period (from $x = 0$ to $x = 2\pi$).
- Find the volume of a torus (“doughnut”) of radius 1 inch that has a hole of radius 2 inches.
- A car starts at rest and has an acceleration of $5 + 3 \sin(t)$ feet per second per second in the northerly direction at time t seconds. Where will the car be, relative to its starting position, after 100 seconds?

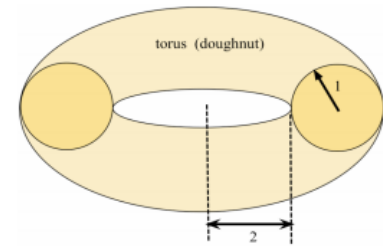
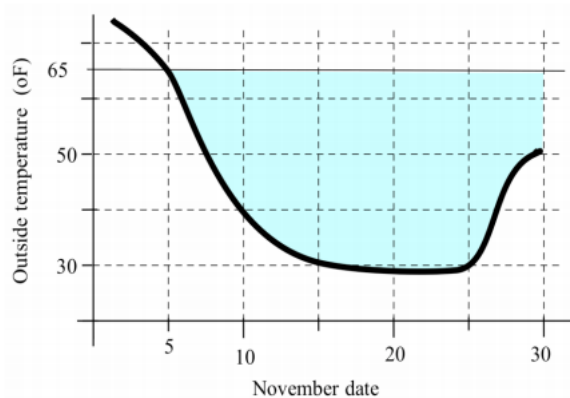


Figure 8: A torus.

Problems

- Approximate the area of the leaf in Fig. 4 using
 - the grid on the left in Fig. 5.
 - the grid on the right in Fig. 5.
- A graph showing temperatures during the month of November appears below.
 - Approximate the shaded area between the temperature curve and the 65° line from Nov. 15 to Nov. 25.
 - The area of the “rectangle” is (base)(height) so what are the units of your answer in part (a)?
 - Approximate the shaded area between the temperature curve and the 65° line from Nov. 5 to Nov. 30.
 - Who might use or care about these results?



- Describe a method for determining the volume of a compact fluorescent light bulb using a ruler, a large can, a scale and a jug of water.

A Unifying Process: Limits

We used similar processes to “solve” both the tangent line problem and the area problem. First, we found a way to get an approximate solution, and then we found a way to improve our approximation. Finally, we asked what would happen if we continued improving our approximations “forever”: that is, we “took a limit.”

For the tangent line problem, we let the point Q get closer and closer and closer to P , the limit as b approached a .

In the area problem, we let the widths of the rectangles get smaller and smaller, the limit as w approached 0. Limiting processes underlie derivatives, integrals and several other fundamental topics in calculus, and we will examine limits and their properties in detail in Chapter 1.

Two Sides of the Same Coin: Differentiation and Integration

Just as the set-up of each of the two basic problems involved a limiting process, the solutions to the two problems are also related. The process of differentiation used to solve the tangent line problem and the process of integration used to solve the area problem turn out to be “opposites” of each other: each process undoes the effect of the other process. The Fundamental Theorem of Calculus in Chapter 4 will show how this “opposite” effect works.

Extensions of the Main Problems

The first five chapters present the two key ideas of calculus, show “easy” ways to calculate derivatives and integrals, and examine some of their applications. And there is more.

Through the ensuing chapters, we will examine new functions and find ways to calculate their derivatives and integrals. We will extend the approximation ideas to use “easy” functions, such as polynomials, to approximate the values of “hard” functions, such as $\sin(x)$ and e^x .

In later chapters, we will extend the notions of “tangent lines” and “areas” to 3-dimensional space as “tangent planes” and “volumes.”

Success in calculus will require time and effort on your part, but such a beautiful and powerful field is worth that time and effort.

0.2 Lines in the Plane

The first graphs and functions you encountered in algebra were straight lines and their equations. These lines were easy to graph, and the equations were easy to evaluate and to solve. They described a variety of physical, biological and financial phenomena such as $d = rt$ relating the distance d traveled to the rate r and time t spent traveling, and $C = \frac{5}{9}(F - 32)$ for converting the temperature in degrees Fahrenheit degrees (F) to degrees Celsius (C).

The first part of calculus—differential calculus—will deal with ideas, techniques and applications of tangent lines to the graphs of functions, so it is important that you understand the graphs, properties and equations of straight lines.

The Real Number Line

The real numbers (consisting of all integers, fractions, rational and irrational numbers) can be represented as a line, called the **real number line**. Once we have selected a starting location, called the **origin**, a positive direction (usually up or to the right), and unit of length, then every number can be located as a point on the number line. If we move from a point $x = a$ to a point $x = b$ on the line, then we will have moved an **increment** of $b - a$. We denote this increment with the symbol Δx (read “delta x ”).

- If b is larger than a , then we will have moved in the positive direction, and $\Delta x = b - a$ will be positive.
- If b is smaller than a , then $\Delta x = b - a$ will be negative and we will have moved in the negative direction.
- Finally, if $\Delta x = b - a = 0$, then $a = b$ and we did not move at all.

We can also use the Δ notation and absolute values to express the **distance** that we have moved. On the number line, the distance from $x = a$ to $x = b$ is

$$\text{dist}(a, b) = \begin{cases} b - a & \text{if } b \geq a \\ a - b & \text{if } b < a \end{cases}$$

or:

$$\text{dist}(a, b) = |b - a| = |\Delta x| = \sqrt{(\Delta x)^2}$$

The **midpoint** of the interval from $x = a$ to $x = b$ is the point M such that $\text{dist}(a, M) = \text{dist}(M, b)$, or $|M - a| = |b - M|$. If $a < M < b$,

$$M - a = b - M \Rightarrow 2M = a + b \Rightarrow M = \frac{a + b}{2}$$

It's not difficult to check that this formula also works when $b < M < a$.

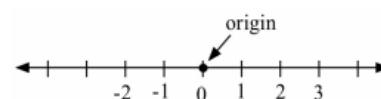


Figure 9: A number line.

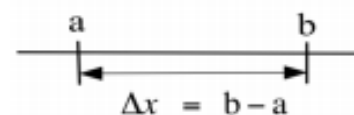


Figure 10: The increment $\Delta x = b - a$.

The capital Greek letter delta (Δ) appears often in calculus to represent the “change” in something.

Caution: Δx does not mean Δ times x , but rather the *difference* between two x -coordinates.

Solutions to Practice problems are at the end of each section.

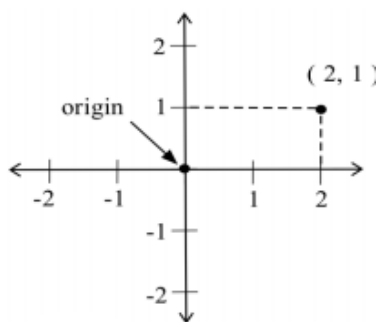


Figure 11: The xy -plane.

In this book, a point in the plane will be labeled either with a name, say P , or with an ordered pair, say (x, y) , or with both: $P = (x, y)$.

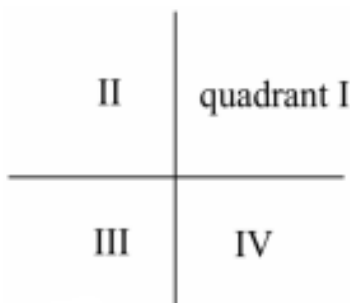


Figure 12: The four quadrants.

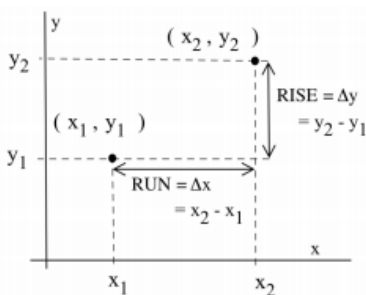


Figure 13: Increments between two points in the plane.

Example 1. Find the length and midpoint of the interval from $x = -3$ to $x = 6$.

Solution. $\text{dist}(-3, 6) = |6 - (-3)| = |9| = 9$; $M = \frac{(-3)+6}{2} = \frac{3}{2}$. ◀

Practice 1. Find the length and midpoint of the interval from $x = 7$ to $x = 2$.

The Cartesian Plane

Two perpendicular number lines, called **coordinate axes**, determine a **real number plane**. The axes intersect at a point called the **origin**. Each point P in the plane can be described by an ordered pair (x, y) of numbers that specify how far, and in which directions, we must move from the origin to reach the point P . We can locate the point $P = (x, y)$ in the plane by starting at the origin and moving x units horizontally and then y units vertically. Similarly, we can label each point in the plane with the ordered pair (x, y) , which directs us how to reach that point from the origin.

This coordinate system is called the **rectangular coordinate system** or the **Cartesian** coordinate system (after René Descartes), and the resulting plane the **Cartesian plane**.

The coordinate axes divide the plane into four **quadrants**, labeled quadrants *I*, *II*, *III* and *IV* moving counterclockwise from the upper-right quadrant.

We will often call the horizontal axis the **x -axis** and the vertical axis the **y -axis** and then refer to the plane as the **xy -plane**. This choice of x and y as labels for the axes is a common choice, but we will sometimes prefer to use different labels—and even different units of measurement on the two axes.

Increments and Distance Between Points In The Plane

If we move from a point $P = (x_1, y_1)$ in the plane to another point $Q = (x_2, y_2)$, then we will need to consider two **increments** or changes.

- The increment in the x (horizontal) direction is $x_2 - x_1$, denoted by $\Delta x = x_2 - x_1$.
- The increment in the y (vertical) direction is $y_2 - y_1$, denoted by $\Delta y = y_2 - y_1$.

Computing the **distance** between the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ involves a simple application of the Pythagorean Theorem:

$$\text{dist}(P, Q) = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The midpoint M of the line segment joining P and Q is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

where we have just used the one-dimension midpoint formula for each coordinate.

Example 2. Find an equation describing all the points $P = (x, y)$ equidistant from $Q = (2, 3)$ and $R = (5, -1)$.

Solution. The points $P = (x, y)$ must satisfy $\text{dist}(P, Q) = \text{dist}(P, R)$ so:

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-5)^2 + (y-(-1))^2}$$

By squaring each side we get:

$$(x-2)^2 + (y-3)^2 = (x-5)^2 + (y+1)^2$$

Expanding we get:

$$x^2 - 4x + 4 + y^2 - 6y + 9 = x^2 - 10x + 25 + y^2 + 2y + 1$$

and canceling like terms yields:

$$-4x - 6y + 13 = -10x + 2y + 26$$

so $y = 0.75x - 1.625$, the equation of a line. Every point on the line $y = 0.75x - 1.625$ is equidistant from both Q and R . ◀

Practice 2. Find an equation describing all points $P = (x, y)$ equidistant from $Q = (1, -4)$ and $R = (0, -3)$.

A circle with radius r and center at the point $C = (a, b)$ consists of all points $P = (x, y)$ at a distance of r from the center C : the points P that satisfy $\text{dist}(P, C) = r$.

Example 3. Find an equation of a circle with radius $r = 4$ and center $C = (5, -3)$.

Solution. A circle consists of the set of points $P = (x, y)$ at a fixed distance r from the center point C , so this circle will be the set of points $P = (x, y)$ at a distance of 4 units from the point $C = (5, -3)$; P will be on this circle if $\text{dist}(P, C) = 4$.

Using the distance formula and rewriting:

$$\sqrt{(x-5)^2 + (y+3)^2} = 4 \Rightarrow (x-5)^2 + (y+3)^2 = 16$$

which we can also express as $x^2 - 10x + 25 + y^2 + 6y + 9 = 16$. ◀

Practice 3. Find the equation of a circle with radius $r = 5$ and center $C = (-2, 6)$.

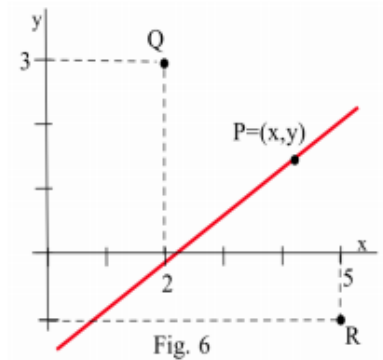


Figure 14: Finding all points equidistant from Q and R .

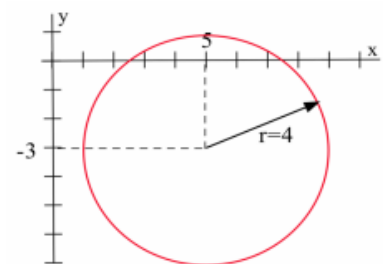


Figure 15: Finding the equation of a circle.

The Slope Between Points in the Plane

In one dimension (on the number line), our only choice was to move in the positive direction (so the x -values were increasing) or in the negative direction. In two dimensions (in the plane), we can move in infinitely many directions, so we need a precise way to describe direction.

The **slope** of the line segment joining $P = (x_1, y_1)$ to $Q = (x_2, y_2)$ is

$$m = \text{slope from } P \text{ to } Q = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

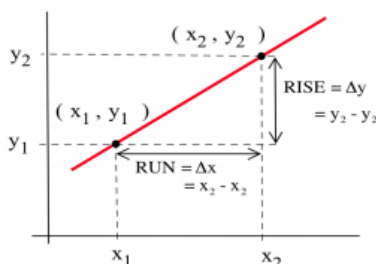


Figure 16: Defining slope.

The slope of a line measures how fast we rise or fall as we move from left to right along the line. It measures the rate of change of the y -coordinate with respect to changes in the x -coordinate. Most of our work will occur in two dimensions, and slope will be a very useful concept that will appear often.

If P and Q have the same x -coordinate, then $x_1 = x_2 \Rightarrow \Delta x = 0$. The line from P to Q is thus **vertical** and the slope $m = \frac{\Delta y}{\Delta x}$ is **undefined** because $\Delta x = 0$.

If P and Q have the same y -coordinate, then $y_1 = y_2 \Rightarrow \Delta y = 0$, so the line is **horizontal** and the slope is $m = \frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$ (assuming $\Delta x \neq 0$).

Practice 4. For $P = (-3, 2)$ and $Q = (5, -14)$, find Δx , Δy , and the slope of the line segment from P to Q .

If the coordinates of P or Q contain variables, then the slope m is still given by $m = \frac{\Delta y}{\Delta x}$, but we will need to use algebra to evaluate and simplify m .

Example 4. Find the slope of the line segment from $P = (1, 3)$ to $Q = (1 + h, 3 + 2h)$.

Solution. $y_1 = 3$ and $y_2 = 3 + 2h$, so $\Delta y = (3 + 2h) - (3) = 2h$; $x_1 = 1$ and $x_2 = 1 + h$, so $\Delta x = (1 + h) - (1) = h$. The slope is:

$$m = \frac{\Delta y}{\Delta x} = \frac{2h}{h} = 2$$

In this example, the value of m is constant (2) and does not depend on the value of h . ◀

Practice 5. Find the slope and midpoint of the line segment from $P = (2, -3)$ to $Q = (2 + h, -3 + 5h)$.

Example 5. Find the slope between the points $P = (x, x^2 + x)$ and $Q = (a, a^2 + a)$ for $a \neq x$.

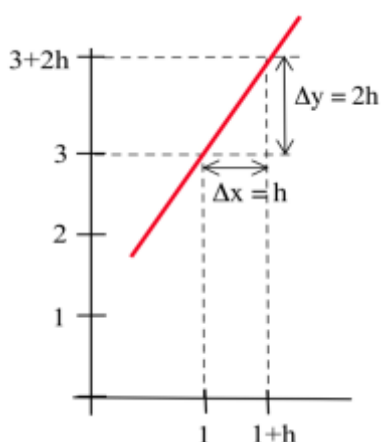


Figure 17: Sketch for the example.

Solution. $y_1 = x^2 + x$ and $y_2 = a^2 + a \Rightarrow \Delta y = (a^2 + a) - (x^2 + x)$; $x_1 = x$ and $x_2 = a$, so $\Delta x = a - x$ and the slope is:

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} = \frac{(a^2 + a) - (x^2 + x)}{a - x} \\ &= \frac{a^2 - x^2 + a - x}{a - x} = \frac{(a - x)(a + x) + (a - x)}{a - x} \\ &= \frac{(a - x)((a + x) + 1)}{a - x} = (a + x) + 1 \end{aligned}$$

Here the value of m depends on the values of both a and x . ◀

Practice 6. Find the slope between the points $P = (x, 3x^2 + 5x)$ and $Q = (a, 3a^2 + 5a)$ for $a \neq x$.

In application problems, it is important to read the information and the questions very carefully—including the units of measurement of the variables can help you avoid “silly” answers.

Example 6. In 1970, the population of Houston was 1,233,535 and in 1980 it was 1,595,138. Find the slope of the line through the points (1970, 1233535) and (1980, 1595138).

Solution. $m = \frac{\Delta y}{\Delta x} = \frac{1595138 - 1233535}{1980 - 1970} = \frac{361603}{10} = 36,160.3$ but 36,160.3 is just a number that may or may not have any meaning to you. If we include the units of measurement along with the numbers we will get a more meaningful result:

$$\frac{1595138 \text{ people} - 1233535 \text{ people}}{\text{year } 1980 - \text{year } 1970} = \frac{361603 \text{ people}}{10 \text{ years}} = 36,160.3 \frac{\text{people}}{\text{year}}$$

which says that during the decade from 1970 to 1980 the population of Houston grew at an average rate of 36,160 people per year. ◀

If the x -unit is time (in hours) and the y -unit is distance (in kilometers), then

$$m = \frac{\Delta y \text{ km}}{\Delta x \text{ hours}}$$

so the units for m are $\frac{\text{km}}{\text{hour}}$ (“kilometers per hour”), a measure of velocity, the rate of change of distance with respect to time.

If the x -unit is the number of employees at a bicycle factory and the y -unit is the number of bicycles manufactured, then

$$m = \frac{\Delta y \text{ bicycles}}{\Delta x \text{ employees}}$$

and the units for m are $\frac{\text{bicycles}}{\text{employee}}$ (“bicycles per employee”), a measure of the rate of production per employee.

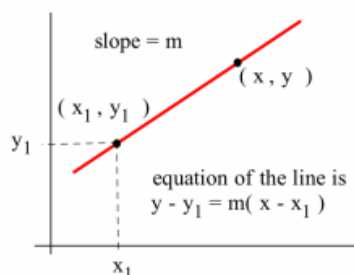


Figure 18: Finding an equation for a line from a point and a slope.

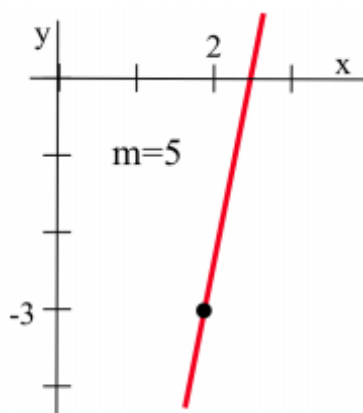


Figure 19: Sketch for the example.

Equations of Lines

Every (non-vertical) line has the property that the slope of the segment between any two points on the line is the same, and this constant slope property of straight lines leads to ways of finding equations to represent non-vertical lines.

Point-Slope Form

In calculus, we will usually know a point on a line and the slope of that line, so the point-slope form will be the easiest to apply. Other forms of equations for lines can be derived from the point-slope form.

If L is a non-vertical line through a known point $P = (x_1, y_1)$ with a known slope m , then the equation of the line L is:

$$\text{Point-Slope: } y - y_1 = m(x - x_1)$$

Example 7. Find an equation of the line through $(2, -3)$ with slope 5.

Solution. We can simply use the point-slope formula: $m = 5$, $y_1 = -3$ and $x_1 = 2$, so $y - (-3) = 5(x - 2)$, which simplifies to $y = 5x - 13$ ◀

An equation for a **vertical line** through a point $P = (a, b)$ is $x = a$. All points $Q = (x, y)$ on the vertical line through the point P have the same x -coordinate as P .

Two-Point Form

If two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are on the line L , then we can calculate the slope between them and use the first point and the point-slope equation to find an equation for L :

$$\text{Two-Point: } y - y_1 = m(x - x_1) \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Once we have the slope, m , it does not matter whether we use P or Q as the point. Either choice will result in the same equation for the line once we simplify it.

Slope-Intercept Form

It is common practice to rewrite an equation of a line into the form $y = mx + b$, the **slope-intercept form** of the line. The line $y = mx + b$ has slope m and crosses the y -axis at the point $(0, b)$.

Practice 7. Use the $\frac{\Delta y}{\Delta x}$ definition of slope to calculate the slope of the line $y = mx + b$.

The point-slope and the two-point forms are usually more useful for *finding* an equation of a line, but the slope-intercept form is usually the most useful form for an *answer* because it allows us to easily picture the graph of the line and to quickly calculate y -values given x -values.

Angles Between Lines

The **angle of inclination** of a line with the x -axis is the smallest angle θ that the line makes with the positive x -axis as measured from the x -axis counterclockwise to the line. Because the slope $m = \frac{\Delta y}{\Delta x}$ and because $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$ in a right triangle, $m = \tan(\theta)$.

The slope of a line is the tangent of its angle of inclination.

Parallel Lines

Two parallel lines L_1 and L_2 make equal angles with the x -axis, so their angles of inclination will be equal and hence so will their slopes.

Similarly, if the slopes, m_1 and m_2 , of two lines are equal, then the equations of the lines (in slope-intercept form) will always differ by a constant:

$$y_1 - y_2 = (m_1x + b_1) - (m_2x + b_2) = (m_1 - m_2)x + (b_1 - b_2) = b_1 - b_2$$

which is a constant, so the lines will be parallel.

The two preceding ideas can be combined into a single statement:

Two non-vertical lines L_1 and L_2 with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$.

Practice 8. Find an equation of the line that contains the point $(-2, 3)$ and is parallel to the line $3x + 5y = 17$.

Perpendicular Lines

If two lines are perpendicular, the situation is a bit more complicated.

Assume L_1 and L_2 are two non-vertical lines that intersect at the origin (for simplicity), with $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ points away from the origin on L_1 and L_2 , respectively. Then the slopes of L_1 and L_2 will be $m_1 = \frac{y_1}{x_1}$ and $m_2 = \frac{y_2}{x_2}$. The line connecting P and Q forms the third side of triangle OPQ , which will be a right triangle if and only if L_1 and L_2 are perpendicular. In particular, L_1 and L_2 are perpendicular if and only if the triangle OPQ satisfies the Pythagorean Theorem:

$$(\text{dist}(O, P))^2 + (\text{dist}(O, Q))^2 = (\text{dist}(P, Q))^2$$

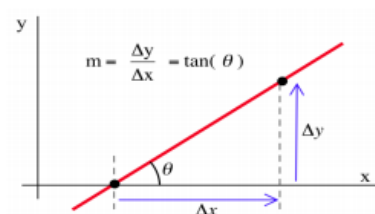


Figure 20: Angle of inclination.

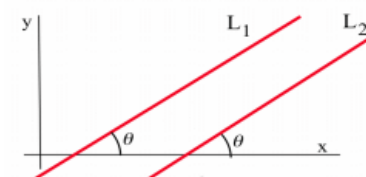


Figure 21: Parallel lines.

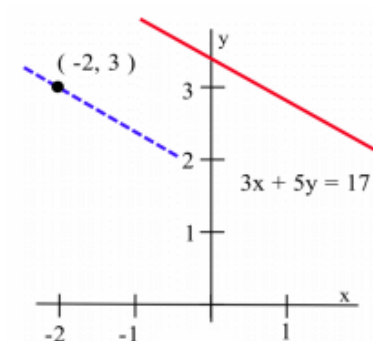


Figure 22: Sketch for practice problem.

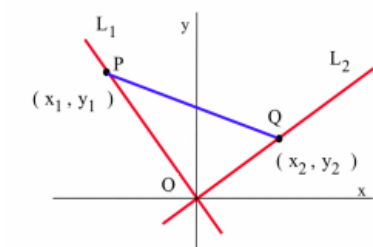


Figure 23: Perpendicular lines.

or:

$$\begin{aligned}(x_1 - 0)^2 + (y_1 - 0)^2 + (x_2 - 0)^2 + (y_2 - 0)^2 \\ = (x_1 - x_2)^2 + (y_1 - y_2)^2\end{aligned}$$

Squaring and simplifying, this reduces to $0 = -2x_1x_2 - 2y_1y_2$, so:

$$\frac{y_2}{x_2} = -\frac{x_1}{y_1} \Rightarrow m_2 = \frac{y_2}{x_2} = -\frac{x_1}{y_1} = -\frac{1}{\frac{x_1}{y_1}} = -\frac{1}{m_1}$$

We have just proved the following result:

Two non-vertical lines L_1 and L_2 with slopes m_1 and m_2 are perpendicular if and only if their slopes are negative reciprocals of each other: $m_2 = -\frac{1}{m_1}$.

Practice 9. Find an equation of the line that goes through the point $(2, -5)$ and is perpendicular to the line $3y - 7x = 2$.

Example 8. Find the distance (that is, the shortest distance) from the point $(1, 8)$ to the line $L : 3y - x = 3$.

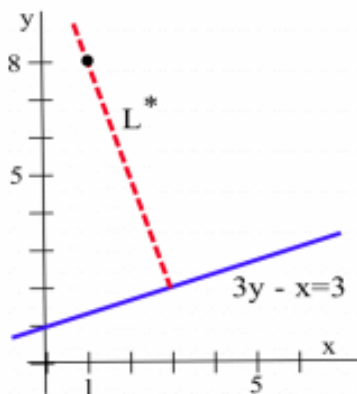


Figure 24: Sketch for example.

Solution. This is a sophisticated problem that requires several steps to solve: First we need a picture of the problem. We will find an equation for the line L^* through the point $(1, 8)$ and perpendicular to L . Then we will find the point P where L and L^* intersect. Finally, we will find the distance from P to $(1, 8)$.

Step 1: L has slope $\frac{1}{3}$ so L^* has slope $m = -\frac{1}{\frac{1}{3}} = -3$, and L^* has equation $y - 8 = -3(x - 1)$, which simplifies to $y = -3x + 11$.

Step 2: We can find the point where L intersects L^* by replacing the y -value in the equation for L with the y -value from our equation for L^* :

$$3(-3x + 11) - x = 3 \Rightarrow x = 3 \Rightarrow y = -3x + 11 = -3(3) + 11 = 2$$

which tells us that L and L^* intersect at $P = (3, 2)$.

Step 3: Finally, the distance from L to $(1, 8)$ is just the distance from the point $(1, 8)$ to the point $P = (3, 2)$, which is

$$\sqrt{(1 - 3)^2 + (8 - 2)^2} = \sqrt{40} \approx 6.325$$

The distance is (exactly) $\sqrt{40}$, or (approximately) 6.325. ◀

Angle Formed by Intersecting Lines

If two lines that are not perpendicular intersect at a point (and neither line is vertical), then we can use some geometry and trigonometry to determine the angles formed by the intersection of those lines.

Because θ_2 (see figure at right) is an exterior angle of the triangle ABC , θ_2 is equal to the sum of the two opposite interior angles, so $\theta_2 = \theta_1 + \theta \Rightarrow \theta = \theta_2 - \theta_1$. From trigonometry, we then know that:

$$\tan(\theta) = \tan(\theta_2 - \theta_1) = \frac{\tan(\theta_2) - \tan(\theta_1)}{1 + \tan(\theta_2)\tan(\theta_1)} = \frac{m_2 - m_1}{1 + m_2m_1}$$

The range of the arctan function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so $\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_2m_1}\right)$ always gives the smaller of the angles. The larger angle is $\pi - \theta$ (or $180^\circ - \theta^\circ$ if we measure the angles in degrees).

The smaller angle θ formed by two non-perpendicular lines

with slopes m_1 and m_2 is: $\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_2m_1}\right)$

Example 9. Find the point of intersection and the angle between the lines $y = x + 3$ and $y = 2x + 1$.

Solution. Solving the first equation for y and then substituting into the second equation:

$$(x + 3) = 2x + 1 \Rightarrow x = 2 \Rightarrow y = 2 + 3 = 5$$

The point of intersection is $(2, 5)$. Because both lines are in slope-intercept form, it is easy to see that $m_1 = 1$ and $m_2 = 2$:

$$\begin{aligned} \theta &= \arctan\left(\frac{m_2 - m_1}{1 + m_2m_1}\right) = \arctan\left(\frac{2 - 1}{1 + 2 \cdot 1}\right) \\ &= \arctan\left(\frac{1}{3}\right) \approx 0.322 \text{ radians} = 18.43^\circ \end{aligned}$$

The lines intersect at an angle of (approximately) 18.43° . ◀

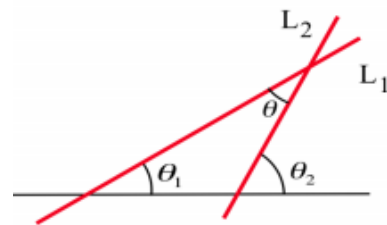


Figure 25: Lines L_1 and L_2 intersect at an angle θ .

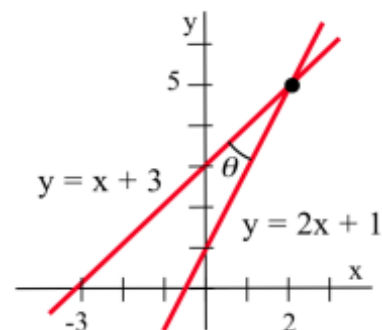
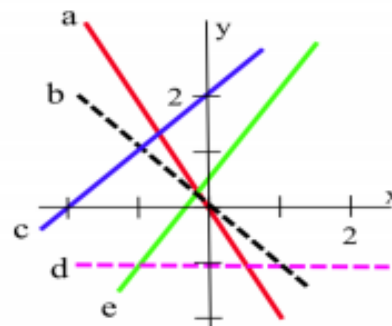
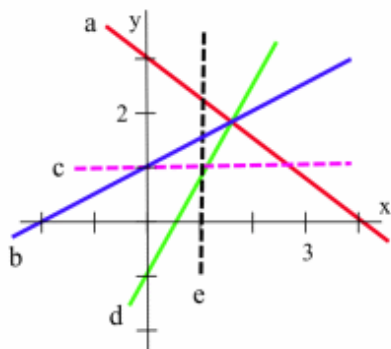


Figure 26: Sketch for example.

Problems

1. Estimate the slope of each line shown below.
2. Estimate the slope of each line shown below.



3. Compute the slope of the line that passes through:

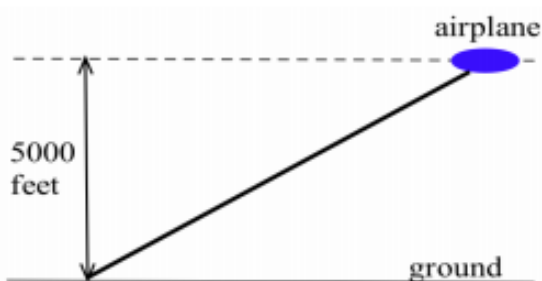
- (a) $(2, 4)$ and $(5, 8)$
- (b) $(-2, 4)$ and $(3, -5)$
- (c) $(2, 4)$ and (x, x^2)
- (d) $(2, 5)$ and $(2 + h, 1 + (2 + h)^2)$
- (e) $(x, x^2 + 3)$ and $(a, a^2 + 3)$

4. Compute the slope of the line that passes through:

- (a) $(5, -2)$ and $(3, 8)$
- (b) $(-2, -4)$ and $(5, -3)$
- (c) $(x, 3x + 5)$ and $(a, 3a + 5)$
- (d) $(4, 5)$ and $(4 + h, 5 - 3h)$
- (e) $(1, 2)$ and $(x, 1 + x^2)$
- (f) $(2, -3)$ and $(2 + h, 1 - (1 + h)^2)$
- (g) (x, x^2) and $(x + h, x^2 + 2xh + h^2)$
- (h) (x, x^2) and $(x - h, x^2 - 2xh + h^2)$

5. A small airplane at an altitude of 5,000 feet is flying east at 300 feet per second (a bit over 200 miles per hour), and you are watching it with a small telescope as it passes directly overhead.

- (a) What is the slope of the telescope 5, 10 and 20 seconds after the plane passes overhead?
- (b) What is the slope of the telescope t seconds after the plane passes overhead?
- (c) After the plane passes overhead, is the slope of the telescope increasing, decreasing or staying the same?



6. You are at the origin, $(0, 0)$, and are watching a small bug at the point $(t, 1 + t^2)$ at time t seconds.

- (a) What is the slope of your line of vision when $t = 5, 10$ and 15 seconds?
- (b) What is the slope of your line of vision at an arbitrary time t ?

7. The blocks in a city are all perfect squares. A friend gives you directions to a good restaurant: "Go north 3 blocks, turn east and go 5 blocks, turn south and go 7 blocks, turn west and go 3 blocks." How far away (straight-line distance) is it?

8. At the restaurant (see previous problem), a fellow diner gives you directions to a hotel: "Go north 5 blocks, turn right and go 6 blocks, turn right and go 3 blocks, turn left and go 2 blocks." How far away is the hotel from the restaurant?

9. The bottom of a 20-foot ladder is 4 feet from the base of a wall.

- (a) How far up the wall does the ladder reach?
- (b) What is the slope of the ladder?
- (c) What angle does it make with the ground?

10. Let $P = (1, -2)$ and $Q = (5, 4)$. Find:

- (a) the midpoint R of the line segment PQ .
- (b) the point T that is $\frac{1}{3}$ of the way from P to Q :

$$\text{dist}(P, T) = \frac{1}{3} \text{dist}(P, Q)$$

- (c) the point S that is $\frac{2}{5}$ of the way from P to Q .

11. If $P = (2, 3)$, $Q = (8, 11)$ and $R = (x, y)$, where:

$$x = 2a + 8(1 - a) \text{ and } y = 3a + 11(1 - a) \quad (0 \leq a \leq 1)$$

- (a) Verify that R is on the line segment PQ .
- (b) Verify that $\text{dist}(P, R) = (1 - a) \cdot \text{dist}(P, Q)$.

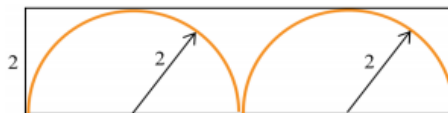
12. A rectangular box is 24 inches long, 18 inches wide and 12 inches high.

- (a) Find the length of the longest (straight) stick that will fit into the box.
- (b) What angle (in degrees) does that stick make with the base of the box?

13. The lines $y = x$ and $y = 4 - x$ intersect at $(2, 2)$.

- (a) Show that the lines are perpendicular.
- (b) Graph the lines together on your calculator using the "window" $[-10, 10] \times [-10, 10]$.
- (c) Why do the lines not appear to be perpendicular on the calculator display?
- (d) Find a suitable window so that the lines do appear perpendicular.

14. Two lines both go through the point $(1, 2)$, one with slope 3 and one with slope $-\frac{1}{3}$.
- Find equations for the lines.
 - Choose a suitable window so that the lines will appear perpendicular, and then graph them together on your calculator.
15. Sketch the line with slope m that goes through the point P , then find an equation for the line.
- $m = 3, P = (2, 5)$
 - $m = -2, P = (3, 2)$
 - $m = -\frac{1}{2}, P = (1, 4)$
16. Sketch the line with slope m that goes through the point P , then find an equation for the line.
- $m = 5, P = (2, 1)$
 - $m = -\frac{2}{3}, P = (1, 3)$
 - $m = \pi, P = (1, -3)$
17. Find an equation for each line.
- L_1 goes through the point $(2, 5)$ and is parallel to $3x - 2y = 9$.
 - L_2 goes through the point $(-1, 2)$ and is perpendicular to $2x = 7 - 3y$.
 - L_3 goes through the point $(3, -2)$ and is perpendicular to $y = 1$.
18. Find a value for the constant (A, B or D) so that:
- the line $y = 2x + A$ goes through $(3, 10)$.
 - the line $y = Bx + 2$ goes through $(3, 10)$.
 - the line $y = Dx + 7$ crosses the y -axis at the point $(0, 4)$.
 - the line $Ay = Bx + 1$ goes through the points $(1, 3)$ and $(5, 13)$.
19. Find the shortest distance between the circles with centers $C_1 = (1, 2)$ and $C_2 = (7, 10)$ with radii r_1 and r_2 when:
- $r_1 = 2$ and $r_2 = 4$
 - $r_1 = 2$ and $r_2 = 7$
 - $r_1 = 5$ and $r_2 = 8$
 - $r_1 = 3$ and $r_2 = 15$
 - $r_1 = 12$ and $r_2 = 1$
20. Find an equation of the circle with center C and radius r when
- $C = (2, 7)$ and $r = 4$
 - $C = (3, -2)$ and $r = 1$
 - $C = (-5, 1)$ and $r = 7$
 - $C = (-3, -1)$ and $r = 4$
21. Explain how to show, without graphing, whether a point $P = (x, y)$ is inside, on, or outside the circle with center $C = (h, k)$ and radius r .
22. A box with a base of dimensions 2 cm and 8 cm is definitely big enough to hold two semicircular rods with radii of 2 cm (see below).
- Will these same two rods fit in a box 2 cm high and 7.6 cm wide?
 - Will they fit in a box 2 cm high and 7.2 cm wide? (Suggestion: Turn one of the rods over.)



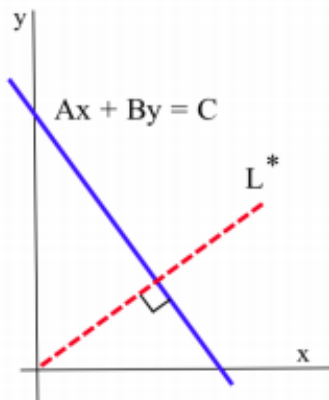
23. Show that an equation of the circle with center $C = (h, k)$ and radius r is $(x - h)^2 + (y - k)^2 = r^2$.
24. Find an equation of the line tangent to the circle $x^2 + y^2 = 25$ at the point P when:
- $P = (3, 4)$
 - $P = (-4, 3)$
 - $P = (0, 5)$
 - $P = (-5, 0)$
25. Find an equation of the line tangent to the circle with center $C = (3, 1)$ at the point P when:
- $P = (8, 13)$
 - $P = (-10, 1)$
 - $P = (-9, 6)$
 - $P = (3, 14)$
26. Find the center $C = (h, k)$ and the radius r of the circle that goes through the three points:
- $(0, 1), (1, 0)$ and $(0, 5)$
 - $(1, 4), (2, 2)$ and $(8, 2)$
 - $(1, 3), (4, 12)$ and $(8, 4)$

27. How close does

- (a) the line $3x - 2y = 4$ come to the point $(2, 5)$?
 (b) the line $y = 5 - 2x$ come to the point $(1, -2)$?
 (c) the circle with radius 3 and center at $(2, 3)$ come to the point $(8, 3)$?

28. How close does

- (a) the line $2x - 5y = 4$ come to the point $(1, 5)$?
 (b) the line $y = 3 - 2x$ come to the point $(5, -2)$?
 (c) the circle with radius 4 and center at $(4, 3)$ come to the point $(10, 3)$?

29. Follow the steps below (and refer to the figure) to find a formula for the distance from the origin to the line $Ax + By = C$.

- (a) Show that the line L given by $Ax + By = C$ has slope $m = -\frac{A}{B}$.
 (b) Find the equation of the line L^* that goes through $(0, 0)$ and is perpendicular to L .
 (c) Show that L and L^* intersect at the point:

$$(x, y) = \left(\frac{AC}{A^2 + B^2}, \frac{BC}{A^2 + B^2} \right)$$

- (d) Show that the distance from the origin to the point (x, y) is:

$$\frac{|C|}{\sqrt{A^2 + B^2}}$$

30. Show that a formula for the distance from the point (p, q) to the line $Ax + By = C$ is:

$$\frac{|Ap + Bq - C|}{\sqrt{A^2 + B^2}}$$

(The steps will be similar to those in the previous problem, but the algebra will be more complicated.)

0.2 Practice Answers

1. Length = $\text{dist}(-7, -2) = |(-7) - (-2)| = |-5| = 5$.

The midpoint is at $\frac{(-7) + (-2)}{2} = \frac{-9}{2} = -4.5$.

2. $\text{dist}(P, Q) = \text{dist}(P, R) \Rightarrow (x - 1)^2 + (y + 4)^2 = (x - 0)^2 + (y + 3)^2$;
squaring each side and simplifying, eventually yields $y = x - 4$.

3. The point $P = (x, y)$ is on the circle when it is 5 units from the center $C = (-2, 6)$, so $\text{dist}(P, C) = 5$. Then $\text{dist}((x, y), (-2, 6)) = 5$, so $(x + 2)^2 + (y - 6)^2 = 5 \Rightarrow (x + 2)^2 + (y - 6)^2 = 25$.

4. $\Delta x = 5 - (-3) = 8$ and $\Delta y = -14 - 2 = -16$, so:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{-16}{8} = -2$$

5. $\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(-3 + 5h) - (-3)}{(2 + h) - 2} = \frac{5h}{h} = 5$. The midpoint is at $\left(\frac{(2) + (2 + h)}{2}, \frac{(-3 + 5h) + (-3)}{2}\right) = \left(2 + \frac{h}{2}, -3 + \frac{5h}{2}\right)$.

6. $\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(3a^2 + 5a) - (3x^2 + 5x)}{a - x} = \frac{3(a^2 - x^2) + 5(a - x)}{a - x} = \frac{3(a + x)(a - x) + 5(a - x)}{a - x} = 3(a + x) + 5$

7. Let $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$. Then:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m$$

8. The line $3x + 5y = 17$ has slope $-\frac{3}{5}$, so the slope of the parallel line is $m = -\frac{3}{5}$. Using the form $y = -\frac{3}{5}x + b$ and the point $(-2, 3)$ on the line, we have $3 = -\frac{3}{5}(-2) + b \Rightarrow b = \frac{9}{5} \Rightarrow y = -\frac{3}{5}x + \frac{9}{5}$, or $5y + 3x = 9$.

9. The line $3y - 7x = 2$ has slope $\frac{7}{3}$, so the slope of the perpendicular line is $m = -\frac{3}{7}$. Using the form $y = -\frac{3}{7}x + b$ and the point $(2, -5)$ on the line, we have $-5 = -\frac{3}{7}(2) + b \Rightarrow b = -\frac{29}{7} \Rightarrow y = -\frac{3}{7}x - \frac{29}{7}$, or $7y + 3x = -29$.

0.3 *Functions and Their Graphs*

When you prepared for calculus, you learned to manipulate functions by adding, subtracting, multiplying and dividing them, as well as evaluating functions of functions (composition). In calculus, we will continue to work with functions and their applications. We will create new functions by operating on old ones. We will gather information from the graphs of functions and from derived functions. We will find ways to describe the point-by-point behavior of functions as well as their behavior “close to” certain points and also over entire intervals. We will find tangent lines to graphs of functions and areas between graphs of functions. And, of course, we will see how these ideas can be used in a variety of fields.

This section and the next one review information and procedures you should already know about functions before we begin calculus.

What is a function?

Let’s begin with a (very) general definition of a function:

A **function** from a set X to a set Y is a rule for assigning to each element of the set X a single element of the set Y . A function assigns a unique (exactly one) output element from the set Y to each input element from the set X .

The rule that defines a function is often given in the form of an equation, but it could also be given in words or graphically or by a table of values. In practice, functions are given in all of these ways, and we will use all of them in this book.

In the definition of a function, the set X of all inputs is called the **domain** of the function. The set Y of all outputs produced from these inputs is called the **range** of the function. Two different inputs (elements in the domain) can be assigned to the same output (an element in the range) but one input cannot lead to two different outputs.

Most of the time we will work with functions whose domains and ranges are real numbers, but there are other types of functions all around us. Final grades for this course provide an example of a function. For each student, the instructor will assign a final grade based on some rule for evaluating that student’s performance. The domain of this function consists of all students registered for the course, and the range consists of grades (either letters like A, B, C, D, F, or numbers like 4.0, 3.7, 2.0, 1.7, 0.0). Two students can receive the same final grade, but only one grade can be assigned to each student.

Function Machines

Functions are abstract structures, but sometimes it is easier to think of them in a more concrete way. One such way is to imagine that a function is a special-purpose computer, a machine that accepts inputs, does something to those inputs according to a defining rule, and produces an output. The output is the value of the function associated with the given input value. If the defining rule for a function f is “multiply the input by itself,” $f(\text{input}) = (\text{input})(\text{input})$, then the figure and table in the margin show the results of putting the inputs x , 5 , 2.5 , a , $c + 3$ and $x + h$ into the machine f .

Practice 1. If we have a function machine g whose rule is “divide 3 by the input and add 1,” $g(x) = \frac{3}{x} + 1$, what outputs do we get from the inputs x , 5 , a , $c + 3$ and $x + h$? What happens if the input is 0?

You expect your calculator to behave as a function: each time you press the same sequence of keys (input) you expect to see the same display (output). In fact, if your calculator did not produce the same output each time, you would need a new calculator.

Functions Defined by Equations

If the domain of a function consists of a collection of real numbers (perhaps all real numbers) and the range is also a collection of real numbers, then the function is called a **numerical function**. We can give the rule for a numerical function in several ways, but usually write it as a formula. If the rule for a numerical function, f , is “the output is the input number multiplied by itself,” then we could write the rule as

$$f(x) = x \cdot x = x^2$$

The use of an “ x ” to represent the input is simply a matter of convenience and custom. We could also represent the same function by:

- $f(a) = a^2$
- $f(\#) = \#^2$ or
- $f(\text{input}) = (\text{input})^2$.

For the function f defined by $f(x) = x^2 - x$, we can see that:

- $f(3) = 3^2 - 3 = 6$
- $f(0.5) = (0.5)^2 - (0.5) = -0.25$
- $f(-2) = (-2)^2 - (-2) = 6$

Notice that the two different inputs 3 and -2 both lead to the output of 6. That is allowable for a function.

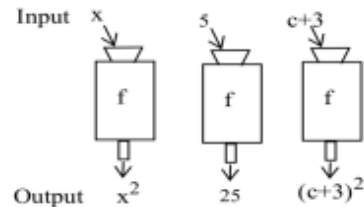


Figure 27: A function machine.

input	output
x	x^2
a	a^2
2.5	6.25
$x + h$	$(x + h)^2$

On many calculators there is a feature that does not produce the same output each time you use it. What is it?

We can also evaluate f if the input contains variables. If we replace the “ x ” with something else in the notation “ $f(x)$,” then we must replace the “ x ” with the same thing everywhere in the formula:

- $f(c) = c^2 - c$
- $f(a + 1) = (a + 1)^2 - (a + 1) = (a^2 + 2a + 1) - (a + 1) = a^2 + a$
- $f(x + h) = (x + h)^2 - (x + h) = (x^2 + 2xh + h^2) - (x + h)$

and, in general: $f(\text{input}) = (\text{input})^2 - (\text{input})$

For more complicated expressions, we can just proceed step-by-step:

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{((x + h)^2 - (x + h)) - (x^2 - x)}{h} \\ &= \frac{((x^2 + 2xh + h^2) - (x + h)) - (x^2 - x)}{h} \\ &= \frac{2xh + h^2 - h}{h} = \frac{h(2x + h - 1)}{h} = 2x + h - 1 \end{aligned}$$

Practise 2. For the function g defined by $g(t) = t^2 - 5t$, evaluate $g(1)$, $g(-2)$, $g(w + 3)$, $g(x + h)$, $g(x + h) - g(x)$, and $\frac{g(x+h)-g(x)}{h}$.

Functions Defined by Graphs and Tables of Values

The graph of a numerical function f consists of a plot of ordered pairs (x, y) where x is in the domain of f and $y = f(x)$, such as the graph of $f(x) = \sin(x)$ for $-4 \leq x \leq 9$ in the margin. A table of values of a numerical function consists of a list of (some of) the ordered pairs (x, y) where $y = f(x)$.

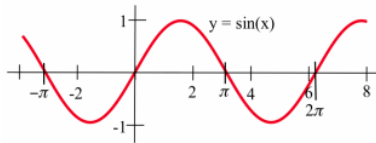


Figure 28: $y = \sin(x)$, $-4 \leq x \leq 9$

A function can be defined by a graph or by a table of values, and these types of definitions are common in applied fields. The outcome of an experiment will depend on the input, but the experimenter may not know the “rule” for predicting the outcome. In that case, the experimenter usually represents the function of interest as a table of measured outcome values versus input values, or as a graph of the outcomes versus the inputs. The table and graph in the margin show the deflections obtained when weights were loaded at the end of a wooden stick. The next graph shows the temperature of a hot cup of tea as a function of the time as it sits in a 68°F room. In these experiments, the “rule” for the function is that $f(\text{input}) = \text{actual outcome of the experiment}$.

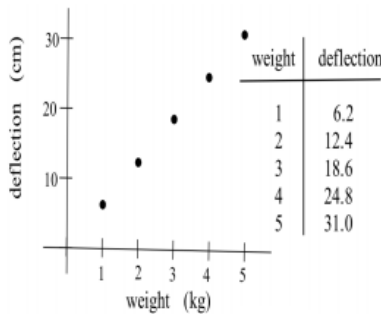


Figure 29: Graph and table showing deflections obtained when weights are loaded at the end of a wooden stick.

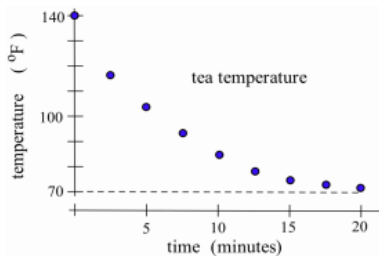


Figure 30: Graph showing temperature of a cup of tea over time.

Tables have the advantage of presenting the data explicitly, but it is often difficult to detect patterns simply from lists of numbers.

Graphs usually obscure some of the precision of the data, but more easily allow us to detect patterns visually—we can actually see what is happening with the numbers

Creating Graphs of Functions

Most people understand and can interpret pictures more quickly than tables of data or equations, so if we have a function defined by a table of values or by an equation, it is often useful (and necessary) to create a picture of the function: a graph.

A Graph from a Table of Values

If we have a table of values for a function, perhaps consisting of measurements obtained from an experiment, then we can plot the ordered pairs in the xy -plane to get a graph consisting of a collection of points.

The table in the margin shows the lengths and weights of trout caught (and released) during several days of fishing. The graph plots those values along with a line that comes “close” to the plotted points. From the graph, you could estimate that a 17-inch trout should weigh slightly more than one pound.

length (in.)	weight (lbs.)
13.5	0.4
14.5	0.9
15.0	0.7
16.0	0.9
18.0	1.2
18.5	1.6
19.5	1.5
20.5	1.7
20.5	2.1

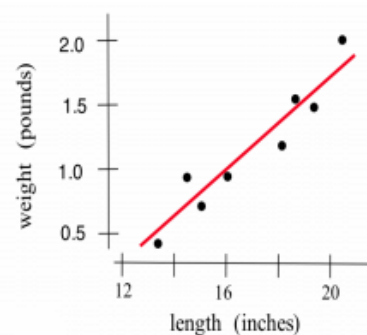


Figure 31: Lengths and weights of trout.

A Graph from an Equation

Creating the graph of a function given by an equation is similar to creating one from a table of values: we need to plot enough points (x, y) where $y = f(x)$ so we can be confident of the shape and location of the graph of the entire function. We can find a point (x, y) that satisfies $y = f(x)$ by picking a value for x and then calculating the value for y by evaluating $f(x)$. Then we can enter the (x, y) value in a table or simply plot the point (x, y) .

If you recognize the form of the equation and know something about the shape of graphs of that form, you may not have to plot many points. If you do not recognize the form of the equation, then you will need to plot more points, maybe 10 or 20 or 234: it depends on how complicated the graph appears and on how important it is to you (or your boss) to have an accurate graph. Evaluating $y = f(x)$ at many different values for x and then plotting the points (x, y) is usually not very difficult, but it can be very time-consuming. Fortunately, calculators and computers can often do the evaluation and plotting for you.

Is Every Graph the Graph of a Function?

The definition of “function” requires that each element of the domain (each input value) be sent by the function to exactly one element of the range (to exactly one output value), so for each input x -value there will be exactly one output y -value, $y = f(x)$. The points (x, y_1) and (x, y_2) cannot both be on the graph of f unless $y_1 = y_2$. The graphical interpretation of this result is called the **Vertical Line Test**.

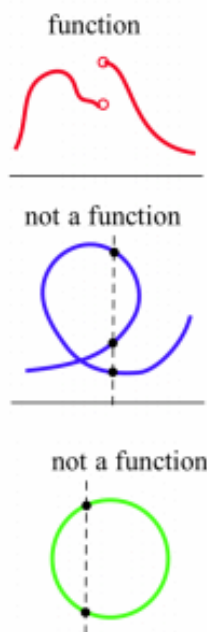


Figure 32: The vertical line test in action.

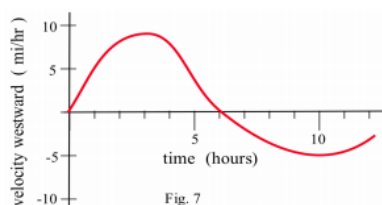


Figure 33: Graph for the example.

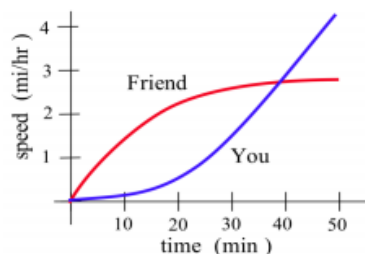


Figure 34: Graph for practice problem.

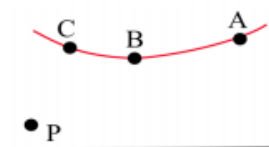


Figure 35: Graph for the example.

Vertical Line Test for a Function: A graph is the graph of a function if and only if a vertical line drawn through any point in the domain intersects the graph at exactly one point.

The figure in the margin shows the graph of a function, followed by two graphs that are not graphs of functions, along with vertical lines that intersect those graphs at more than one point. Non-functions are not “bad,” and are often necessary to describe complicated phenomena.

Reading Graphs Carefully

Calculators and computers can help students, reporters, financial analysts and scientists create graphs quickly and easily. Because of this, graphs are being used more often than ever to present information and justify arguments. This text takes a distinctly graphical approach to the ideas and meaning of calculus. Calculators and computers can help us create graphs, but we need to be able to read them carefully. The next examples illustrate some types of information that can be obtained by carefully reading and understanding graphs.

Example 1. A boat starts from St. Thomas and sails due west with the velocity shown in the figure.

- When is the boat traveling the fastest?
- What does a negative velocity away from St. Thomas mean?
- When is the boat the farthest away from St. Thomas?

Solution. (a) The greatest speed is 10 mph at $t = 3$ hours. (b) It means that the boat is heading back toward St. Thomas. (c) The boat is farthest from St. Thomas at $t = 6$ hours. For $t < 6$ the boat’s velocity is positive, and the distance from the boat to St. Thomas is increasing. For $t > 6$ the boat’s velocity is negative, and the distance from the boat to St. Thomas is decreasing. ◀

Practice 3. You and a friend start out together and hike along the same trail but walk at different speeds, as shown in the figure.

- Who is walking faster at $t = 20$?
- Who is ahead at $t = 20$?
- When are you and your friend farthest apart?
- Who is ahead when $t = 50$?

Example 2. Which has the largest slope: the line through the points A and P , the line through B and P , or the line through C and P ?

Solution. The line through C and P : $m_{PC} > m_{PB} > m_{PA}$. ◀

Practice 4. In the figure, the point Q on the curve is fixed, and the point P is moving to the right along the curve toward the point Q . As P moves toward Q , is the indicated value increasing, decreasing, remaining constant, or doing something else?

- (a) x -coordinate of P
- (b) x -increment from P to Q
- (c) slope from P to Q

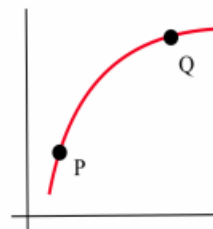


Figure 36: Graph for practice problem.

Example 3. The graph of $y = f(x)$ appears in the margin. Let $g(x)$ be the slope of the line tangent to the graph of $f(x)$ at the point $(x, f(x))$.

- (a) Estimate the values $g(1)$, $g(2)$ and $g(3)$.
- (b) For what value(s) of x is $g(x) = 0$?
- (c) At what value(s) of x is $g(x)$ largest?
- (d) Sketch the graph of $y = g(x)$.

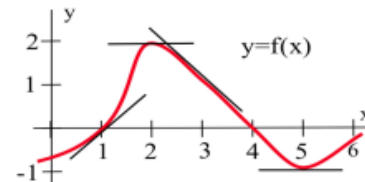


Figure 37: Graph for the example.

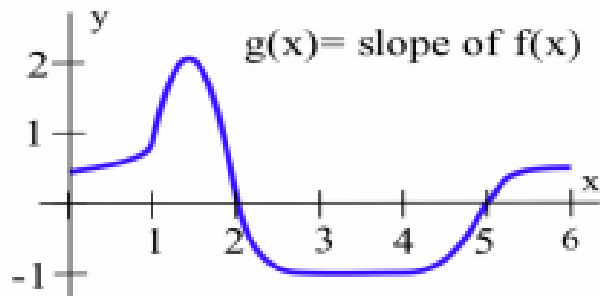
Solution. (a) The figure in the margin shows the graph of $y = f(x)$ with several tangent lines to the graph of f . From this graph, we can estimate that $g(1)$ (the slope of the line tangent to the graph of f at $(1, 0)$) is approximately equal to 1. Similarly, $g(2) \approx 0$ and $g(3) \approx -1$.

(b) The slope of the tangent line appears to be horizontal (slope = 0) at $x = 2$ and at $x = 5$.

(c) The tangent line to the graph of f appears to have greatest slope (be steepest) near $x = 1.5$.

(d) We can build a table of values of $g(x)$ and then sketch the graph of these values. A graph of $y = g(x)$ appears below. ◀

x	$f(x)$	$g(x)$
0	-1.0	0.5
1	0.0	1.0
2	2.0	0.0
3	1.0	-1.0
4	0.0	-1.0
5	-1.0	0.0
6	-0.5	0.5



Practice 5. Water flows into a container (see figure) at a constant rate of 3 gallons per minute. Starting with an empty container, sketch the graph of the height of the water in the container as a function of time.

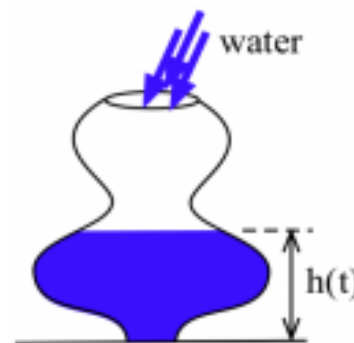
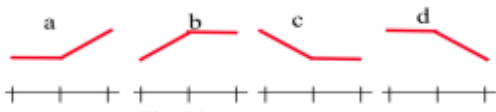


Figure 38: Water flows into a container.

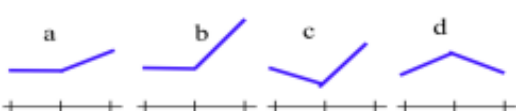
Problems

In problems 1-4, match the numerical triples to the graphs. For example, in Problem 1, A: 3, 3, 6 is "over and up" so it matches graph (a); B is "down and over" so it matches graph (c).

1. A: 3, 3, 6; B: 12, 6, 6; C: 7, 7, 3 D: 2, 4, 4



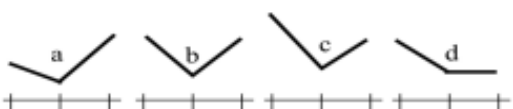
2. A: 7, 10, 7; B: 17, 17, 25; C: 4, 4, 8 D: 12, 8, 16



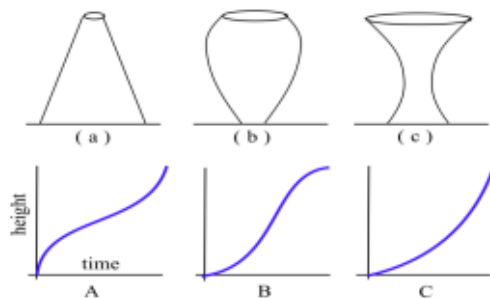
3. A: 7, 14, 10; B: 23, 45, 22; C: 8, 12, 8 D: 6, 9, 3



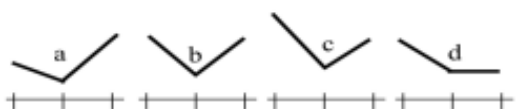
4. A: 6, 3, 9; B: 8, 1, 1; C: 12, 6, 9 D: 3.7, 1.9, 3.6



5. Water is flowing at a steady rate into each of the bottles shown below. Match each bottle shape with the graph of the height of the water as a function of time



6. Sketch shapes of bottles that will have the water height versus time graphs shown below.



7. If $f(x) = x^2 + 3$, $g(x) = \sqrt{x - 5}$ and $h(x) = \frac{x}{x-2}$:
- evaluate $f(1)$, $g(1)$ and $h(1)$.
 - graph $f(x)$, $g(x)$ and $h(x)$ for $-5 \leq x \leq 10$.
 - evaluate $f(3x)$, $g(3x)$ and $h(3x)$.
 - evaluate $f(x+h)$, $g(x+h)$ and $h(x+h)$.

8. Find the slope of the line through the points P and Q when:

- $P = (1, 3)$, $Q = (2, 7)$
- $P = (x, x^2 + 2)$, $Q = (x+h, (x+h)^2 + 2)$
- $P = (1, 3)$, $Q = (x, x^2 + 2)$
- P, Q as in (c) with $x = 2$, $x = 1.1$, $x = 1.002$

9. Find the slope of the line through the points P and Q when:

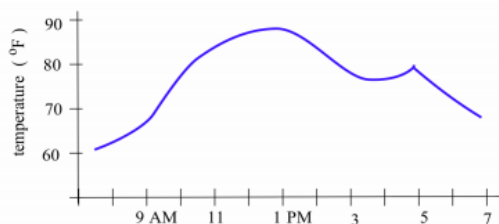
- $P = (1, 5)$, $Q = (2, 7)$
- $P = (x, x^2 + 3x - 1)$,
 $Q = (x+h, (x+h)^2 + 3(x+h) - 1)$
- P, Q as in (c) with $x = 1.3$, $x = 1.1$, $x = 1.002$

10. If $f(x) = x^2 + x$ and $g(x) = \frac{3}{x}$, evaluate and simplify $\frac{f(a+h) - f(a)}{h}$ and $\frac{g(a+h) - g(a)}{h}$ when $a = 1$, $a = 2$, $a = -1$, $a = x$.

11. If $f(x) = x^2 - 2x$ and $g(x) = \sqrt{x}$, evaluate and simplify $\frac{f(a+h) - f(a)}{h}$ and $\frac{g(a+h) - g(a)}{h}$ when $a = 1$, $a = 2$, $a = 3$, $a = x$.

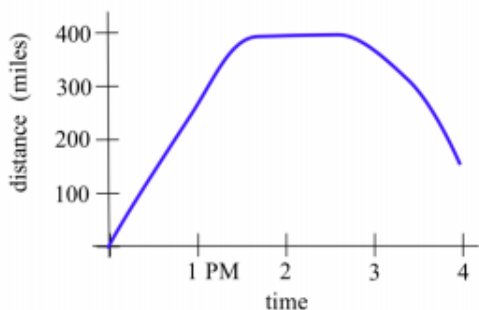
12. The temperatures shown below were recorded during a 12-hour period in Chicago.

- At what time was the temperature the highest? Lowest?
- How fast was the temperature rising at 10 a.m.? At 1 p.m.?
- What could have caused the drop in temperature between 1 p.m. and 3 p.m.?



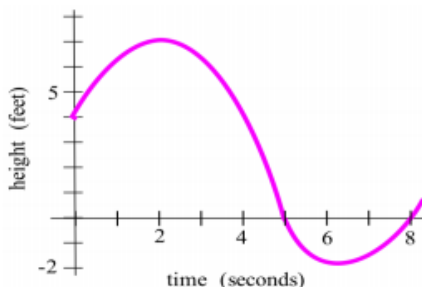
13. The graph below shows the distance of an airplane from an airport during a long flight.

- (a) How far was the airplane from the airport at 1 p.m.? At 2 p.m.?
- (b) How fast was the distance changing at 1 p.m.?
- (c) How could the distance from the plane to the airport remain unchanged from 1:45 p.m. until 2:30 p.m. without the airplane falling?



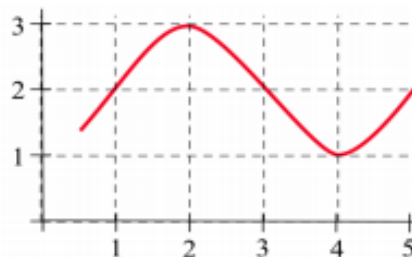
14. The graph below shows the height of a diver above the water level at time t seconds.

- (a) What was the height of the diving board?
- (b) When did the diver hit the water?
- (c) How deep did the diver get?
- (d) When did the diver return to the surface?



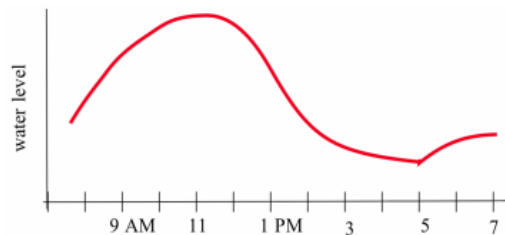
15. Refer to the curve shown below.

- (a) Sketch the lines tangent to the curve at $x = 1, 2, 3, 4$ and 5 .
- (b) For what value(s) of x is the value of the function largest? Smallest?
- (c) For what value(s) of x is the slope of the tangent line largest? Smallest?

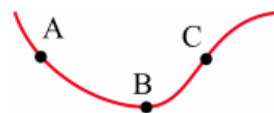


16. The figure below shows the height of the water (above and below mean sea level) at a Maine beach.

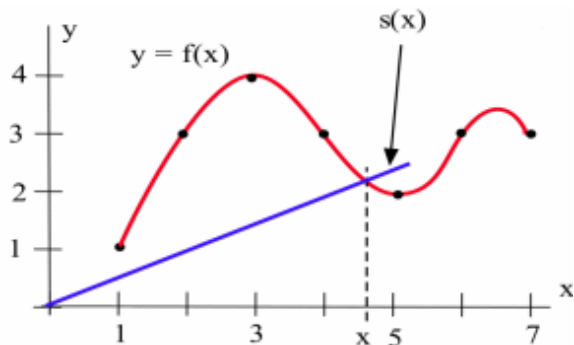
- (a) At which time(s) was the most beach exposed? The least?
- (b) At which time(s) was the current the strongest?



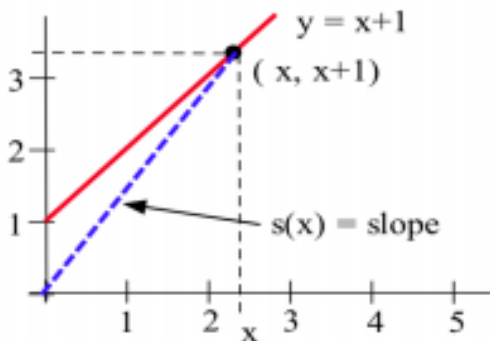
17. Imagine that you are ice skating, from left to right, along the path shown below. Sketch the path you will follow if you fall at points A, B and C .



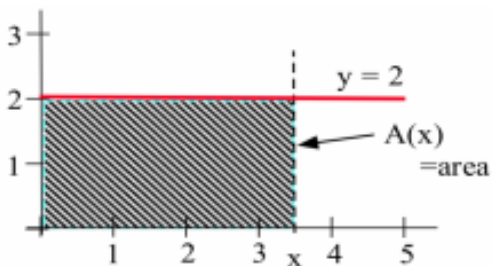
18. Define $s(x)$ to be the slope of the line through the points $(0,0)$ and $(x, f(x))$ where $f(x)$ is the function graphed below. For example, $s(3) =$ slope of the line through $(0,0)$ and $(3, f(3)) = \frac{4}{3}$.
- (a) Evaluate $s(1), s(2)$ and $s(4)$.
- (b) For which integer value of x between 1 and 7 is $s(x)$ smallest?



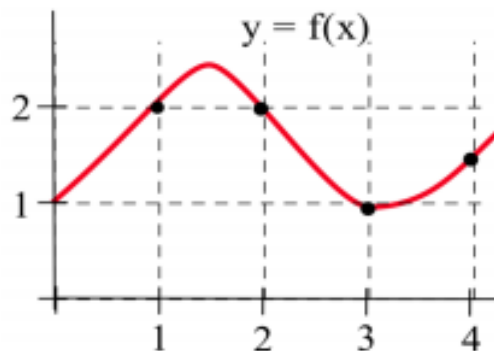
19. Let $f(x) = x + 1$ and define $s(x)$ to be the slope of the line through the points $(0,0)$ and $(x, f(x))$, as shown below. For example, $s(2) =$ slope of the line through $(0,0)$ and $(2,3) = \frac{3}{2}$.
- (a) Evaluate $s(1), s(3)$ and $s(4)$.
- (b) Find a formula for $s(x)$.



20. Define $A(x)$ to be the area of the rectangle bounded by the coordinate axes, the line $y = 2$ and a vertical line at x , as shown below. For example, $A(3) =$ area of a 2×3 rectangle $= 6$.
- (a) Evaluate $A(1), A(2)$ and $A(5)$.
- (b) Find a formula for $A(x)$.

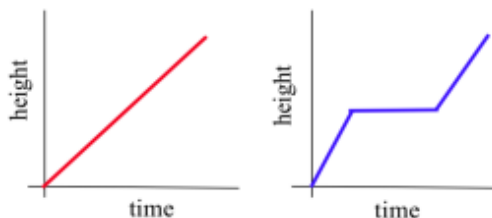


21. Using the graph of $y = f(x)$ below, let $g(x)$ be the slope of the line tangent to the graph of $f(x)$ at the point $(x, f(x))$. Complete the table, estimating values of the slopes as best you can.



x	$f(x)$	$g(x)$
0	1	1
1		
2		
3		
4		

22. Sketch the graphs of water height versus time for water pouring into a bottle shaped like:
- (a) a milk carton
- (b) a spherical glass vase
- (c) an oil drum (cylinder) lying on its side
- (d) a giraffe
- (e) you
23. Design bottles whose graphs of (highest) water height versus time will look like those shown below.



0.3 Practice Answers

1. Create an input-output table using the function rule:

input	output
x	$\frac{3}{x} + 1$
5	$\frac{3}{5} + 1 = 1.6$
a	$\frac{3}{a} + 1$
$c + 3$	$\frac{3}{c+3} + 1$
$x + h$	$\frac{3}{x+h} + 1$

If $x = 0$, then $g(0) = \frac{3}{0} + 1$ is not defined (because of division by 0).

2. $g(t) = t^2 - 5t$

$g(1) = 1^2 - 5(1) = -4$

$g(-2) = (-2)^2 - 5(-2) = 14$

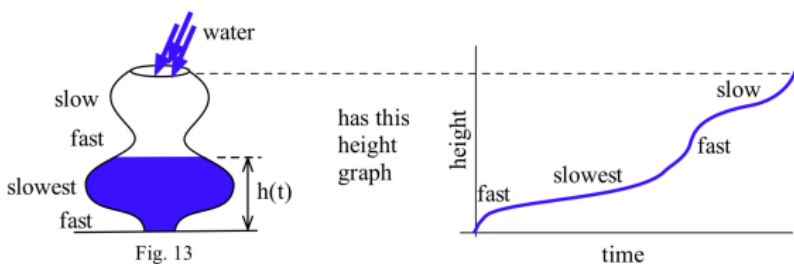
$g(w+3) = (w+3)^2 - 5(w+3) = w^2 + 6w + 9 - 5w - 15 = w^2 + w - 6$

$g(x+h) = (x+h)^2 - 5(x+h) = x^2 + 2xh + h^2 - 5x - 5h$

$g(x+h) - g(x) = (x^2 + 2xh + h^2 - 5x - 5h) - (x^2 - 5x) = 2xh + h^2 - 5h$

$$\frac{g(x+h) - g(x)}{h} = \frac{2xh + h^2 - 5h}{h} = 2x + h - 5$$

3. (a) Friend (b) Friend (c) At $t = 40$. Before that, your friend is walking faster and increasing the distance between you. Then you start to walk faster than your friend and start to catch up. (d) Friend. You are walking faster than your friend at $t = 50$, but you still have not caught up.
4. (a) The x -coordinate is increasing. (b) The x -increment Δx is decreasing. (c) The slope of the line through P and Q is decreasing.
5. See the figure:



0.4 Combinations of Functions

Sometimes a physical or economic situation behaves differently depending on various circumstances. In these situations, a more complicated formula may be needed to describe the situation.

Multiline Definitions of Functions: Putting Pieces Together

Sales Tax: Some states have different rates of sales tax depending on the type of item purchased. As an example, for many years food purchased at restaurants in Seattle was taxed at a rate of 10%, while most other items were taxed at a rate of 9.5% and food purchased at grocery stores had no tax assessed. We can describe this situation by using a multiline function: a function whose defining rule consists of several pieces. Which piece of the rule we need to use will depend on what we buy. In this example, we could define the tax T on an item that costs x to be:

$$T(x) = \begin{cases} 0 & \text{if } x \text{ is the cost of a food at a grocery store} \\ 0.10x & \text{if } x \text{ is the cost of food at a restaurant} \\ 0.095x & \text{if } x \text{ is the cost of any other item} \end{cases}$$

To find the tax on a \$2 can of stew, we would use the first piece of the rule and find that the tax is \$0. To find the tax on a \$30 restaurant bill, we would use the second piece of the rule and find that the tax is \$3.00. The tax on a \$150 textbook requires using the third rule: the tax would be \$14.25.

Wind Chill Index: The rate at which a person's body loses heat depends on the temperature of the surrounding air and on the speed of the air. You lose heat more quickly on a windy day than you do on a day with little or no wind. Scientists have experimentally determined this rate of heat loss as a function of temperature and wind speed, and the resulting function is called the Wind Chill Index, WCI. The WCI is the temperature on a still day (no wind) at which your body would lose heat at the same rate as on the windy day. For example, the WCI value for 30°F air moving at 15 miles per hour is 9°F: your body loses heat as quickly on a 30°F day with a 15 mph wind as it does on a 9°F day with no wind.

If T is the Fahrenheit temperature of the air and v is the speed of the wind in miles per hour, then the WCI can be expressed as a multiline function of the wind speed v (and of the temperature T):

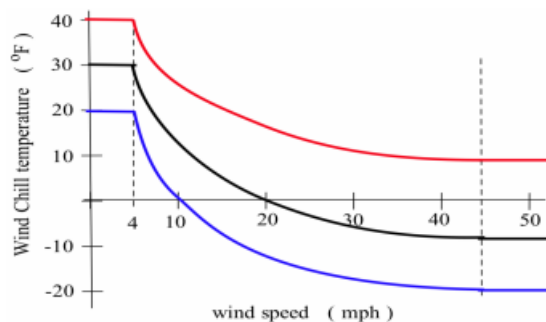
$$\text{WCI} = \begin{cases} T & \text{if } 0 \leq v \leq 4 \\ 91.4 - \frac{10.45 + 6.69\sqrt{v} - 0.447v}{22}(91.5 - T) & \text{if } 4 < v \leq 45 \\ 1.60T - 55 & \text{if } v > 45 \end{cases}$$

The WCI value for a still day ($0 \leq v \leq 4$ mph) is just the air temperature. The WCI for wind speeds above 45 mph are the same as the WCI for a wind speed of 45 mph. The WCI for wind speeds between 4 mph and 45 mph decrease as the wind speeds increase.

This WCI function depends on two variables: the temperature and the wind speed. However, if the temperature is constant, then the resulting formula for the WCI will only depend on the speed of the wind. If the air temperature is 30°F ($T = 30$), then the formula for the Wind Chill Index is

$$\text{WCI}_{30} = \begin{cases} 30^\circ & \text{if } 0 \leq v \leq 4 \text{ mph} \\ 62.19 - 18.70\sqrt{v} + 1.25v & \text{if } 4 \text{ mph} < v \leq 45 \text{ mph} \\ -7^\circ & \text{if } v > 45 \text{ mph} \end{cases}$$

The graphs of the the Wind Chill Indices for temperatures of 40°F , 30°F and 20°F appear below:



From UMAP Module 658, "Windchill," by William Bosch and L.G. Cobb, 1984.

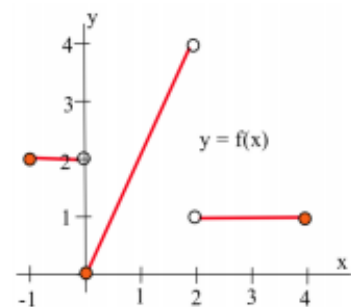
Practice 1. A Hawaiian condo rents for \$380 per night during the tourist season (from December 15 through April 30), and for \$295 per night otherwise. Define a multiline function that describes these rates.

Example 1. Define $f(x)$ by:

$$f(x) = \begin{cases} 2 & \text{if } x < 0 \\ 2x & \text{if } 0 \leq x < 2 \\ 1 & \text{if } 2 < x \end{cases}$$

Evaluate $f(-3)$, $f(0)$, $f(1)$, $f(4)$ and $f(2)$. Graph $y = f(x)$ on the interval $-1 \leq x \leq 4$.

Solution. To evaluate the function at different values of x , we must first decide which line of the rule applies. If $x = -3 < 0$, then we need to use the first line, so $f(-3) = 2$. When $x = 0$ or $x = 1$, we need the second line of the function definition, so $f(0) = 2(0) = 0$ and $f(1) = 2(1) = 2$. At $x = 4$ we need the third line, so $f(4) = 1$. Finally, at $x = 2$, none of the lines apply: the second line requires $x < 2$ and the third line requires $2 < x$, so $f(2)$ is undefined. The graph of $f(x)$ appears in the margin. Note the "hole" above $x = 2$, which indicates $f(2)$ is not defined by the rule for f . ◀



Practice 2. Define $g(x)$ by:

$$g(x) = \begin{cases} x & \text{if } x < -1 \\ 2 & \text{if } -1 \leq x < 1 \\ -x & \text{if } 1 < x \leq 3 \\ 1 & \text{if } 4 < x \end{cases}$$

Graph $y = g(x)$ for $-3 \leq x \leq 6$ and evaluate $g(-3)$, $g(-1)$, $g(0)$, $g(\frac{1}{2})$, $g(1)$, $g(\frac{\pi}{3})$, $g(2)$, $g(3)$, $g(4)$ and $g(5)$.

Practice 3. Write a multiline definition for the function whose graph appears in the margin.

We can think of a multiline function definition as a machine that first examines the input value to decide which line of the function rule to apply:

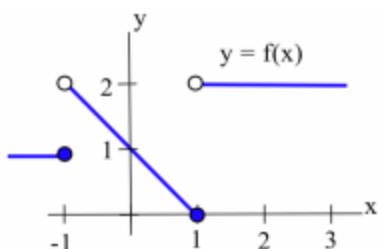


Figure 39: Graph for Practice problem.

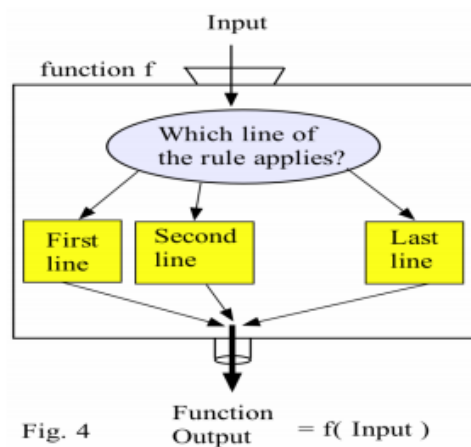


Fig. 4

Composition of Functions: Functions of Functions

Basic functions are often combined with each other to describe more complicated situations. Here we will consider the composition of functions—functions of functions.

The **composite** of two functions f and g , written $f \circ g$, is:

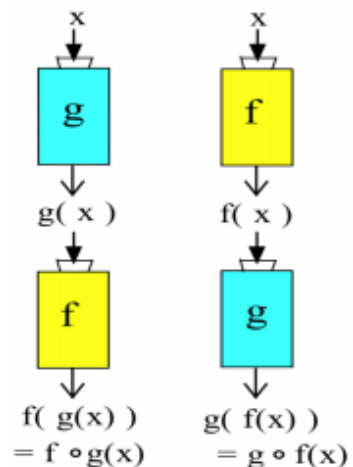
$$f \circ g(x) = f(g(x))$$

The domain of the composite function $f \circ g(x) = f(g(x))$ consists of those x -values for which $g(x)$ and $f(g(x))$ are both defined: we can evaluate the composition of two functions at a point x only if each step in the composition is defined.

If we think of our functions as machines, then composition is simply a new machine consisting of an arrangement of the original machines. The composition $f \circ g$ of the function machines f and g shown in the

margin is an arrangement of the machines so that the original input x goes into machine g , the output from machine g becomes the input into machine f , and the output from machine f is our final output. The composition of the function machines $f \circ g(x) = f(g(x))$ is only valid if x is an allowable input into g (that is, x is in the domain of g) and if $g(x)$ is then an allowable input into f .

The composition $g \circ f$ involves arranging the machines so the original input goes into f , and the output from f then becomes the input for g (see right side of margin figure).



Example 2. For $f(x) = \sqrt{x-2}$, $g(x) = x^2$ and

$$h(x) = \begin{cases} 3x & \text{if } x < 2 \\ x-1 & \text{if } 2 \leq x \end{cases}$$

evaluate $f \circ g(3)$, $g \circ f(6)$, $f \circ h(2)$ and $h \circ g(-3)$. Find the formulas and domains of $f \circ g(x)$ and $g \circ f(x)$.

Solution. $f \circ g(3) = f(g(3)) = f(3^2) = f(9) = \sqrt{9-2} = \sqrt{7} \approx 2.646$; $g \circ f(6) = g(f(6)) = g(\sqrt{6-2}) = g(\sqrt{4}) = g(2) = 2^2 = 4$; $f \circ h(2) = f(h(2)) = f(2-1) = f(1) = \sqrt{1-2} = \sqrt{-1}$, which is undefined; $h \circ g(-3) = h(g(-3)) = h(9) = 9-1 = 8$; $f \circ g(x) = f(g(x)) = f(x^2) = \sqrt{x^2-2}$, and the domain of $f \circ g$ consists of those x -values for which $x^2-2 \geq 0$, so the domain of $f \circ g$ is all x such that $x \geq \sqrt{2}$ or $x \leq -\sqrt{2}$; $g \circ f(x) = g(f(x)) = g(\sqrt{x-2}) = (\sqrt{x-2})^2 = x-2$, but this last equality is true only when $x-2 \geq 0 \Rightarrow x \geq 2$, so the domain of $g \circ f$ is all $x \geq 2$. ◀

Practice 4. For $f(x) = \frac{x}{x-3}$, $g(x) = \sqrt{1+x}$ and

$$h(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ 5-x & \text{if } 1 < x \end{cases}$$

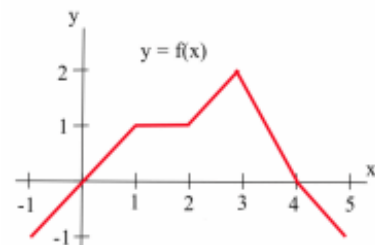
evaluate $f \circ g(3)$, $f \circ g(8)$, $g \circ f(4)$, $f \circ h(1)$, $f \circ h(3)$, $f \circ h(2)$ and $h \circ g(-1)$. Find formulas for $f \circ g(x)$ and $g \circ f(x)$.

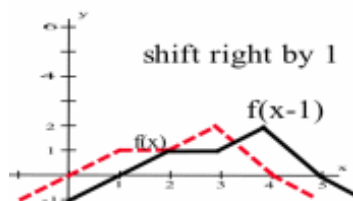
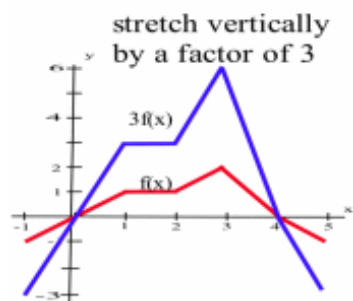
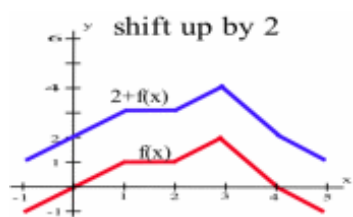
Shifting and Stretching Graphs

Some relatively common compositions are fairly straightforward, and you should recognize the effect of these compositions on the graphs of the functions.

Example 3. The margin figure shows the graph of $y = f(x)$. Graph

- (a) $2 + f(x)$
- (b) $3 \cdot f(x)$
- (c) $f(x-1)$





Solution. All of the new graphs appear in the margin.

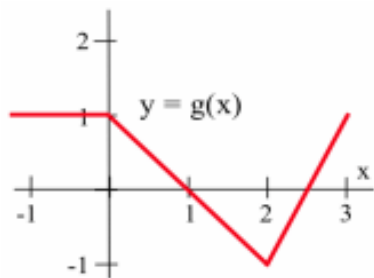
- (a) Adding 2 to all of the values of $f(x)$ rigidly shifts the graph of $f(x)$ upward 2 units.
- (b) Multiplying all of the values of $f(x)$ by 3 leaves all of the roots (zeros) of f fixed: if x is a root of f then $f(x) = 0 \Rightarrow 3 \cdot f(x) = 3(0) = 0$ so x is also a root of $3 \cdot f(x)$. If x is not a root of f , then the graph of $3f(x)$ looks like the graph of $f(x)$ stretched vertically by a factor of 3.
- (c) The graph of $f(x - 1)$ is the graph of $f(x)$ rigidly shifted 1 unit to the right. We could also get these results by examining the graph of $y = f(x)$, creating a table of values for $f(x)$ and the new functions:

x	$f(x)$	$2 + f(x)$	$3f(x)$	$x - 1$	$f(x - 1)$
-1	-1	1	-3	-2	$f(-2)$ not defined
0	0	2	0	-1	$f(0 - 1) = -1$
1	1	3	3	0	$f(1 - 1) = 0$
2	1	3	3	1	$f(2 - 1) = 1$
3	2	4	6	2	$f(3 - 1) = 1$
4	0	2	0	3	$f(4 - 1) = 2$
5	-1	1	-3	4	$f(5 - 1) = 0$

and then graphing those new functions. ◀

If k is a positive constant, then

- the graph of $y = k + f(x)$ will be the graph of $y = f(x)$ rigidly shifted up by k units
- the graph of $y = kf(x)$ will have the same roots as the graph of $f(x)$ and will be the graph of $y = f(x)$ vertically stretched by a factor of k
- the graph of $y = f(x - k)$ will be the graph of $y = f(x)$ rigidly shifted right by k units
- the graph of $y = f(x + k)$ will be the graph of $y = f(x)$ rigidly shifted left by k units



Practice 5. The figure in the margin shows the graph of $g(x)$. Graph:

- (a) $1 + g(x)$
 (b) $2g(x)$
 (c) $g(x - 1)$
 (d) $-3g(x)$

Iteration of Functions

Certain applications feed the output from a function machine back into the same machine as the new input. Each time through the machine is called an **iteration** of the function.

Example 4. Suppose $f(x) = \frac{5}{x} + x$ and we start with the input $x = 4$ and repeatedly feed the output from f back into f . What happens?

Solution. Creating a table:

iteration	input	output
1	4	$f(4) = \frac{5}{4} + 4 = 2.62500000$
2	2.62500000	$f(f(4)) = \frac{5}{2.625} + 2.625 = 2.264880952$
3	2.264880952	$f(f(f(4))) = 2.236251251$
4	2.236251251	2.236067985
5	2.236067985	2.236067977
6	2.236067977	2.236067977

Once we have obtained the output 2.236067977, we will just keep getting the same output (to 9 decimal places). You might recognize this output value as an approximation of $\sqrt{5}$.

This algorithm always finds $\pm\sqrt{5}$. If we start with any positive input, the values will eventually get as close to $\sqrt{5}$ as we want. Starting with any negative value for the input will eventually get us close to $-\sqrt{5}$. We cannot start with $x = 0$, as $\frac{5}{0}$ is undefined. ◀

Practice 6. What happens if we start with the input value $x = 1$ and iterate the function $f(x) = \frac{9}{x} + x$ several times? Do you recognize the resulting number? What do you think will happen to the iterates of $g(x) = \frac{A}{x} + x$? (Try several positive values of A .)

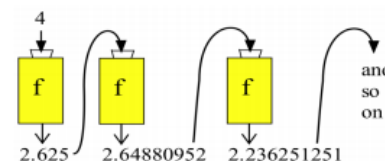
Two Useful Functions: Absolute Value and Greatest Integer

Two functions (one of which should be familiar to you, the other perhaps not) possess useful properties that let us describe situations in which an object abruptly changes direction or jumps from one value to another value. Their graphs will have corners and breaks, respectively.

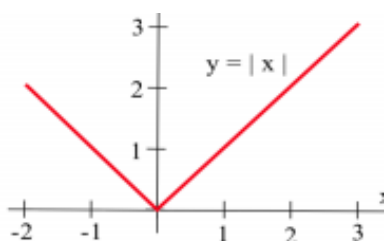
The **absolute value function** evaluated at a number x , $y = f(x) = |x|$, is the distance between the number x and 0.

If x is greater than or equal to 0, then $|x|$ is simply $x - 0 = x$. If x is negative, then $|x|$ is $0 - x = -x = -1 \cdot x$, which is positive because:

$$-1 \cdot (\text{negative number}) = \text{a positive number}$$



Some calculators and computer programming languages represent the absolute value function by $\text{abs}(x)$ or $\text{ABS}(x)$.



Definition of $|x|$:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We can also write: $|x| = \sqrt{x^2}$.

The domain of $y = f(x) = |x|$ consists of all real numbers. The range of $f(x) = |x|$ consists of all numbers larger than or equal to zero (all non-negative numbers). The graph of $y = f(x) = |x|$ (see margin) has no holes or breaks, but it does have a sharp corner at $x = 0$.

The absolute value will be useful for describing phenomena such as reflected light and bouncing balls that change direction abruptly or whose graphs have corners. The absolute value function has a number of properties we will use later.

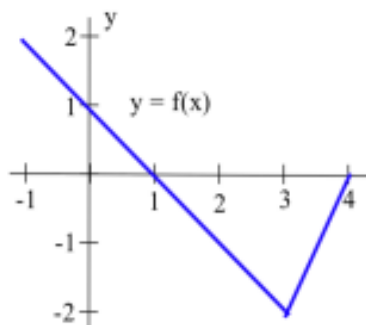
Properties of $|x|$: For all real numbers a and b ,

- $|a| = 0 \cdot |a| = 0$ if and only if $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a + b| \leq |a| + |b|$

Taking the absolute value of a function has an interesting effect on the graph of the function: for any function $f(x)$, we have

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

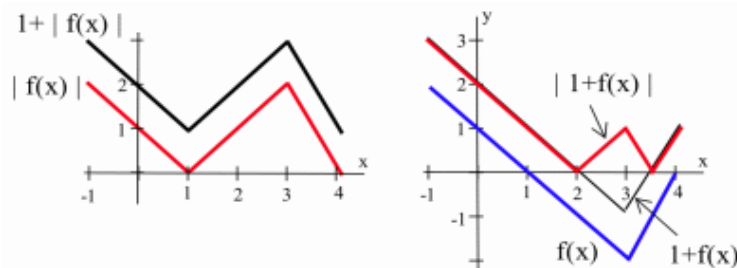
In other words, if $f(x) \geq 0$, then $|f(x)| = f(x)$, so the graph of $|f(x)|$ is the same as the graph of $f(x)$. If $f(x) < 0$, then $|f(x)| = -f(x)$, so the graph of $|f(x)|$ is just the graph of $f(x)$ “flipped” about the x -axis, and it lies above the x -axis. The graph of $|f(x)|$ will always be on or above the x -axis.



Example 5. The figure in the margin shows the graph of $f(x)$. Graph:

- (a) $|f(x)|$
- (b) $|1 + f(x)|$
- (c) $1 + |f(x)|$

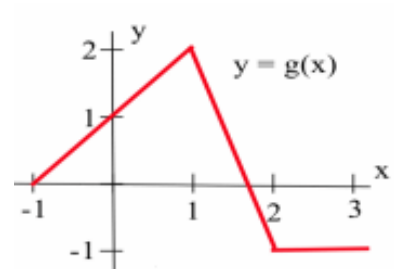
Solution. The graphs appear below:



In (b), we rigidly shift the graph of f up 1 unit before taking the absolute value. In (c), we take the absolute value before rigidly shifting the graph up 1 unit. ◀

Practice 7. The figure in the margin shows the graph of $g(x)$. Graph:

- (a) $|g(x)|$
- (b) $|g(x - 1)|$
- (c) $g(|x|)$



The **greatest integer function** evaluated at a number x , $y = f(x) = \lfloor x \rfloor$, is the largest integer less than or equal to x .

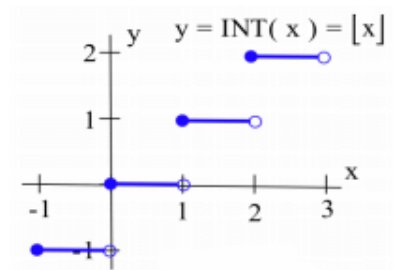
The value of $\lfloor x \rfloor$ is always an integer and $\lfloor x \rfloor$ is always less than or equal to x . For example, $\lfloor 3.2 \rfloor = 3$, $\lfloor 3.9 \rfloor = 3$ and $\lfloor 3 \rfloor = 3$. If x is positive, then $\lfloor x \rfloor$ **truncates** x (drops the fractional part of x). If x is negative, the situation is different: $\lfloor -4.2 \rfloor \neq -4$ because -4 is not less than or equal to -4.2 : $\lfloor -4.2 \rfloor = -5$, $\lfloor -4.7 \rfloor = -5$ and $\lfloor -4 \rfloor = -4$.

Historically, many textbooks have used the square brackets $\lfloor \rfloor$ to represent the greatest integer function, while calculators and many programming languages use $\text{INT}(x)$.

Definition of $\lfloor x \rfloor$:

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{largest integer strictly less than } x & \text{if } x \text{ is not an integer} \end{cases}$$

The domain of $f(x) = \lfloor x \rfloor$ is all real numbers. The range of $f(x) = \lfloor x \rfloor$ is only the integers. The graph of $y = f(x) = \lfloor x \rfloor$ appears in the margin. It has a jump break—a “step”—at each integer value of x , so $f(x) = \lfloor x \rfloor$ is called a **step function**. Between any two consecutive integers, the graph is horizontal with no breaks or holes.

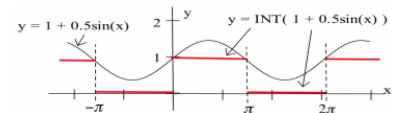


The greatest integer function is useful for describing phenomena that change values abruptly, such as postage rates as a function of weight. As of January 27, 2013, the cost to mail a first-class retail “flat” (such as a manila envelope) was \$0.92 for the first ounce and another \$0.20 for each additional ounce.

The $\lfloor x \rfloor$ function can also be used for functions whose graphs are “square waves,” such as the on and off of a flashing light.

Example 6. Graph $y = \lfloor 1 + 0.5 \sin(x) \rfloor$.

Solution. One way to create this graph is to first graph $y = 1 + 0.5 \sin(x)$, the thin curve in margin figure, and then apply the greatest integer function to y to get the thicker “square wave” pattern. ◀



Practice 8. Sketch the graph of $y = \lfloor x^2 \rfloor$ for $-2 \leq x \leq 2$.

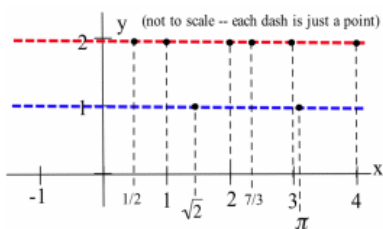
A Really “Holey” Function

The graph of $\lfloor x \rfloor$ has a break or jump at each integer value, but how many breaks can a function have? The next function illustrates just how broken or “holey” the graph of a function can be.

Define a function $h(x)$ as:

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

Then $h(3) = 2$, $h(\frac{5}{3}) = 2$ and $h(-\frac{2}{5}) = 2$, because 3 , $\frac{5}{3}$ and $-\frac{2}{5}$ are all rational numbers. Meanwhile, $h(\pi) = 1$, $h(\sqrt{7}) = 1$, and $h(\sqrt{2}) = 1$, because π , $\sqrt{7}$ and $\sqrt{2}$ are all irrational numbers. These and some other points are plotted in margin figure.



In order to analyze the behavior of $h(x)$ the following fact about rational and irrational numbers is useful.

Fact: Every interval contains both rational and irrational numbers.

Equivalently: If a and b are real numbers and $a < b$, then there is

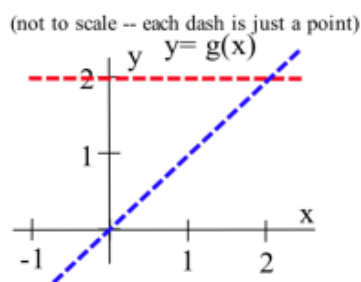
- a rational number R between a and b ($a < R < b$)
- an irrational number I between a and b ($a < I < b$).

The above fact tells us that between any two places where $y = h(x) = 1$ (because x is rational) there is a place where $y = h(x)$ is 2, because there is an irrational number between any two distinct rational numbers. Similarly, between any two places where $y = h(x) = 2$ (because x is irrational) there is a place where $y = h(x) = 1$, because there is a rational number between any two distinct irrational numbers.

The graph of $y = h(x)$ is impossible to actually draw, because every two points on the graph are separated by a hole. This is also an example of a function that your computer or calculator cannot graph, because in general it can not determine whether an input value of x is irrational.

Example 7. Sketch the graph of

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ x & \text{if } x \text{ is an irrational number} \end{cases}$$



Solution. A sketch of the graph of $y = g(x)$ appears in the margin. When x is rational, the graph of $y = g(x)$ looks like the “holey” horizontal line $y = 2$. When x is irrational, the graph of $y = g(x)$ looks like the “holey” line $y = x$. ◀

Practice 9. Sketch the graph of

$$h(x) = \begin{cases} \sin(x) & \text{if } x \text{ is a rational number} \\ x & \text{if } x \text{ is an irrational number} \end{cases}$$

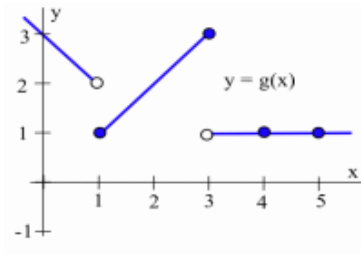
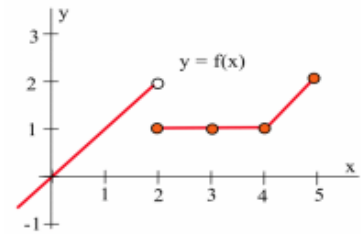
Problems

1. If T is the Celsius temperature of the air and v is the speed of the wind in kilometers per hour, then

$$\text{WCI} = \begin{cases} T & \text{if } 0 \leq v \leq 6.5 \\ 33 - \frac{10.45 + 5.29\sqrt{v} - 0.279v}{22} (33 - T) & \text{if } 6.5 < v \leq 72 \\ 1.6T - 19.8 & \text{if } v > 72 \end{cases}$$

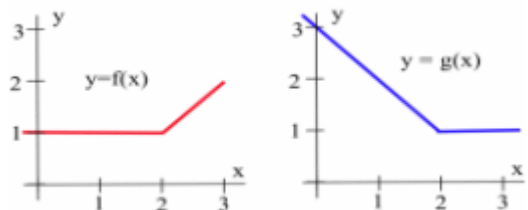
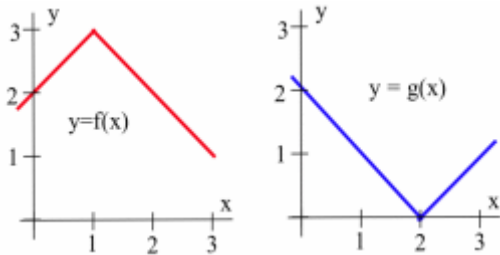
- (a) Determine the Wind Chill Index for
- a temperature of 0°C and a wind speed of 49 km/hr
 - a temperature of 11°C and a wind speed of 80 km/hr.
- (b) Write a multiline function definition for the WCI if the temperature is 11°C .

2. Use the graph of $y = f(x)$ in the margin to evaluate $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$. Write a multiline function definition for f .
3. Use the graph of $y = g(x)$ in the margin to evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$ and $g(5)$. Write a multiline function definition for g .
4. Use the values given in the table below, along with $h(x) = 2x + 1$, to determine the missing values of $f \circ g$, $g \circ f$ and $h \circ g$.



x	$f(x)$	$g(x)$	$f \circ g(x)$	$g \circ f(x)$	$h \circ g(x)$
-1	2	0			
0	1	2			
1	-1	1			
2	0	2			

5. Use the graphs shown below and the function $h(x) = x - 2$ to determine the values of
- $f(f(1))$, $f(g(2))$, $f(g(0))$, $f(g(1))$
 - $g(f(2))$, $g(f(3))$, $g(g(0))$, $g(f(0))$
 - $f(h(3))$, $f(h(4))$, $h(g(0))$, $h(g(1))$
6. Use the graphs shown below and the function $h(x) = 5 - 2x$ to determine the values of
- $h(f(0))$, $f(h(1))$, $f(g(2))$, $f(f(3))$
 - $g(f(0))$, $g(f(1))$, $g(h(2))$, $h(f(3))$
 - $f(g(0))$, $f(g(1))$, $f(h(2))$, $h(g(3))$



7. Defining
- $h(x) = x - 2$
- ,
- $f(x)$
- as:

$$f(x) = \begin{cases} 3 & \text{if } x < 1 \\ x - 2 & \text{if } 1 \leq x < 3 \\ 1 & \text{if } 3 \leq x \end{cases}$$

and $g(x)$ as:

$$g(x) = \begin{cases} x^2 - 3 & \text{if } x < 0 \\ \lfloor x \rfloor & \text{if } 0 \leq x \end{cases}$$

- (a) evaluate $f(x)$, $g(x)$ and $h(x)$ for $x = -1, 0, 1, 2, 3$ and 4 .
- (b) evaluate $f(g(1))$, $f(h(1))$, $h(f(1))$, $f(f(2))$, $g(g(3.5))$.
- (c) graph $f(x)$, $g(x)$ and $h(x)$ for $-5 \leq x \leq 5$.
8. Defining $h(x) = 3$, $f(x)$ as:

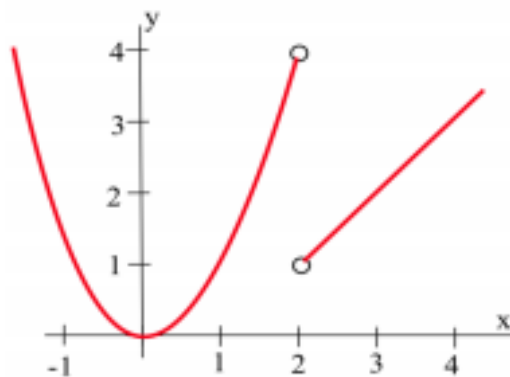
$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ 1 & \text{if } 1 \leq x < 3 \\ 2 - x & \text{if } 3 \leq x \end{cases}$$

and $g(x)$ as:

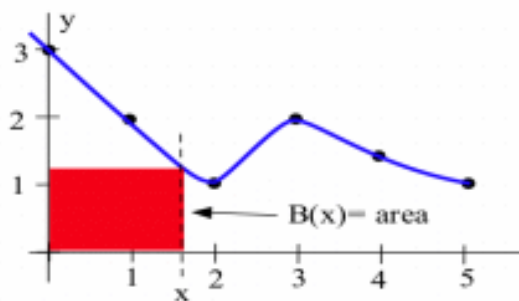
$$g(x) = \begin{cases} |x + 1| & \text{if } x < 0 \\ 2x & \text{if } 0 \leq x \end{cases}$$

- (a) evaluate $f(x)$, $g(x)$ and $h(x)$ for $x = -1, 0, 1, 2, 3$ and 4 .
- (b) evaluate $f(g(1))$, $f(h(1))$, $h(f(1))$, $f(f(2))$, $g(g(3.5))$.
- (c) graph $f(x)$, $g(x)$ and $h(x)$ for $-5 \leq x \leq 5$.
9. You are planning to take a one-week vacation in Europe, and the tour brochure says that Monday and Tuesday will be spent in England, Wednesday in France, Thursday and Friday in Germany, and Saturday and Sunday in Italy. Let $L(d)$ be the location of the tour group on day d and write a multiline function definition for $L(d)$.
10. A state has just adopted the following state income tax system: no tax on the first \$10,000 earned, 1% of the next \$10,000 earned, 2% of the next \$20,000 earned, and 3% of all additional earnings. Write a multiline function definition for $T(x)$, the state income tax due on earnings of x dollars.

11. Write a multiline function definition for the curve
- $y = f(x)$
- shown below.



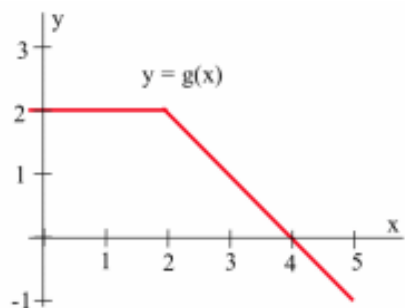
12. Define
- $B(x)$
- to be the area of the rectangle whose lower left corner is at the origin and whose upper right corner is at the point
- $(x, f(x))$
- for the function
- f
- shown below. For example,
- $B(3) = 6$
- . Evaluate
- $B(1)$
- ,
- $B(2)$
- ,
- $B(4)$
- and
- $B(5)$
- .



13. Define $B(x)$ to be the area of the rectangle whose lower left corner is at the origin and whose upper right corner is at the point $(x, \frac{1}{x})$.
- (a) Evaluate $B(1)$, $B(2)$ and $B(3)$.
- (b) Show that $B(x) = 1$ for all $x > 0$.
14. For $f(x) = |9 - x|$ and $g(x) = x - 1$:
- (a) evaluate $f \circ g(1)$, $f \circ g(3)$, $f \circ g(5)$, $f \circ g(7)$, $f \circ g(0)$.
- (b) evaluate $f \circ f(2)$, $f \circ f(5)$, $f \circ f(-2)$.
- (c) Does $f \circ f(x) = |x|$ for all values of x ?

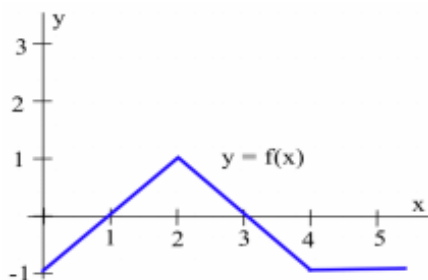
15. The function $g(x)$ is graphed below. Graph

- (a) $g(x) - 1$
- (b) $g(x - 1)$
- (c) $|g(x)|$
- (d) $\lfloor g(x) \rfloor$



16. The function $f(x)$ is graphed below. Graph

- (a) $f(x) - 2$
- (b) $f(x - 2)$
- (c) $|f(x)|$
- (d) $\lfloor f(x) \rfloor$



17. Find A and B so that $f(g(x)) = g(f(x))$ when:

- (a) $f(x) = 3x + 2$ and $g(x) = 2x + A$
- (b) $f(x) = 3x + 2$ and $g(x) = Bx - 1$

18. Find C and D so that $f(g(x)) = g(f(x))$ when:

- (a) $f(x) = Cx + 3$ and $g(x) = Cx - 1$
- (b) $f(x) = 2x + D$ and $g(x) = 3x + D$

19. Graph $y = f(x) = x - \lfloor x \rfloor$ for $-1 \leq x \leq 3$. This function is called the “fractional part of x ” and its graph an example of a “sawtooth” graph.

20. The function $f(x) = \lfloor x + 0.5 \rfloor$ rounds off x to the **nearest integer**, while $g(x) = \frac{\lfloor 10x + 0.5 \rfloor}{10}$ rounds off x to the nearest tenth (the first decimal place). What function will round off x to:

- (a) the nearest hundredth (two decimal places)?
- (b) the nearest thousandth (three decimal places)?

21. Modify the function in Example 6 to produce a “square wave” graph with a “long on, short off, long on, short off” pattern.

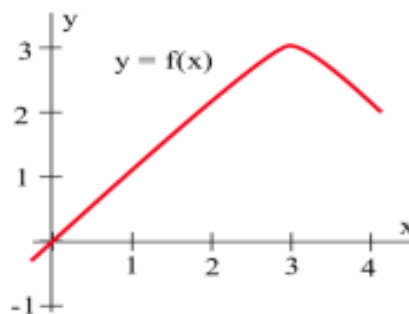
22. Many computer languages contain a “signum” or “sign” function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

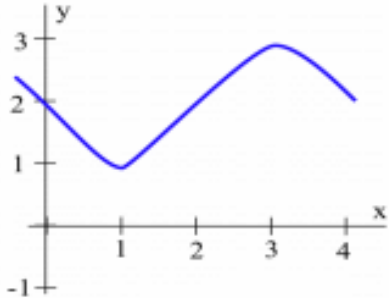
- (a) Graph $\operatorname{sgn}(x)$.
- (b) Graph $\operatorname{sgn}(x - 2)$.
- (c) Graph $\operatorname{sgn}(x - 4)$.
- (d) Graph $\operatorname{sgn}(x - 2) \cdot \operatorname{sgn}(x - 4)$.
- (e) Graph $1 - \operatorname{sgn}(x - 2) \cdot \operatorname{sgn}(x - 4)$.
- (f) For real numbers a and b with $a < b$, describe the graph of $1 - \operatorname{sgn}(x - a) \cdot \operatorname{sgn}(x - b)$.

23. Define $g(x)$ to be the slope of the line tangent to the graph of $y = f(x)$ (shown below) at (x, y) .

- (a) Estimate $g(1)$, $g(2)$, $g(3)$ and $g(4)$.
- (b) Graph $y = g(x)$ for $0 \leq x \leq 4$.



24. Define $h(x)$ to be the slope of the line tangent to the graph of $y = f(x)$ (see figure below) at (x, y) .
- (a) Estimate $h(1)$, $h(2)$, $h(3)$ and $h(4)$.
- (b) Graph $y = h(x)$ for $0 \leq x \leq 4$.

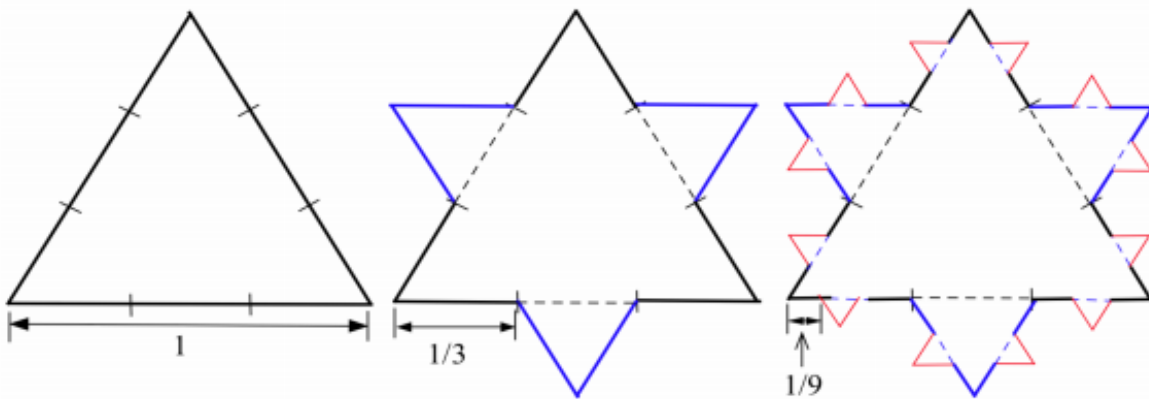


25. Using the **cos** (cosine) button on your calculator several times produces iterates of $f(x) = \cos(x)$. What number will the iterates approach if you use the **cos** button 20 or 30 times starting with
- (a) $x = 1$?
- (b) $x = 2$?
- (c) $x = 10$?
- (Be sure your calculator is in radian mode.)

30. Some iterative procedures are geometric rather than numerical. Start with an equilateral triangle with sides of length 1, as shown at left in the figure below.

- Remove the middle third of each line segment.
- Replace the removed portion with two segments with the same length as the removed segment.

The first two iterations of this procedure are shown at center and right in the figure below. Repeat these steps several more times, each time removing the middle third of each line segment and replacing it with two new segments. What happens to the length of the shape with each iteration? (The result of iterating over and over with this procedure is called Koch's Snowflake, named for Helga von Koch.)



26. Let $f(x) = 1 + \sin(x)$.

- (a) What happens if you start with $x = 1$ and repeatedly feed the output from f back into f ?
- (b) What happens if you start with $x = 2$ and examine the iterates of f ?

(Be sure your calculator is in radian mode.)

27. Starting with $x = 1$, do the iterates of $f(x) = \frac{x^2+1}{2x}$ approach a number? What happens if you start with $x = 0.5$ or $x = 4$?

28. Let $f(x) = \frac{x}{2} + 3$.

- (a) What are the iterates of f if you start with $x = 2$? $x = 4$? $x = 6$?
- (b) Find a number c so that $f(c) = c$. This value of c is called a **fixed point** of f .
- (c) Find a fixed point of $g(x) = \frac{x}{2} + A$.

29. Let $f(x) = \frac{x}{3} + 4$.

- (a) What are the iterates of f if you start with $x = 2$? $x = 4$? $x = 6$?
- (b) Find a number c so that $f(c) = c$.
- (c) Find a fixed point of $g(x) = \frac{x}{3} + A$.

31. Sketch the graph of

$$p(x) = \begin{cases} 3 - x & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

32. Sketch the graph of

$$q(x) = \begin{cases} x^2 & \text{if } x \text{ is a rational number} \\ x + 11 & \text{if } x \text{ is an irrational number} \end{cases}$$

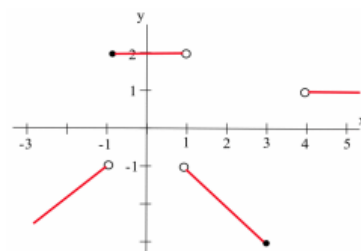
0.4 Practice Answers

1. $C(x)$ is the cost for one night on date x :

$$C(x) = \begin{cases} \$380 & \text{if } x \text{ is between December 15 and April 30} \\ \$295 & \text{if } x \text{ is any other date} \end{cases}$$

2. For the graph, see the figure in the margin.

x	$g(x)$	x	$g(x)$
-3	-3	$\frac{\pi}{3}$	$-\frac{\pi}{3}$
-1	2	2	-2
0	2	3	-3
$\frac{1}{2}$	2	4	undefined
1	undefined	5	1

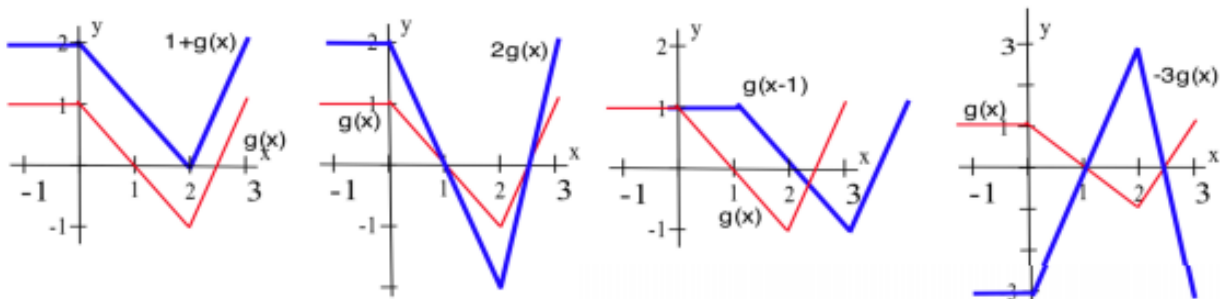


3. Define $f(x)$ as:

$$f(x) = \begin{cases} 1 & \text{if } x \leq -1 \\ 1 - x & \text{if } -1 < x \leq 1 \\ 2 & \text{if } 1 < x \end{cases}$$

4. $f \circ g(3) = f(2) = \frac{2}{-1} = -2$; $f \circ g(8) = f(3)$ is undefined; $g \circ f(4) = g(4) = 5$; $f \circ h(1) = f(2) = \frac{2}{-1} = -2$; $f \circ h(3) = f(2) = -2$; $f \circ h(2) = f(3)$ is undefined; $h \circ g(-1) = h(0) = 0$; $f \circ g(x) = f(\sqrt{1+x}) = \frac{1+x}{\sqrt{1+x}-3}$, $g \circ f(x) = g(\frac{x}{x-3}) = \sqrt{1 + \frac{x}{x-3}}$

5. See the figure below:

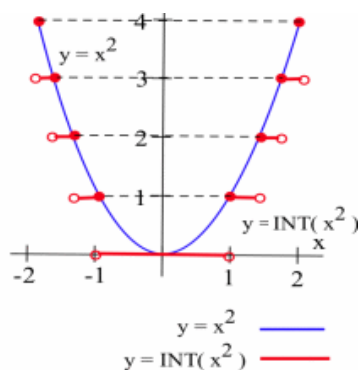
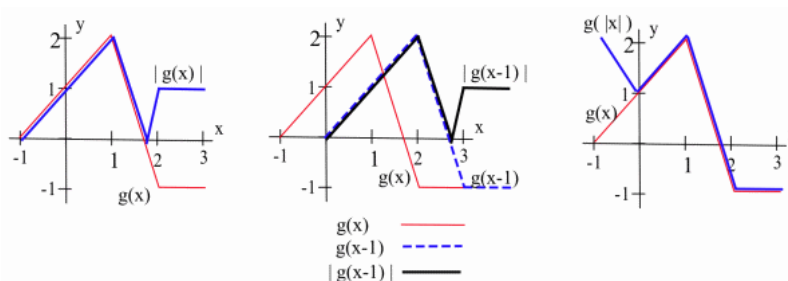


6. Using $f(x) = \frac{9}{x} + x$, $f(1) = \frac{9}{1} + 1 = 10$, $f(5) = \frac{9}{5} + 5 = 6.8$, $f(3.4) \approx 3.023529412$ and $f(3.023529412) \approx 3.000091554$. The next iteration gives $f(3.000091554) \approx 3.000000001$: these values are approaching 3, the square root of 9.

With $A = 6$, $f(x) = \frac{6}{x} + x$, so $f(1) = \frac{6}{1} + 1 = 7$, $f(3.5) = \frac{6}{3.5} + 3.5 \approx 2.607142857$, and the next iteration gives $f(2.607142857) \approx 2.45425636$. Then $f(2.45425636) \approx 2.449494372$, $f(2.449494372) \approx 2.449489743$ and $f(2.449489743) \approx 2.449489743$ (the output is the same as the input to 9 decimal places): these values are approaching 2.449489743, an approximation of $\sqrt{6}$.

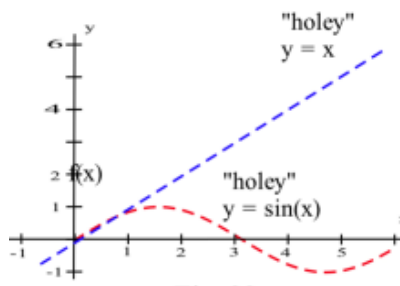
For any positive value A , the iterates of $f(x) = \frac{A}{x} + x$ (starting with any positive x) will approach \sqrt{A} .

7. The figure below shows some intermediate steps and final graphs:



8. The figure in the margin shows the graph of $y = x^2$ and the (thicker) graph of $y = \lfloor x^2 \rfloor$.

9. The figure below shows the “holey” graph of $y = x$ with a hole at each rational value of x and the “holey” graph of $y = \sin(x)$ with a hole at each irrational value of x . Together they form the graph of $r(x)$.



(This is a very crude image, since we can't really see the individual holes, which have zero width.)

0.5 Mathematical Language

The calculus concepts we will explore in this book are simple and powerful, but sometimes subtle. To succeed in calculus you will need to master some techniques, but (more importantly) you will need to understand ideas and be able to work with the ideas in words and pictures—very clear words and pictures.

You also need to understand some of the common linguistic constructions used in mathematics. In this section, we will discuss a few of the most common mathematical phrases, the meanings of these phrases, and some of their equivalent forms.

Your calculus instructor is going to use these types of statements, and it is very important that you understand exactly what your instructor means. You have reached a level in mathematics where the precise use of language is important.

Equivalent Statements

Two statements are **equivalent** if they always have the same logical value (a logical value is either “true” or “false”): that is, they are both true or are both false.

The statements “ $x = 3$ ” and “ $x + 2 = 5$ ” are equivalent statements, because if one of them is true then so is the other—and if one of them is false then so is the other.

The statements “ $x = 3$ ” and “ $x^2 - 4x + 3 = 0$ ” are not equivalent, because $x = 1$ makes the second statement true but the first one false.

AND and OR

In everyday language, we use the words “and” and “or” all the time, but in mathematics we must use them very carefully.

The compound statement “ A **and** B are true” is equivalent to “both of A and B are true.”

If A or if B or if both are false, then the statement “ A and B are true” is false. The statement “ $x^2 = 4$ and $x > 0$ ” is true when $x = 2$ and is false for every other value of x .

The compound statement “ A **or** B is true” is equivalent to “at least one of A or B is true.”

If both A and B are false, then the statement “ A or B is true” is false.

The statement " $x^2 = 4$ or $x > 0$ " is true if $x = -2$ or x is any positive number. The statement is false when $x = -3$ (and for lots of other values of x).

Practice 1. Which values of x make each statement true?

- (a) $x < 5$
- (b) $x + 2 = 6$
- (c) $x^2 - 10x + 24 = 0$
- (d) "(a) and (b)"
- (e) "(a) or (c)"

Negation of a Statement

For some simple statements, we can construct the **negation** just by adding the word "not."

statement	negation of the statement
x is equal to 3 ($x = 3$)	x is not equal to 3 ($x \neq 3$)
x is less than 5 ($x < 5$)	x is not less than 5 ($x \not< 5$) x is greater than or equal to 5 ($x \geq 5$)

When the statement contains words such as "all," "no" or "some," then the negation becomes more complicated.

statement	negation of the statement
All x satisfy A . Every x satisfies A .	At least one x does not satisfy A . There is an x that does not satisfy A . Some x does not satisfy A .
No x satisfies A . Every x does not satisfy A .	At least one x satisfies A . Some x satisfies A .
There is an x that satisfies A . At least one x satisfies A . Some x satisfies A .	No x satisfies A . Every x does not satisfy A .

We can also negate compound statements containing "and" and "or."

statement	negation of the statement
A and B are both true.	At least one of A or B is not true.
A and B and C are all true.	At least one of A or B or C is not true.
A or B is true.	Both A and B are not true.

Practice 2. Write the negation of each statement.

- (a) $x + 5 = 3$
- (b) All prime numbers are odd.
- (c) $x^2 < 4$
- (d) x divides 2 and x divides 3.
- (e) No mathematician can sing well.

If...Then...: A Very Common Structure in Mathematics

The most common and basic structure used in mathematical language is the “If {some hypothesis} then {some conclusion}” sentence. Almost every result in mathematics can be stated using one or more “If...then...” sentences.

“If A then B ” means that when the hypothesis A is true, the conclusion B must also be true.

When the hypothesis is false, the “If...then...” sentence makes no claim about the truth or falsity of the conclusion—the conclusion may be either true or false.

Even in everyday life you have probably encountered “If...then...” statements for a long time. A parent might try to encourage a child with a statement like “If you clean your room then I will buy you an ice cream cone.”

To show that an “If...then...” statement is not valid (not true), all we need to do is find a single example where the hypothesis is true and the conclusion is false. Such an example with a true hypothesis and false conclusion is called a **counterexample** for the “If...then...” statement. A valid “If...then...” statement has no counterexample.

A **counterexample** to the statement “If A then B ” is an example in which A is true and B is false.

The only way for the statement “If you clean your room then I will buy you an ice cream cone” to be false is if the child cleaned the room and the parent did not buy the ice cream cone. If the child did not clean the room but the parent bought the ice cream cone anyway, we would say that the statement was true.

The statement “If n is a positive integer, then $n^2 + 5n + 5$ is a prime number” has hypothesis “ n is a positive integer” and conclusion “ $n^2 + 5n + 5$ is a prime number.” This “If...then...” statement is false, because replacing n with the number 5 will make the hypothesis true

and the conclusion false. The number 5 is a counterexample for the statement.

Every invalid “If . . . then . . .” statement has at least one counterexample, and the most convincing way to show that a statement is not valid is to find a counterexample to the statement.

Several other language structures are equivalent to the “If . . . then . . .” form. The statements below all mean the same as “If $\{A\}$ then $\{B\}$ ”:

- “All A are B .”
- “Every A is B .”
- “Each A is B .”
- “Whenever A , then B .”
- “ B whenever A .”
- “ A only if B .”
- “ A implies B .”
- “ $A \Rightarrow B$ ” (the symbol “ \Rightarrow ” means “implies”)

Practice 3. Restate “If {a shape is a square} then {the shape is a rectangle}” as many ways as you can.

“If . . . then . . .” statements occur hundreds of times in every mathematics book, including this one. It is important that you are able to recognize the various forms of “If . . . then . . .” statements and that you are able to distinguish the hypotheses from the conclusions.

Contrapositive Form of an “If . . . Then . . .” Statement

The statement “If A then B ” means that if the hypothesis A is true, then the conclusion B is guaranteed to be true.

Suppose we know that in a certain town the statement “If {a building is a church} then {the building is green}” is a true statement. What can we validly conclude about a red building? Based on the information we have, we can validly conclude that the red building is “not a church.” because every church is green. We can also conclude that a blue building is not a church. In fact, we can conclude that every “not green” building is “not a church.” That is, if the conclusion of a valid “If . . . then . . .” statement is false, then the hypothesis must also be false.

The **contrapositive** form of “If A then B ” is
 “If {negation of B } then {negation of A }”
 or “If $\{B$ is false} then $\{A$ is false}.”

The statement “If A then B ” and its contrapositive
“If {not B } then {not A }” are equivalent.

What about a green building in the aforementioned town? The green building may or may not be a church—perhaps every post office is also painted green. Or perhaps every building in town is green, in which case the statement “If {a building is a church} then {the building is green}” is certainly true.

Practice 4. Write the contrapositive form of each statement.

- (a) If a function is differentiable then it is continuous.
- (b) All men are mortal.
- (c) If $x = 3$ then $x^2 - 5x + 6 = 0$
- (d) If {2 divides x and 3 divides x } then {6 divides x }.

Converse of an “If... then...” Statement

If we switch the hypothesis and the conclusion of an “If A then B ” statement, we get the **converse** “If B then A .” For example, the converse of “If {a building is a church} then {the building is green}” is “If {a building is green} then {the building is a church}.”

The converse of an “If... then...” statement is **not equivalent** to the original “If... then...” statement. For example, the statement “If $x = 2$ then $x^2 = 4$ ” is true, but the converse statement “If $x^2 = 4$ then $x = 2$ ” is not true because $x = -2$ makes the hypothesis of the converse true and the conclusion false

The **converse** of “If A then B ” is “If B then A .”

The statement “If A then B ” and its converse “If B then A .”
are **not equivalent**.

Wrap-up

The precise use of language by mathematicians (and mathematics books) is an attempt to clearly communicate ideas from one person to another, but this requires that both people understand the use and rules of the language. If you don’t understand this usage, the communication of the ideas will almost certainly fail.

Problems

In problems 1 and 2, define the sets A , B and C as $A = \{1, 2, 3, 4, 5\}$, $B = \{0, 2, 4, 6\}$ and $C = \{-2, -1, 0, 1, 2, 3\}$. List all values of x that satisfy each statement.

1. (a) x is in A and x is in B
 (b) x is in A or x is in C
 (c) x is not in B and x is in C
2. (a) x is not in B or C
 (b) x is in B and C but not in A
 (c) x is not in A but is in B or C

In problems 3–5, list or describe all the values of x that make each statement true.

3. (a) $x^2 + 3 > 1$
 (b) $x^3 + 3 > 1$
 (c) $\lfloor x \rfloor = |x|$
4. (a) $\frac{x^2+3x}{x} = x + 3$
 (b) $x > 4$ and $x < 9$
 (c) $|x| = 3$ and $x < 0$
5. (a) $x + 5 = 3$ or $x^2 = 9$
 (b) $x + 5 = 3$ and $x^2 = 9$
 (c) $|x + 3| = |x| + 3$

In problems 6–8, write the contrapositive of each statement. If the statement is false, give a counterexample.

6. (a) If $x > 3$ then $x^2 > 9$.
 (b) Every solution of $x^2 - 6x + 8 = 0$ is even.
7. (a) If $x^2 + x - 6 = 0$ then $x = 2$ or $x = -3$.
 (b) All triangles have 3 sides.
8. (a) Every polynomial has at least one zero.
 (b) If I exercise and eat right then I will be healthy.

In problems 9–11, write the contrapositive of each statement.

9. (a) If your car is properly tuned, it will get at least 24 miles per gallon.
 (b) You can have pie if you eat your vegetables.
10. (a) A well-prepared student will miss less than 15 points on the final exam.
 (b) I feel good when I jog.

11. (a) If you love your country, you will vote for me.
 (b) If guns are outlawed then only outlaws will have guns.

In 12–15, write the negation of each statement.

12. (a) It is raining.
 (b) Some equations have solutions.
 (c) $f(x)$ and $g(x)$ are polynomials.
13. (a) $f(x)$ or $g(x)$ is positive.
 (b) x is positive.
 (c) 8 is a prime number.
14. (a) Some months have six Mondays.
 (b) All quadratic equations have solutions.
 (c) The absolute value of a number is positive.
15. (a) For all numbers a and b , $|a + b| = |a| + |b|$.
 (b) All snakes are poisonous.
 (c) No dog can climb trees.
16. Write an “If... then...” statement that is true but whose converse is false.
17. Write an “If... then...” statement that is true and whose converse is true.
18. Write an “If... then...” statement that is false and whose converse is false.

In 19–22, determine whether each statement is true or false. If the statement is false, give a counterexample.

19. (a) If a and b are real numbers then:

$$(a + b)^2 = a^2 + b^2$$
 (b) If $a > b$ then $a^2 > b^2$.
 (c) If $a > b$ then $a^3 > b^3$.
20. (a) For all real numbers a and b , $|a + b| = |a| + |b|$
 (b) For all real numbers a and b , $\lfloor a \rfloor + \lfloor b \rfloor = \lfloor a + b \rfloor$.
 (c) If $f(x)$ and $g(x)$ are linear functions, then $f(g(x))$ is a linear function.
21. (a) If $f(x)$ and $g(x)$ are linear functions then $f(x) + g(x)$ is a linear function.
 (b) If $f(x)$ and $g(x)$ are linear functions then $f(x) \cdot g(x)$ is a linear function.
 (c) If x divides 6 then x divides 30.

22. (a) If x divides 50 then x divides 10.
 (b) If x divides yz then x divides y or z .
 (c) If x divides a^2 then x divides a .
- In 23–26, rewrite each statement in the form of an “If... then...” statement and determine whether it is true or false. If the statement is false, give a counterexample.
23. (a) The sum of two prime numbers is a prime.
 (b) The sum of two prime numbers is never a prime number.
 (c) Every prime number is odd.
 (d) Every prime number is even.
24. (a) Every square has 4 sides.
 (b) All 4-sided polygons are squares.
 (c) Every triangle has 2 equal sides.
 (d) Every 4-sided polygon with equal sides is a square.
25. (a) Every solution of $x + 5 = 9$ is odd.
 (b) Every 3-sided polygon with equal sides is a triangle.
 (c) Every calculus student studies hard.
 (d) All (real) solutions of $x^2 - 5x + 6 = 0$ are even.
26. (a) Every line in the plane intersects the x -axis.
 (b) Every (real) solution of $x^2 + 3 = 0$ is even.
 (c) All birds can fly.
 (d) No mammal can fly.

0.5 Practice Answers

1. (a) All values of x less than 5. (b) $x = 4$ (c) Both $x = 4$ and $x = 6$.
 (d) $x = 4$ (e) $x = 6$ and all x less than 5.
2. (a) $x + 5 < 3$ (b) At least one prime number is even. (There is an even prime number.) (c) $x^2 = 4$ (d) x does not divide 2 or x does not divide 3. (e) At least one mathematician can sing well. (There is a mathematician who can sing well.)
3. Here are several ways:
- All squares are rectangles.
 - Every square is a rectangle.
 - Each square is a rectangle.
 - Whenever a shape is a square, then it is a rectangle.
 - A shape is a rectangle whenever it is a square.
 - A shape is a square only if it is a rectangle.
 - A shape is a square implies that it is a rectangle.
 - Being a square implies being a rectangle.
4. (a) If a function is not continuous then it is not differentiable.
 (b) All immortals are not men.
 (c) $x^2 - 5x + 6 \neq 0 \Rightarrow x \neq 3$
 (d) If {6 does not divide x } then {2 does not divide x or 3 does not divide x }.