

# Business Precalculus

Edition 0.1

David Lippman

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## About the Author



David Lippman received his master's degree in mathematics from Western Washington University and has been teaching at Pierce College since Fall 2000.

David has been long time advocate of open learning, open materials, and basically any idea that will reduce the cost of education for students. It started by supporting the college's calculator rental program, and running a book loan scholarship program. Eventually the frustration with the escalating costs of commercial text books and the online homework systems that charged for access led him and his colleagues to take action.

First, David developed IMathAS, open source online math homework software that runs WAMAP.org and MyOpenMath.com. Through this platform, he became an integral part of a vibrant sharing and learning community of teachers from around Washington State that support and contribute to WAMAP. Those pioneering efforts, supported by dozens of other dedicated faculty and financial support from the Transition Math Project, led to a system used by thousands of students every quarter, saving hundreds of thousands of dollars over comparable commercial offerings.

David continued further and wrote his first open textbook, *Math in Society*, a math for liberal arts majors book, after being frustrated by students having to pay \$100+ for a textbook for a terminal course. Frustrated by both cost and the style of commercial texts, David and colleague Melonie Rasmussen began writing *PreCalculus: An Investigation of Functions* in 2010.

This text pulls in portions from *Precalculus*, *Math in Society*, and other open textbooks, and adapts them to have the business focus desired for a business precalculus or finite math course. This text serves as preparation for *Applied Calculus*, a business-focused brief calculus text coauthored by Shana Calaway, Dale Hoffman, and David.

## Supplements

Online homework and video lesson sets to accompany this text are available on MyOpenMath.com.

## **Table of Contents**

<b>Chapter 1: Functions and Lines .....</b>	<b>1</b>
Section 1.1 Functions and Function Notation.....	1
Section 1.2 Domain and Range.....	10
Section 1.3 Rates of Change and Behavior of Graphs.....	19
Section 1.4 Linear Functions .....	27
Section 1.5 Graphs of Linear Functions .....	34
Section 1.6 Modeling with Linear Functions.....	45
Section 1.7 Fitting Linear Models to Data.....	53
<b>Chapter 2: Systems of Equations and Matrices .....</b>	<b>59</b>
Section 2.1 Systems of Equations.....	59
Section 2.2 Solving Systems using Matrices .....	76
Section 2.3 Matrix Operations .....	90
Section 2.4 Solving Systems with Inverses .....	101
<b>Chapter 3: Linear Programming .....</b>	<b>115</b>
Section 3.1 Inequalities in One Variable .....	115
Section 3.2 Linear Inequalities .....	125
Section 3.3 Graphical Solutions.....	131
Section 3.4 Simplex Method.....	140
Section 3.5 Applications of Linear Programming .....	146
<b>Chapter 4: Polynomial and Rational Functions.....</b>	<b>153</b>
Section 4.1 Quadratic Functions.....	153
Section 4.2 Polynomial Functions .....	164
Section 4.3 Rational Functions .....	175
<b>Chapter 5: Exponential and Logarithmic Functions.....</b>	<b>185</b>
Section 5.1 Exponential Functions .....	185
Section 5.2 Logarithmic Functions.....	199
Section 5.3 Exponential and Logarithmic Models.....	211
<b>Chapter 6: Finance .....</b>	<b>221</b>
Section 6.1 Simple and Compound Interest.....	221
Section 6.2 Annuities .....	230
Section 6.3 Payout Annuities.....	235
Section 6.4 Loans.....	239
Section 6.5 Multistage Finance Problems.....	246
<b>Chapter 7: Sets .....</b>	<b>253</b>
Section 7.1 Sets.....	253
Section 7.2 Venn Diagrams and Cardinality .....	258
<b>Chapter 8: Probability.....</b>	<b>265</b>
Section 8.1 Concepts of Probability .....	265
Section 8.2 Conditional Probability and Bayes Theorem.....	275
Section 8.3 Counting.....	283
Section 8.4 Expected Value .....	296

## Chapter 1: Functions and Lines

Section 1.1 Functions and Function Notation.....	1
Section 1.2 Domain and Range.....	10
Section 1.3 Rates of Change and Behavior of Graphs.....	19
Section 1.4 Linear Functions .....	27
Section 1.5 Graphs of Linear Functions .....	34
Section 1.6 Modeling with Linear Functions.....	45
Section 1.7 Fitting Linear Models to Data.....	53

### Section 1.1 Functions and Function Notation

#### What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask “If I know one quantity, can I then determine the other?” This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

#### Function

**Function:** A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say “the output is a function of the input.”

#### Example 1

In the height and age example above, is height a function of age? Is age a function of height?

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18<sup>th</sup> birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 2

At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.

Example 3

In many classes the overall percentage you earn in the course corresponds to a decimal grade point. Is decimal grade a function of percentage? Is percentage a function of decimal grade?

For any percentage earned, there would be a decimal grade associated, so we could say that the decimal grade is a function of percentage. That is, if you input the percentage, your output would be a decimal grade. Percentage may or may not be a function of decimal grade, depending upon the teacher's grading scheme. With some grading systems, there are a range of percentages that correspond to the same decimal grade.

Try it Now

Let's consider bank account information.

1. Is your balance a function of your bank account number?  
*(if you input a bank account number does it make sense that the output is your balance?)*
2. Is your bank account number a function of your balance?  
*(if you input a balance does it make sense that the output is your bank account number?)*

**Function Notation**

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable  $h$  to represent height and we could use the descriptive variable  $a$  to represent age.

"height is a function of age" if we name the function  $f$  we write  
"h is f of a" or more simply  
 $h = f(a)$  we could instead name the function  $h$  and write  
 $h(a)$  which is read "h of a"

Remember we can use any variable to name the function; the notation  $h(a)$  shows us that  $h$  depends on  $a$ . The value "a" must be put into the function "h" to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

### Function Notation

The notation  $\text{output} = f(\text{input})$  defines a function named  $f$ . This would be read “output is  $f$  of input”

#### Example 4

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function  $f$ , we could write “days =  $f(\text{month})$ ” or  $d = f(m)$ . If we simply name the function  $d$ , we could write  $d(m)$

For example,  $d(\text{March}) = 31$ , since March has 31 days. The notation  $d(m)$  reminds us that the number of days,  $d$  (the output) is dependent on the name of the month,  $m$  (the input)

#### Example 5

A function  $N = f(y)$  gives the number of police officers,  $N$ , in a town in year  $y$ . What does  $f(2005) = 300$  tell us?

When we read  $f(2005) = 300$ , we see the input quantity is 2005, which is a value for the input quantity of the function, the year ( $y$ ). The output value is 300, the number of police officers ( $N$ ), a value for the output quantity. Remember  $N = f(y)$ . So this tells us that in the year 2005 there were 300 police officers in the town.

### Tables as Functions

Functions can be represented in many ways: Words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year)

(input) Month number, $m$	1	2	3	4	5	6	7	8	9	10	11	12
(output) Days in month, $D$	31	28	31	30	31	30	31	31	30	31	30	31

## 4 Chapter 1

Table 2: The table below defines a function  $Q = g(n)$ . Remember this notation tells us  $g$  is the name of the function that takes the input  $n$  and gives the output  $Q$ .

$n$	1	2	3	4	5
$Q$	8	6	7	6	8

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children.

(input) $a$ , age in years	5	5	6	7	8	9	10
(output) $h$ , height inches	40	42	44	47	50	52	54

### Example 6

Which of these tables define a function (if any)?

Input	Output	Input	Output	Input	Output
2	1	-3	5	1	0
5	3	0	1	5	2
8	6	4	5	5	4

The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

### Try it Now

3. If each percentage earned translated to one letter grade, would this be a function?

### Solving and Evaluating Functions:

When we work with functions, there are two typical things we do: evaluate and solve.

Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

**Example 7**

Using the table shown, where  $Q=g(n)$

a) Evaluate  $g(3)$

$n$	1	2	3	4	5
$Q$	8	6	7	6	8

Evaluating  $g(3)$  (read: “ $g$  of 3”)

means that we need to determine the output value,  $Q$ , of the function  $g$  given the input value of  $n=3$ . Looking at the table, we see the output corresponding to  $n=3$  is  $Q=7$ , allowing us to conclude  $g(3) = 7$ .

b) Solve  $g(n) = 6$

Solving  $g(n) = 6$  means we need to determine what input values,  $n$ , produce an output value of 6. Looking at the table we see there are two solutions:  $n = 2$  and  $n = 4$ .

When we input 2 into the function  $g$ , our output is  $Q = 6$

When we input 4 into the function  $g$ , our output is also  $Q = 6$

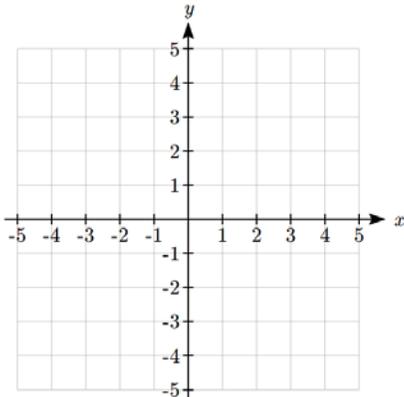
**Try it Now**

4. Using the function in Example 7, evaluate  $g(4)$

**Graphs as Functions**

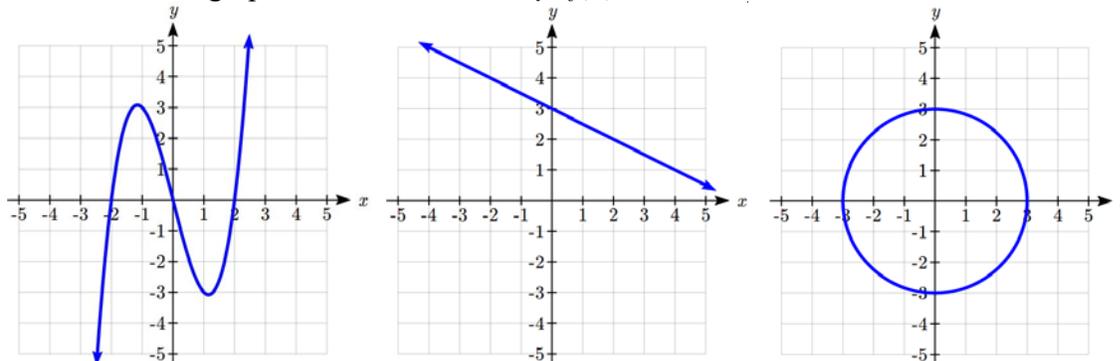
Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

The most common graph has  $y$  on the vertical axis and  $x$  on the horizontal axis, and we say  $y$  is a function of  $x$ , or  $y = f(x)$  when the function is named  $f$ .



## Example 8

Which of these graphs defines a function  $y=f(x)$ ?



Looking at the three graphs above, the first two define a function  $y=f(x)$ , since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the  $y$ -value of the graph. The 3<sup>rd</sup> graph does not define a function  $y=f(x)$  since some input values, such as  $x=2$ , correspond with more than one output value.

## Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

## Example 9

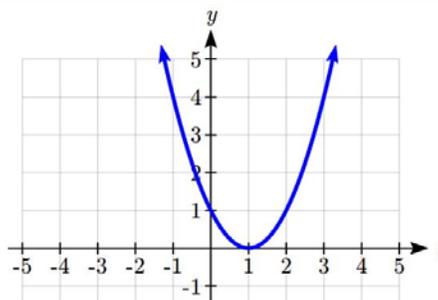
Given the graph shown,

- Evaluate  $f(2)$
- Solve  $f(x) = 4$

a) To evaluate  $f(2)$ , we find the input of  $x=2$  on the horizontal axis. Moving up to the graph gives the point  $(2, 1)$ , giving an output of  $y=1$ . So  $f(2) = 1$

b) To solve  $f(x) = 4$ , we find the value 4 on the vertical axis because if  $f(x) = 4$  then 4 is the output. Moving horizontally across the graph gives two points with the output of 4:  $(-1, 4)$  and  $(3, 4)$ . These give the two solutions to  $f(x) = 4$ :  $x = -1$  or  $x = 3$

This means  $f(-1)=4$  and  $f(3)=4$ , or when the input is -1 or 3, the output is 4.



## Try it Now

5. Using the graph from example 9, solve  $f(x)=1$ .

**Formulas as Functions**

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

## Example 10

Express the relationship  $2n + 6p = 12$  as a function  $p = f(n)$  if possible.

To express the relationship in this form, we need to be able to write the relationship where  $p$  is a function of  $n$ , which means writing it as  $p = [\text{something involving } n]$ .

$$\begin{array}{ll} 2n + 6p = 12 & \text{subtract } 2n \text{ from both sides} \\ 6p = 12 - 2n & \text{divide both sides by 6 and simplify} \end{array}$$

$$p = \frac{12 - 2n}{6} = \frac{12}{6} - \frac{2n}{6} = 2 - \frac{1}{3}n$$

Having rewritten the formula as  $p =$ , we can now express  $p$  as a function:

$$p = f(n) = 2 - \frac{1}{3}n$$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations - no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

## Example 11

Given the function  $k(t) = t^3 + 2$

- a) Evaluate  $k(2)$   
 b) Solve  $k(t) = 1$

a) To evaluate  $k(2)$ , we plug in the input value 2 into the formula wherever we see the input variable  $t$ , then simplify

$$k(2) = 2^3 + 2$$

$$k(2) = 8 + 2$$

$$\text{So } k(2) = 10$$

b) To solve  $k(t) = 1$ , we set the formula for  $k(t)$  equal to 1, and solve for the input value that will produce that output

$$k(t) = 1 \quad \text{substitute the original formula } k(t) = t^3 + 2$$

$$t^3 + 2 = 1 \quad \text{subtract 2 from each side}$$

$$t^3 = -1 \quad \text{take the cube root of each side}$$

$$t = -1$$

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if  $k(t) = 1$  is true when  $t = -1$ .

$$k(-1) = (-1)^3 + 2$$

$$= -1 + 2$$

$$= 1 \text{ which was the desired result.}$$

## Example 12

Given the function  $h(p) = p^2 + 2p$

- a) Evaluate  $h(4)$   
 b) Solve  $h(p) = 3$

To evaluate  $h(4)$  we substitute the value 4 for the input variable  $p$  in the given function.

$$\text{a) } h(4) = (4)^2 + 2(4)$$

$$= 16 + 8$$

$$= 24$$

$$\text{b) } h(p) = 3 \quad \text{Substitute the original function } h(p) = p^2 + 2p$$

$$p^2 + 2p = 3 \quad \text{This is quadratic, so we can rearrange the equation to get it = 0}$$

$$p^2 + 2p - 3 = 0 \quad \text{subtract 3 from each side}$$

$$p^2 + 2p - 3 = 0 \quad \text{this is factorable, so we factor it}$$

$$(p+3)(p-1) = 0$$

By the zero factor theorem since  $(p+3)(p-1) = 0$ , either  $(p+3) = 0$  or  $(p-1) = 0$  (or both of them equal 0) and so we solve both equations for  $p$ , finding  $p = -3$  from the first equation and  $p = 1$  from the second equation.

This gives us the solution:  $h(p) = 3$  when  $p = 1$  or  $p = -3$

### Try it Now

6. Given the function  $g(m) = \sqrt{m-4}$

- Evaluate  $g(5)$
- Solve  $g(m) = 2$

### Important Topics of this Section

Definition of a function  
Input (independent variable)  
Output (dependent variable)  
Function notation  
Descriptive variables  
Functions in words, tables, graphs & formulas  
Vertical line test  
Evaluating a function at a specific input value  
Solving a function given a specific output value

### Try it Now Answers

- Yes
- No
- Yes it's a function
- $Q = g(4) = 6$
- $x = 0$  or  $x = 2$
- a.  $g(5) = 1$  b.  $m = 8$

## Section 1.2 Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, $c$	1.7	2.5	5.5	8.2	13.7
Height, $h$	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

### Domain and Range

**Domain:** The set of possible input values to a function

**Range:** The set of possible output values of a function

### Example 1

Using the tree table above, determine a reasonable domain and range.

We could combine the data provided with our own experiences and reason to approximate the domain and range of the function  $h = f(c)$ . For the domain, possible values for the input circumference  $c$ , it doesn't make sense to have negative values, so  $c > 0$ . We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is about 119 feet<sup>1</sup>. With this information we would say a reasonable domain is  $0 < c \leq 119$  feet.

Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is  $0 < h \leq 379$  feet.

<sup>1</sup> <http://en.wikipedia.org/wiki/Tree>, retrieved July 19, 2010

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**Example 2**

When sending a letter through the United States Postal Service, the price depends upon the weight of the letter<sup>2</sup>, as shown in the table below. Determine the domain and range.

Letters	
Weight not Over	Price
1 ounce	\$0.44
2 ounces	\$0.61
3 ounces	\$0.78
3.5 ounces	\$0.95

Suppose we notate Weight by  $w$  and Price by  $p$ , and set up a function named  $P$ , where Price,  $p$  is a function of Weight,  $w$ .  $p = P(w)$ .

Since acceptable weights are 3.5 ounces or less, and negative weights don't make sense, the domain would be  $0 < w \leq 3.5$ . Technically 0 could be included in the domain, but logically it would mean we are mailing nothing, so it doesn't hurt to leave it out.

Since possible prices are from a limited set of values, we can only define the range of this function by listing the possible values. The range is  $p = \$0.44, \$0.61, \$0.78, \text{ or } \$0.95$ .

**Try it Now**

1. The population of a small town in the year 1960 was 100 people. Since then the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

**Notation**

In the previous examples, we used inequalities to describe the domain and range of the functions. This is one way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as  $0 < c \leq 163$ ,  $0 < w \leq 3.5$ , and  $0 < h \leq 379$  imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters,  $p = \$0.44, \$0.61, \$0.78, \text{ or } \$0.95$ . These numbers represent a set of specific values:  $\{0.44, 0.61, 0.78, 0.95\}$

<sup>2</sup> <http://www.usps.com/prices/first-class-mail-prices.htm>, retrieved July 19, 2010

Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable  $x$  that are 10 or greater, but less than 30. In inequalities, we would write  $10 \leq x < 30$ .

When describing domains and ranges, we sometimes extend this into **set-builder notation**, which would look like this:  $\{x \mid 10 \leq x < 30\}$ . The curly brackets  $\{\}$  are read as “the set of”, and the vertical bar  $\mid$  is read as “such that”, so altogether we would read  $\{x \mid 10 \leq x < 30\}$  as “the set of  $x$ -values such that 10 is less than or equal to  $x$  and  $x$  is less than 30.”

When describing ranges in set-builder notation, we could similarly write something like  $\{f(x) \mid 0 < f(x) < 100\}$ , or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range  $\{h \mid 0 < h \leq 379\}$ . In set-builder notation, if a domain or range is not limited, we could write  $\{t \mid t \text{ is a real number}\}$ , or  $\{t \mid t \in \mathbb{R}\}$ , read as “the set of  $t$ -values such that  $t$  is an element of the set of real numbers.

A more compact alternative to set-builder notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for “strictly less than,” and square brackets are used for “less than or equal to.” Since infinity is not a number, we can’t include it in the interval, so we always use curved parentheses with  $\infty$  and  $-\infty$ . The table below will help you see how inequalities correspond to set-builder notation and interval notation:

<b>Inequality</b>	<b>Set Builder Notation</b>	<b>Interval notation</b>
$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
all real numbers	$\{h \mid h \in \mathbb{R}\}$	$(-\infty, \infty)$

To combine two intervals together, using inequalities or set-builder notation we can use the word “or”. In interval notation, we use the union symbol,  $\cup$ , to combine two unconnected intervals together.

**Example 3**

Describe the intervals of values shown on the line graph below using set builder and interval notations.



To describe the values,  $x$ , that lie in the intervals shown above we would say, “ $x$  is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

As an inequality it is:  $1 \leq x \leq 3$  or  $x > 5$

In set builder notation:  $\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$

In interval notation:  $[1, 3] \cup (5, \infty)$

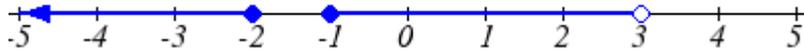
Remember when writing or reading interval notation:

Using a square bracket  $[$  means the start value is included in the set

Using a parenthesis  $($  means the start value is not included in the set

**Try it Now**

2. Given the following interval, write its meaning in words, set builder notation, and interval notation.

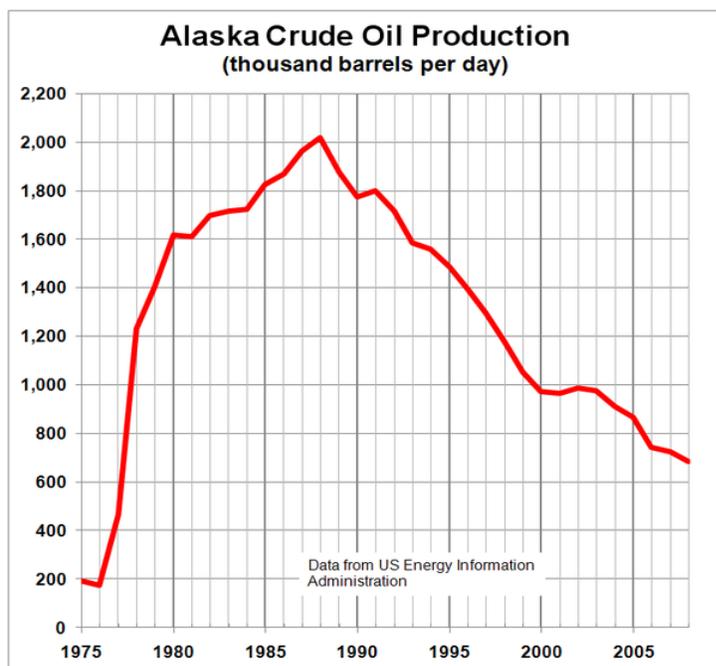
**Domain and Range from Graphs**

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

## Example 4

Determine the domain and range of the graph below.



In the graph above<sup>3</sup>, the input quantity along the horizontal axis appears to be “year”, which we could notate with the variable  $y$ . The output is “thousands of barrels of oil per day”, which we might notate with the variable  $b$ , for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is  $1975 \leq y \leq 2008$ , and the range is approximately  $180 \leq b \leq 2010$ .

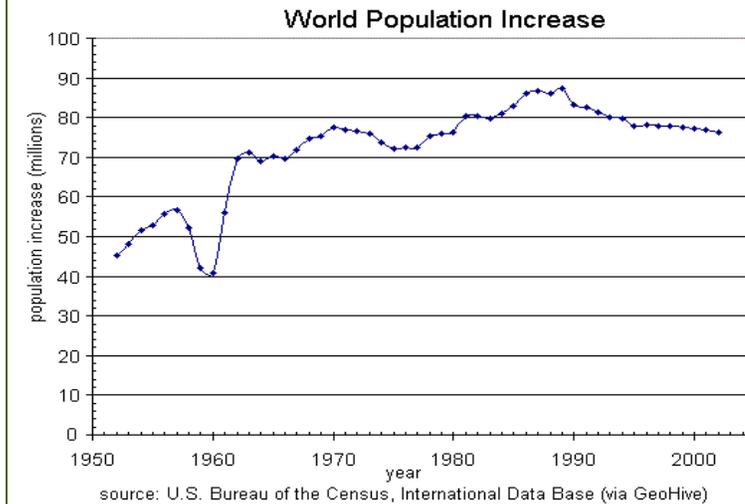
In interval notation, the domain would be  $[1975, 2008]$  and the range would be about  $[180, 2010]$ . For the range, we have to approximate the smallest and largest outputs since they don’t fall exactly on the grid lines.

Remember that, as in the previous example,  $x$  and  $y$  are not always the input and output variables. Using descriptive variables is an important tool to remembering the context of the problem.

<sup>3</sup> [http://commons.wikimedia.org/wiki/File:Alaska\\_Crude\\_Oil\\_Production.PNG](http://commons.wikimedia.org/wiki/File:Alaska_Crude_Oil_Production.PNG), CC-BY-SA, July 19, 2010

## Try it Now

3. Given the graph below write the domain and range in interval notation

**Domain and Range from Formulas**

Most basic formulas can be evaluated at an input. Two common restrictions are:

- The square root of negative values is non-real.
- We cannot divide by zero.

## Example 5

Find the domain of each function: a)  $f(x) = 2\sqrt{x+4}$     b)  $g(x) = \frac{3}{6-3x}$

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative.

$$x + 4 \geq 0 \text{ when } x \geq -4.$$

The domain of  $f(x)$  is  $[-4, \infty)$ .

b) We cannot divide by zero, so we need the denominator to be non-zero.

$$6 - 3x = 0 \text{ when } x = 2, \text{ so we must exclude } 2 \text{ from the domain.}$$

The domain of  $g(x)$  is  $(-\infty, 2) \cup (2, \infty)$ .

## Piecewise Functions

Some functions cannot be described by a single formula.

### Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1} & \text{if domain to use formula 1} \\ \text{formula 2} & \text{if domain to use formula 2} \\ \text{formula 3} & \text{if domain to use formula 3} \end{cases}$$

### Example 6

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Set up a function relating the number of people,  $n$ , to the cost,  $C$ .

To set up this function, two different formulas would be needed.  $C = 5n$  would work for  $n$  values under 10, and  $C = 50$  would work for values of  $n$  ten or greater. Notating this:

$$C(n) = \begin{cases} 5n & \text{if } 0 < n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

### Example 7

A cell phone company uses the function below to determine the cost,  $C$ , in dollars for  $g$  gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g - 2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

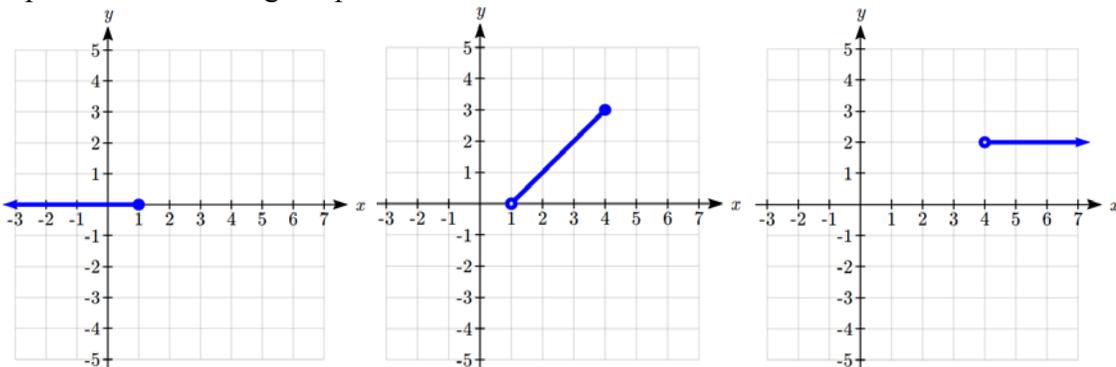
To find the cost of using 1.5 gigabytes of data,  $C(1.5)$ , we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving  $C(1.5) = \$25$ .

To find the cost of using 4 gigabytes of data,  $C(4)$ , we see that our input of 4 is greater than 2, so we'll use the second formula.  $C(4) = 25 + 10(4 - 2) = \$45$ .

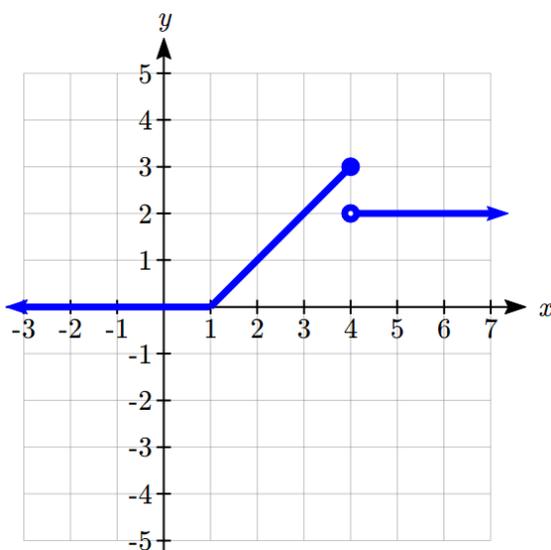
## Example 8

Sketch a graph of the function  $f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ x-1 & \text{if } 1 < x \leq 4 \\ 2 & \text{if } x > 4 \end{cases}$

We can imagine graphing each function, then limiting the graph to the indicated domain. At the endpoints of the domain, we put open circles to indicate where the endpoint is not included, due to a strictly-less-than inequality, and a closed circle where the endpoint is included, due to a less-than-or-equal-to inequality. The first and last parts are constant functions, where the output is the same for all inputs. The middle part we might recognize as a line, and could graph by evaluating the function at a couple inputs and connecting the points with a line.



Now that we have each piece individually, we combine them onto the same graph. When the first and second parts meet at  $x = 1$ , we can imagine the closed dot filling in the open dot. Since there is no break in the graph, there is no need to show the dot.



## Try it Now

4. At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for over 18 credits the rate is \$73 per credit<sup>4</sup>. Write a piecewise defined function for the total tuition,  $T$ , at Pierce College during 2009-2010 as a function of the number of credits taken,  $c$ . Be sure to consider a reasonable domain and range.

## Important Topics of this Section

Definition of domain  
 Definition of range  
 Inequalities  
 Interval notation  
 Set builder notation  
 Domain and Range from graphs  
 Piecewise defined functions

## Try it Now Answers

1. Domain;  $y = \text{years}$  [1960,2010] ; Range,  $p = \text{population}$ , [100,1400]
2. a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3  
 b.  $\{x \mid x \leq -2 \text{ or } -1 \leq x < 3\}$   
 c.  $(-\infty, -2] \cup [-1, 3)$

3. Domain;  $y = \text{years}$ , [1952,2002] ; Range,  $p = \text{population in millions}$ , [40,88]

$$4. T(c) = \begin{cases} 89.5c & \text{if } c \leq 10 \\ 895 + 33(c - 10) & \text{if } 10 < c \leq 18 \\ 1159 + 73(c - 18) & \text{if } c > 18 \end{cases} \text{ Tuition, } T, \text{ as a function of credits, } c.$$

Reasonable domain should be whole numbers 0 to (answers may vary), e.g. [0, 23]

Reasonable range should be \$0 – (answers may vary), e.g. [0,1524]

<sup>4</sup> [https://www.pierce.ctc.edu/dist/tuition/ref/files/0910\\_tuition\\_rate.pdf](https://www.pierce.ctc.edu/dist/tuition/ref/files/0910_tuition_rate.pdf), retrieved August 6, 2010

## Section 1.3 Rates of Change and Behavior of Graphs

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing.

For example, the function  $C(t)$  below gives the average cost, in dollars, of a gallon of gasoline  $t$  years after 2000.

$t$	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from \$1.47 to \$2.14, an increase of \$0.67. While this is interesting, it might be more useful to look at how much the price changed *per year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by \$0.67 from 2002 to 2009, over 7 years, for an average of  $\frac{\$0.67}{7 \text{ years}} \approx 0.096$  dollars per year. On average, the price of gas increased by about 9.6 cents each year.

### Rate of Change

A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are “output units per input units”

Some other examples of rates of change would be quantities like:

- A population of rats increases by 40 rats per week
- A barista earns \$9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by \$4,000 per quarter

### Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

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## Example 1

Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009

From the table, in 2007 the cost of gas was \$2.64. In 2009 the cost was \$2.14.

The input (years) has changed by 2. The output has changed by  $\$2.14 - \$2.64 = -\$0.50$ .

The average rate of change is then  $\frac{-\$0.50}{2 \text{ years}} = -0.25$  dollars per year

## Try it Now

1. Using the same cost-of-gas function, find the average rate of change between 2003 and 2008

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

## Example 2

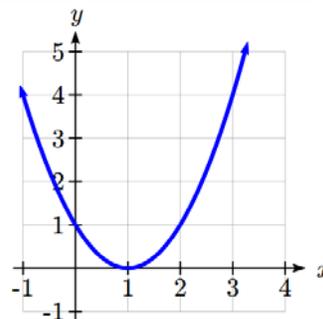
Given the function  $g(t)$  shown here, find the average rate of change on the interval  $[0, 3]$ .

At  $t = 0$ , the graph shows  $g(0) = 1$

At  $t = 3$ , the graph shows  $g(3) = 4$

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0} = \frac{3}{3} = 1$$



## Example 3

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

$t$ (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Here, your average speed is the average rate of change.

You traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0} = \frac{282}{6} = 47 \text{ miles per hour}$$

We can more formally state the average rate of change calculation using function notation.

#### Average Rate of Change using Function Notation

Given a function  $f(x)$ , the average rate of change on the interval  $[a, b]$  is

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b) - f(a)}{b - a}$$

#### Example 4

Compute the average rate of change of  $f(x) = x^2 - \frac{1}{x}$  on the interval  $[2, 4]$

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$

$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

$$\text{Average rate of change} = \frac{f(4) - f(2)}{4 - 2} = \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$$

#### Try it Now

2. Find the average rate of change of  $f(x) = x - 2\sqrt{x}$  on the interval  $[1, 9]$

#### Example 5

The magnetic force  $F$ , measured in Newtons, between two magnets is related to the distance between the magnets  $d$ , in centimeters, by the formula  $F(d) = \frac{2}{d^2}$ . Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

We are computing the average rate of change of  $F(d) = \frac{2}{d^2}$  on the interval  $[2, 6]$

$$\text{Average rate of change} = \frac{F(6) - F(2)}{6 - 2} \qquad \text{Evaluating the function}$$

$$\frac{F(6) - F(2)}{6 - 2} =$$

$$\frac{2 - 2}{6^2 - 2^2}$$

Simplifying

$$\frac{2 - 2}{36 - 4}$$

Combining the numerator terms

$$\frac{-16}{36}$$

Simplifying further

$$\frac{-16}{36}$$

$$\frac{-1}{9}$$

Newtons per centimeter

This tells us the magnetic force decreases, on average, by  $1/9$  Newtons per centimeter over this interval.

### Example 6

Find the average rate of change of  $g(t) = t^2 + 3t + 1$  on the interval  $[0, a]$ . Your answer will be an expression involving  $a$ .

Using the average rate of change formula

$$\frac{g(a) - g(0)}{a - 0}$$

Evaluating the function

$$\frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0}$$

Simplifying

$$\frac{a^2 + 3a + 1 - 1}{a}$$

Simplifying further, and factoring

$$\frac{a(a + 3)}{a}$$

Cancelling the common factor  $a$ 

$$a + 3$$

This result tells us the average rate of change between  $t = 0$  and any other point  $t = a$ . For example, on the interval  $[0, 5]$ , the average rate of change would be  $5 + 3 = 8$ .

### Try it Now

3. Find the average rate of change of  $f(x) = x^3 + 2$  on the interval  $[a, a + h]$ .

### Application to Business

In business, average rate of change is related to the idea of marginal cost, marginal revenue, and marginal profit.

#### Marginal Cost/Revenue/Profit

**Marginal Cost** is typically defined as the change in total cost if the quantity of items produced increases by one. Likewise, **marginal revenue** and **marginal profit** are the change in revenue and profit if the quantity of items increases by one.

In practice, marginal cost is usually calculated another way, using calculus techniques you will learn in future courses. In this course, though, we will stick with calculating marginal cost by looking at the actual change of total cost when production increases by one.

#### Example 7

A company has determined the total cost of producing  $q$  items is  $TC(q) = 2000 + 50\sqrt{q}$ . Find the marginal cost, when production is currently 5000 items.

We want to find the increase in total cost when increasing production from 5000 items to 5001 items. This is equivalent to finding the average rate of change on the interval  $[5000, 5001]$ .

The total cost at 5000 items is:  $TC(5000) = 2000 + 50\sqrt{5000} \approx 5535.53$

The total cost at 5001 items is:  $TC(5001) = 2000 + 50\sqrt{5001} \approx 5535.89$

The marginal cost is  $5535.89 - 5535.53 = \$0.36$ .

In other words, after producing 5000 items, the additional cost of producing the 5001<sup>st</sup> item is \$0.36

## Graphical Behavior of Functions

As part of exploring how functions change, it is interesting to explore the graphical behavior of functions.

### Increasing/Decreasing

A function is **increasing** on an interval if the function values increase as the inputs increase. More formally, a function is increasing if  $f(b) > f(a)$  for any two input values  $a$  and  $b$  in the interval with  $b > a$ . The average rate of change of an increasing function is **positive**.

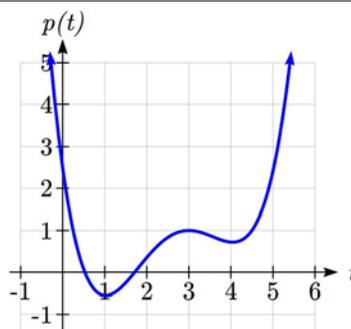
A function is **decreasing** on an interval if the function values decrease as the inputs increase. More formally, a function is decreasing if  $f(b) < f(a)$  for any two input values  $a$  and  $b$  in the interval with  $b > a$ . The average rate of change of a decreasing function is **negative**.

### Example 8

Given the function  $p(t)$  graphed here, on what intervals does the function appear to be increasing?

The function appears to be increasing from  $t = 1$  to  $t = 3$ , and from  $t = 4$  on.

In interval notation, we would say the function appears to be increasing on the interval  $(1, 3)$  and the interval  $(4, \infty)$ .



Notice in the last example that we used open intervals (intervals that don't include the endpoints) since the function is neither increasing nor decreasing at  $t = 1, 3$ , or  $4$ .

### Local Extrema

A point where a function changes from increasing to decreasing is called a **local maximum**.

A point where a function changes from decreasing to increasing is called a **local minimum**.

Together, local maxima and minima are called the **local extrema**, or local extreme values, of the function.

## Example 9

Using the cost of gasoline function from the beginning of the section, find an interval on which the function appears to be decreasing. Estimate any local extrema using the table.

$t$	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

It appears that the cost of gas increased from  $t = 2$  to  $t = 8$ . It appears the cost of gas decreased from  $t = 8$  to  $t = 9$ , so the function appears to be decreasing on the interval  $(8, 9)$ .

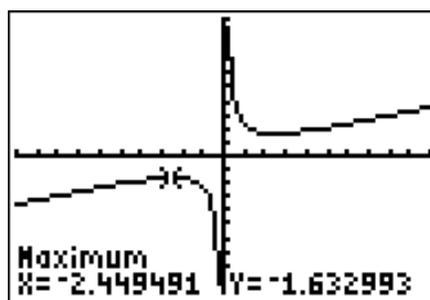
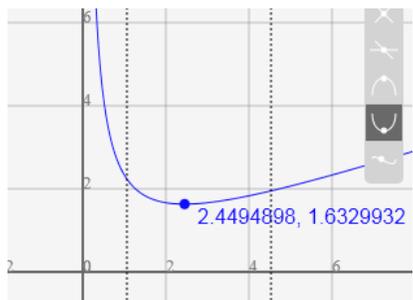
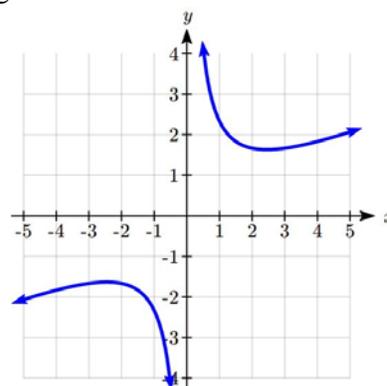
Since the function appears to change from increasing to decreasing at  $t = 8$ , there is local maximum at  $t = 8$ .

## Example 10

Use a graph to estimate the local extrema of the function  $f(x) = \frac{2}{x} + \frac{x}{3}$ . Use these to determine the intervals on which the function is increasing.

Using technology to graph the function, it appears there is a local minimum somewhere between  $x = 2$  and  $x = 3$ , and a symmetric local maximum somewhere between  $x = -3$  and  $x = -2$ .

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Below are screen images from two different technologies, showing the estimate for the local maximum and minimum.



Based on these estimates, the function is increasing on the intervals  $(-\infty, -2.449)$  and  $(2.449, \infty)$ . Notice that while we expect the extrema to be symmetric, the two different technologies agree only up to 4 decimals due to the differing approximation algorithms used by each.

## Try it Now

4. Use a graph of the function  $f(x) = x^3 - 6x^2 - 15x + 20$  to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

## Important Topics of This Section

Rate of Change  
 Average Rate of Change  
 Calculating Average Rate of Change using Function Notation  
 Marginal cost/revenue/profit  
 Increasing/Decreasing  
 Local Maxima and Minima (Extrema)

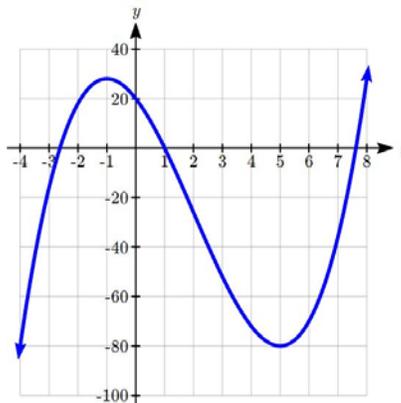
## Try it Now Answers

1.  $\frac{\$3.01 - \$1.69}{5 \text{ years}} = \frac{\$1.32}{5 \text{ years}} = 0.264$  dollars per year.

2. Average rate of change =  $\frac{f(9) - f(1)}{9 - 1} = \frac{(9 - 2\sqrt{9}) - (1 - 2\sqrt{1})}{9 - 1} = \frac{(3) - (-1)}{9 - 1} = \frac{4}{8} = \frac{1}{2}$

3.  $\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{((a+h)^3 + 2) - (a^3 + 2)}{h} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 + 2 - a^3 - 2}{h} =$   
 $\frac{3a^2h + 3ah^2 + h^3}{h} = \frac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2$

4. Based on the graph, the local maximum appears to occur at  $(-1, 28)$ , and the local minimum occurs at  $(5, -80)$ . The function is increasing on  $(-\infty, -1) \cup (5, \infty)$  and decreasing on  $(-1, 5)$ .



## Section 1.4 Linear Functions

As you hop into a taxicab in Las Vegas, the meter will immediately read \$3.30; this is the “drop” charge made when the taximeter is activated. After that initial fee, the taximeter will add \$2.60 for each mile the taxi drives<sup>5</sup>. In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function. Using descriptive variables, we choose  $m$  for miles and  $C$  for Cost in dollars as a function of miles:  $C(m)$ .

We know for certain that  $C(0) = 3.30$ , since the \$3.30 drop charge is assessed regardless of how many miles are driven. Since \$2.60 is added for each mile driven, then  $C(1) = 3.30 + 2.60 = 5.90$

If we then drove a second mile, another \$2.60 would be added to the cost:

$$C(2) = 3.30 + 2.60 + 2.60 = 3.30 + 2.60(2) = 8.50$$

If we drove a third mile, another \$2.40 would be added to the cost:

$$C(3) = 3.30 + 2.60 + 2.60 + 2.60 = 3.30 + 2.60(3) = 11.10$$

From this we might observe the pattern, and conclude that if  $m$  miles are driven,  $C(m) = 3.30 + 2.60m$  because we start with a \$3.30 drop fee and then for each mile increase we add \$2.60.

It is good to verify that the units make sense in this equation. The \$3.30 drop charge is measured in dollars; the \$2.60 charge is measured in dollars per mile. So

$$C(m) = 3.30 \text{dollars} + \left( 2.60 \frac{\text{dollars}}{\text{mile}} \right) (m \text{ miles})$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.30, and matching the desired units for the  $C$  function.

Notice this equation  $C(m) = 3.30 + 2.60m$  consisted of two quantities. The first is the fixed \$3.30 charge which does not change based on the value of the input. The second is the \$2.60 dollars per mile value, which is a **rate of change**. In the equation this rate of change is multiplied by the input value.

Looking at this same problem in table format we can also see the cost changes by \$2.60 for every 1 mile increase.

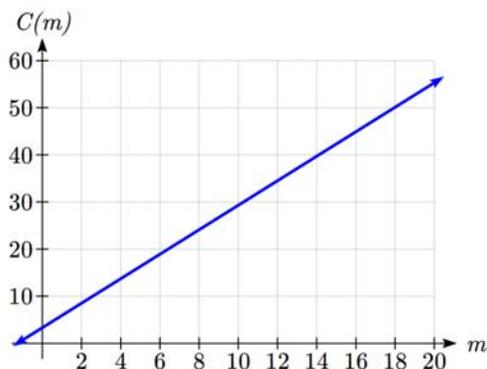
$m$	0	1	2	3
$C(m)$	3.30	5.90	8.50	11.10

It is important here to note that in this equation, the **rate of change is constant**; over any interval, the rate of change is the same.

<sup>5</sup> There is also a waiting fee assessed when the taxi is waiting at red lights, but we’ll ignore that in this discussion.

Graphing this equation,  $C(m) = 3.30 + 2.60m$  we see the shape is a line, which is how these functions get their name: **linear functions**

When the number of miles is zero the cost is \$3.30, giving the point  $(0, 3.30)$  on the graph. This is the vertical or  $C(m)$  intercept. The graph is increasing in a straight line from left to right because for each mile the cost goes up by \$2.40; this rate remains consistent.



In this example you have seen the taxicab cost modeled in words, an equation, a table and in graphical form. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all 4 representations for a problem and so being able to work with all 4 forms is very important.

### Linear Function

A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$$f(x) = b + mx \quad \text{or} \quad f(x) = mx + b; \text{ they're equivalent}$$

where

$b$  is the initial or starting value of the function (when input,  $x = 0$ ), and

$m$  is the constant rate of change of the function

Many people like to write linear functions in the form  $f(x) = b + mx$  because it corresponds to the way we tend to speak: “The output starts at  $b$  and increases at a rate of  $m$ .”

For this reason alone we will use the  $f(x) = b + mx$  form for many of the examples, but remember they are equivalent and can be written correctly both ways.

### Slope and Increasing/Decreasing

$m$  is the constant rate of change of the function (also called **slope**). The slope determines if the function is an increasing function or a decreasing function.

$f(x) = b + mx$  is an **increasing** function if  $m > 0$

$f(x) = b + mx$  is a **decreasing** function if  $m < 0$

If  $m = 0$ , the rate of change is zero, and the function  $f(x) = b + 0x = b$  is just a horizontal line passing through the point  $(0, b)$ , neither increasing nor decreasing.

## Example 1

To start producing a new model of custom wheels, a company will have to purchase \$30,000 in new equipment. Each wheel made costs them \$40 in supplies and labor. Write a formula for the total cost,  $TC$ , of producing  $q$  wheels. What will be their total costs in the first year if they expect to sell 240 wheels?

The initial value for this function is 30,000, since that is the startup costs, so  $TC(0) = 30,000$ . The cost increases by \$40 for each wheel made, so the rate of change is \$40 per wheel. With this information, we can write the formula:

$$TC(q) = 30,000 + 40q.$$

$TC(q)$  is an increasing linear function.

With this formula we can predict the total cost of 240 wheels:

$$TC(240) = 30,000 + 40(240) = 30,000 + 9,600 = 39,600. \text{ The total cost will be } \$39,600.$$

## Try it Now

1. If you earn \$30,000 per year and you spend \$29,000 per year write an equation for the amount of money you save after  $y$  years, if you start with nothing.

*“The most important thing, spend less than you earn!”<sup>6</sup>*

## Calculating Rate of Change

Given two values for the input,  $x_1$  and  $x_2$ , and two corresponding values for the output,  $y_1$  and  $y_2$ , or a set of points,  $(x_1, y_1)$  and  $(x_2, y_2)$ , if we wish to find a linear function that contains both points we can calculate the rate of change,  $m$ :

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Rate of change of a linear function is also called the **slope** of the line.

Note in function notation,  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$ , so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

<sup>6</sup> <http://www.thesimpledollar.com/2009/06/19/rule-1-spend-less-than-you-earn/>

## Example 2

The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

The rate of change will relate the change in population to the change in time. The population increased by  $27800 - 23400 = 4400$  people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$\frac{4400 \text{ people}}{4 \text{ years}} = 1100 \frac{\text{people}}{\text{year}} = 1100 \text{ people per year}$$

Notice that we knew the population was increasing, so we would expect our value for  $m$  to be positive. This is a quick way to check to see if your value is reasonable.

## Example 3

The pressure,  $P$ , in pounds per square inch (PSI) on a diver depends upon their depth below the water surface,  $d$ , in feet, following the equation  $P(d) = 14.696 + 0.434d$ .

Interpret the components of this function.

The rate of change, or slope, 0.434 would have units  $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{PSI}}{\text{ft}}$ . This tells us the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

## Example 4

If  $f(x)$  is a linear function,  $f(3) = -2$ , and  $f(8) = 1$ , find the rate of change.

$f(3) = -2$  tells us that the input 3 corresponds with the output -2, and  $f(8) = 1$  tells us that the input 8 corresponds with the output 1. To find the rate of change, we divide the change in output by the change in input:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1 - (-2)}{8 - 3} = \frac{3}{5}. \text{ If desired we could also write this as } m = 0.6$$

Note that it is not important which pair of values comes first in the subtractions so long as the first output value used corresponds with the first input value used.

## Try it Now

2. Given the two points (2, 3) and (0, 4), find the rate of change. Is this function increasing or decreasing?

We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

### Example 5

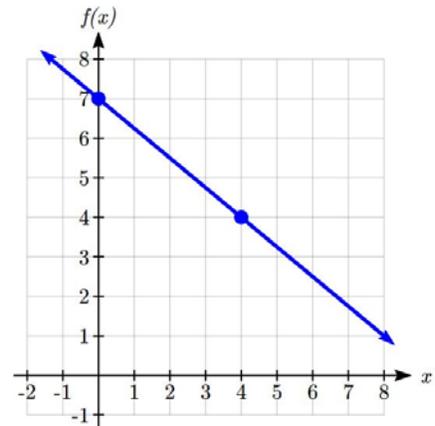
Write an equation for the linear function graphed to the right.

Looking at the graph, we might notice that it passes through the points  $(0, 7)$  and  $(4, 4)$ . From the first value, we know the initial value of the function is  $b = 7$ , so in this case we will only need to calculate the rate of change:

$$m = \frac{4 - 7}{4 - 0} = \frac{-3}{4}$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x$$



### Example 6

If  $f(x)$  is a linear function,  $f(3) = -2$ , and  $f(8) = 1$ , find an equation for the function.

In example 3, we computed the rate of change to be  $m = \frac{3}{5}$ . In this case, we do not know the initial value  $f(0)$ , so we will have to solve for it. Using the rate of change, we know the equation will have the form  $f(x) = b + \frac{3}{5}x$ . Since we know the value of the function when  $x = 3$ , we can evaluate the function at 3.

$$f(3) = b + \frac{3}{5}(3) \quad \text{Since we know that } f(3) = -2, \text{ we can substitute on the left side}$$

$$-2 = b + \frac{3}{5}(3) \quad \text{This leaves us with an equation we can solve for the initial value}$$

$$b = -2 - \frac{9}{5} = \frac{-19}{5}$$

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = \frac{-19}{5} + \frac{3}{5}x$$

## Example 7

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya's weekly income,  $I$ , depends on the number of new policies,  $n$ , he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for  $I(n)$ , and interpret the meaning of the components of the equation.

The given information gives us two input-output pairs:  $(3, 760)$  and  $(5, 920)$ . We start by finding the rate of change.

$$m = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy; Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value

$$I(n) = b + 80n \quad \text{then when } n = 3, I(3) = 760, \text{ giving}$$

$$760 = b + 80(3) \quad \text{this allows us to solve for } b$$

$$b = 760 - 80(3) = 520$$

This value is the starting value for the function. This is Ilya's income when  $n = 0$ , which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Writing the final equation:

$$I(n) = 520 + 80n$$

Our final interpretation is: Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold each week.

## Flashback

Looking at Example 7:

Determine the independent and dependent variables.

What is a reasonable domain and range?

## Try it Now

3. The balance in your college payment account,  $C$ , is a function of the number of quarters,  $q$ , you attend. Interpret the function  $C(q) = 20000 - 4000q$  in words. How many quarters of college can you pay for until this account is empty?

## Example 8

Given the table below write a linear equation that represents the table values

$w$ , number of weeks	0	2	4	6
$P(w)$ , number of rats	1000	1080	1160	1240

We can see from the table that the initial value of rats is 1000 so in the linear format  $P(w) = b + mw$ ,  $b = 1000$ .

Rather than solving for  $m$ , we can notice from the table that the population goes up by 80 for every 2 weeks that pass. This rate is consistent from week 0, to week 2, 4, and 6. The rate of change is 80 rats per 2 weeks. This can be simplified to 40 rats per week and we can write

$$P(w) = b + mw \text{ as } P(w) = 1000 + 40w$$

If you didn't notice this from the table you could still solve for the slope using any two points from the table. For example, using (2, 1080) and (6, 1240),

$$m = \frac{1240 - 1080}{6 - 2} = \frac{160}{4} = 40 \text{ rats per week}$$

## Important Topics of this Section

- Definition of a linear function
- Structure of a linear function
- Increasing & Decreasing functions
- Finding the vertical intercept (0,  $b$ )
- Finding the slope/rate of change,  $m$
- Interpreting linear functions

## Try it Now Answers

1.  $S(y) = 30,000y - 29,000y = 1000y$  \$1000 is saved each year.
2.  $m = \frac{4 - 3}{0 - 2} = \frac{1}{-2} = -\frac{1}{2}$  ; Decreasing because  $m < 0$
3. Your College account starts with \$20,000 in it and you withdraw \$4,000 each quarter (or your account contains \$20,000 and decreases by \$4000 each quarter.) You can pay for 5 quarters before the money in this account is gone.

## Flashback Answers

$n$  (number of policies sold) is the independent variable  
 $I(n)$  (weekly income as a function of policies sold) is the dependent variable.  
 A reasonable domain is (0, 15)\* A reasonable range is (\$540, \$1740)\*  
 \*answers may vary given reasoning is stated; 15 is an arbitrary upper limit based on selling 3 policies per day in a 5 day work week and \$1740 corresponds with the domain.

## Section 1.5 Graphs of Linear Functions

When we are working with a new function, it is useful to know as much as we can about the function: its graph, where the function is zero, and any other special behaviors of the function.

When graphing a linear function, there are two basic ways to graph it:

- 1) By plotting points (at least 2) and drawing a line through the points
- 2) Using the initial value (output when  $x = 0$ ) and rate of change (slope)

### Example 1

Graph  $f(x) = 5 - \frac{2}{3}x$  by plotting points

In general, we evaluate the function at two or more inputs to find at least two points on the graph. Usually it is best to pick input values that will “work nicely” in the equation.

In this equation, multiples of 3 will work nicely due to the  $\frac{2}{3}$  in the equation, and of

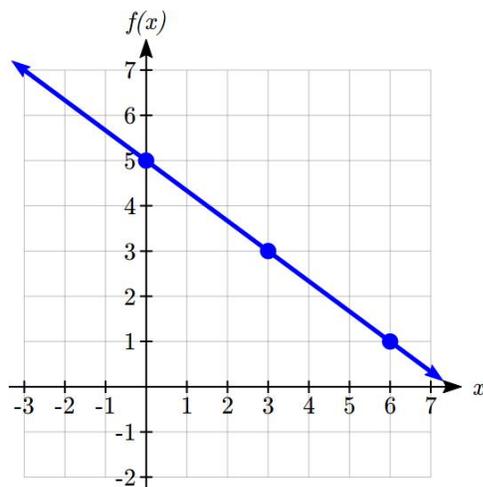
course using  $x = 0$  to get the vertical intercept. Evaluating  $f(x)$  at  $x = 0, 3$  and  $6$ :

$$f(0) = 5 - \frac{2}{3}(0) = 5$$

$$f(3) = 5 - \frac{2}{3}(3) = 3$$

$$f(6) = 5 - \frac{2}{3}(6) = 1$$

These evaluations tell us that the points  $(0,5)$ ,  $(3,3)$ , and  $(6,1)$  lie on the graph of the line. Plotting these points and drawing a line through them gives us the graph.



When using the initial value and rate of change to graph, we need to consider the graphical interpretation of these values. Remember the initial value of the function is the output when the input is zero, so in the equation  $f(x) = b + mx$ , the graph includes the point  $(0, b)$ . On the graph, this is the vertical intercept – the point where the graph crosses the vertical axis.

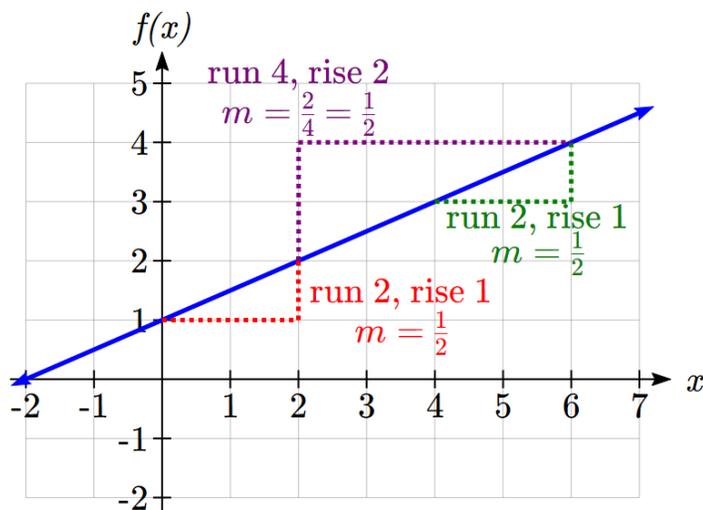
For the rate of change, it is helpful to recall that we calculated this value as

$$m = \frac{\text{change of output}}{\text{change of input}}$$

From a graph of a line, this tells us that if we divide the vertical difference, or rise, of the function outputs by the horizontal difference, or run, of the inputs, we will obtain the rate of change, also called slope of the line.

$$m = \frac{\text{change of output}}{\text{change of input}} = \frac{\text{rise}}{\text{run}}$$

Notice that this ratio is the same regardless of which two points we use



### Graphical Interpretation of a Linear Equation

Graphically, in the equation  $f(x) = b + mx$

$b$  is the **vertical intercept** of the graph and tells us we can start our graph at  $(0, b)$

$m$  is the **slope of the line** and tells us how far to rise & run to get to the next point

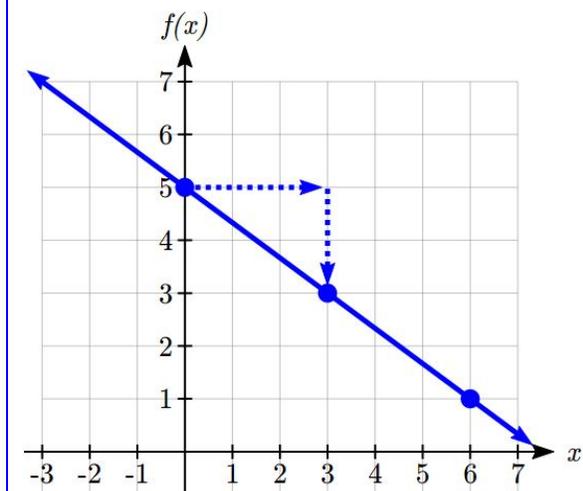
Once we have at least 2 points, we can extend the graph of the line to the left and right.

## Example 2

Graph  $f(x) = 5 - \frac{2}{3}x$  using the vertical intercept and slope.

The vertical intercept of the function is  $(0, 5)$ , giving us a point on the graph of the line.

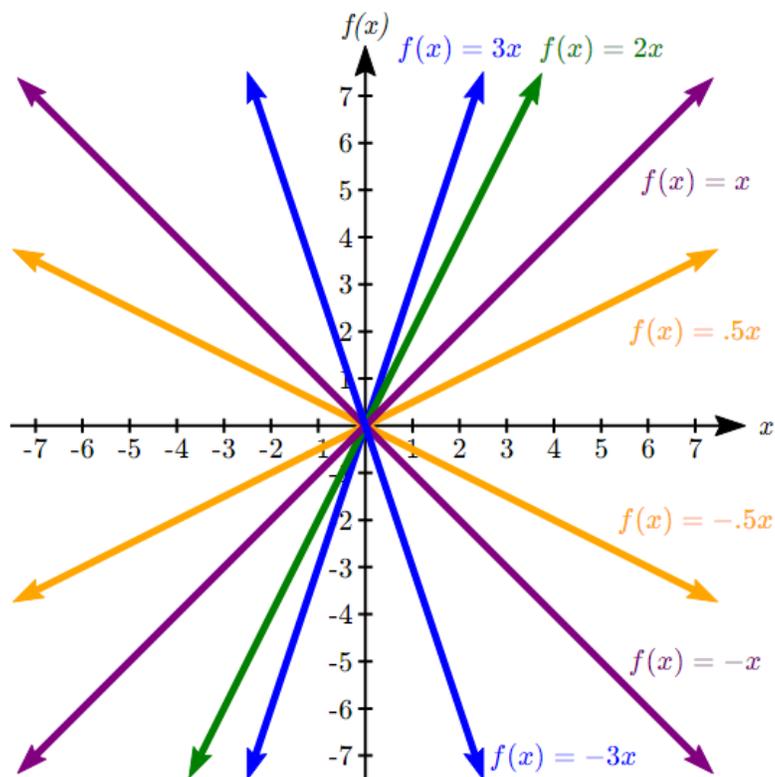
The slope is  $-\frac{2}{3}$ . This tells us that for every 3 units the graph “runs” in the horizontal, the vertical “rise” decreases by 2 units. In graphing, we can use this by first plotting our vertical intercept on the graph, then using the slope to find a second point. From the initial value  $(0, 5)$  the slope tells us that if we move to the right 3, we will move down 2, moving us to the point  $(3, 3)$ . We can continue this again to find a third point at  $(6, 1)$ . Finally, extend the line to the left and right, containing these points.



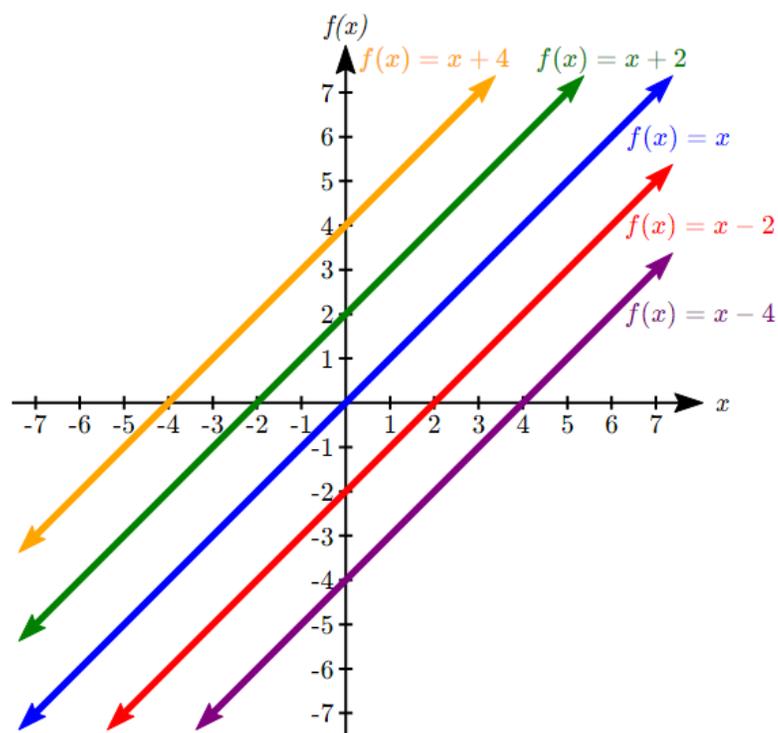
## Try it Now

1. Consider that the slope  $-2/3$  could also be written as  $\frac{2}{-3}$ . Using  $\frac{2}{-3}$ , find another point on the graph that has a negative  $x$  value.

In  $f(x) = mx + b$ , the value of  $m$  determines the steepness of the line. When  $m$  is positive the graph will be increasing, and when  $m$  is negative the graph will be decreasing. Looking at some examples:



In  $f(x) = mx + b$ , the  $b$  acts as a vertical shift, moving the graph up and down without affecting the slope of the line. Some examples:



## Example 3

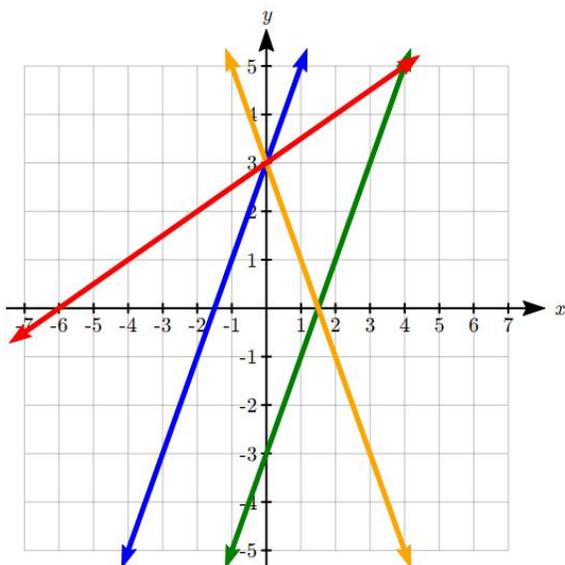
Match each equation with one of the lines in the graph below

$$f(x) = 2x + 3$$

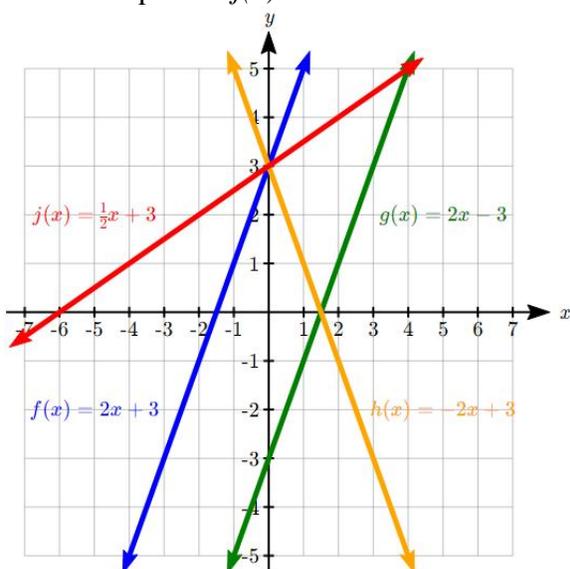
$$g(x) = 2x - 3$$

$$h(x) = -2x + 3$$

$$j(x) = \frac{1}{2}x + 3$$



Only one graph has a vertical intercept of -3, so we can immediately match that graph with  $g(x)$ . For the three graphs with a vertical intercept at 3, only one has a negative slope, so we can match that line with  $h(x)$ . Of the other two, the steeper line would have a larger slope, so we can match that graph with equation  $f(x)$ , and the flatter line with the equation  $j(x)$ .



In addition to understanding the basic behavior of a linear function (increasing or decreasing, recognizing the slope and vertical intercept), it is often helpful to know the horizontal intercept of the function – where it crosses the horizontal axis.

### Finding Horizontal Intercept

The **horizontal intercept** of the function is where the graph crosses the horizontal axis. If a function has a horizontal intercept, you can always find it by solving  $f(x) = 0$ .

### Example 4

Find the horizontal intercept of  $f(x) = -3 + \frac{1}{2}x$

Setting the function equal to zero to find what input will put us on the horizontal axis,

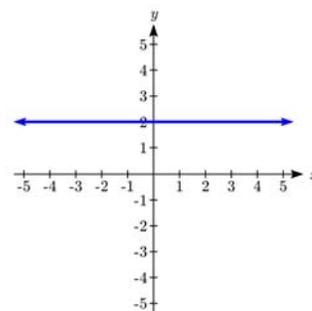
$$0 = -3 + \frac{1}{2}x$$

$$3 = \frac{1}{2}x$$

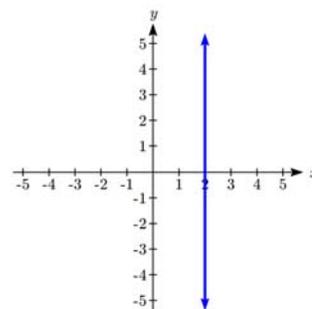
$$x = 6$$

The graph crosses the horizontal axis at (6,0)

There are two special cases of lines: a horizontal line and a vertical line. In a horizontal line like the one graphed to the right, notice that between any two points, the change in the outputs is 0. In the slope equation, the numerator will be 0, resulting in a slope of 0. Using a slope of 0 in the  $f(x) = b + mx$ , the equation simplifies to  $f(x) = b$ . Notice a horizontal line has a vertical intercept, but no horizontal intercept (unless it's the line  $f(x) = 0$ ).



In the case of a vertical line, notice that between any two points, the change in the inputs is zero. In the slope equation, the denominator will be zero, and you may recall that we cannot divide by the zero; the slope of a vertical line is undefined. You might also notice that a vertical line is not a function. To write the equation of vertical line, we simply write input=value, like  $x = b$ .



Notice a vertical line has a horizontal intercept, but no vertical intercept (unless it's the line  $x = 0$ ).

### Horizontal and Vertical Lines

**Horizontal lines** have equations of the form  $f(x) = b$

**Vertical lines** have equations of the form  $x = a$

#### Example 5

Write an equation for the horizontal line graphed above.

This line would have equation  $f(x) = 2$

#### Example 6

Write an equation for the vertical line graphed above.

This line would have equation  $x = 2$

### Parallel and Perpendicular Lines

When two lines are graphed together, the lines will be **parallel** if they are increasing at the same rate – if the rates of change are the same. In this case, the graphs will never cross (unless they're the same line).

#### Parallel Lines

Two lines are **parallel** if the slopes are equal (or, if both lines are vertical). In other words, given two linear equations  $f(x) = b + m_1x$  and  $g(x) = b + m_2x$

The lines will be parallel if  $m_1 = m_2$

#### Example 7

Find a line parallel to  $f(x) = 6 + 3x$  that passes through the point  $(3, 0)$

We know the line we're looking for will have the same slope as the given line,  $m = 3$ . Using this and the given point, we can solve for the new line's vertical intercept:

$$g(x) = b + 3x \quad \text{then at } (3, 0),$$

$$0 = b + 3(3)$$

$$b = -9$$

The line we're looking for is  $g(x) = -9 + 3x$

If two lines are not parallel, one other interesting possibility is that the lines are perpendicular, which means the lines form a right angle (90 degree angle – a square corner) where they meet. In this case, the slopes when multiplied together will equal -1. Solving for one slope leads us to the definition:

### Perpendicular Lines

Given two linear equations  $f(x) = b + m_1x$  and  $g(x) = b + m_2x$

The lines will be **perpendicular** if  $m_1m_2 = -1$ , and so  $m_2 = \frac{-1}{m_1}$

We often say the slope of a perpendicular line is the “negative reciprocal” of the other line’s slope.

### Example 8

Find the slope of a line perpendicular to a line with:

a) a slope of 2.      b) a slope of -4.      c) a slope of  $\frac{2}{3}$ .

If the original line had slope 2, the perpendicular line’s slope would be  $m_2 = \frac{-1}{2}$

If the original line had slope -4, the perpendicular line’s slope would be  $m_2 = \frac{-1}{-4} = \frac{1}{4}$

If the original line had slope  $\frac{2}{3}$ , the perpendicular line’s slope would be  $m_2 = \frac{-1}{\frac{2}{3}} = \frac{-3}{2}$

### Example 9

Find the equation of a line perpendicular to  $f(x) = 6 + 3x$  and passing through the point (3, 0)

The original line has slope  $m = 3$ . The perpendicular line will have slope  $m = \frac{-1}{3}$ .

Using this and the given point, we can find the equation for the line.

$g(x) = b - \frac{1}{3}x$                       then at (3, 0),

$$0 = b - \frac{1}{3}(3)$$

$$b = 1$$

The line we’re looking for is  $g(x) = 1 - \frac{1}{3}x$

## Try it Now

2. Given the line  $h(t) = -4 + 2t$ , find an equation for the line passing through  $(0, 0)$  that is: a) parallel to  $h(t)$ . b) perpendicular to  $h(t)$ .

## Example 10

A line passes through the points  $(-2, 6)$  and  $(4, 5)$ . Find the equation of a perpendicular line that passes through the point  $(4, 5)$ .

From the two given points on the reference line, we can calculate the slope of that line:

$$m_1 = \frac{5 - 6}{4 - (-2)} = \frac{-1}{6}$$

The perpendicular line will have slope

$$m_2 = \frac{-1}{-1/6} = 6$$

We can then solve for the vertical intercept that makes the line pass through the desired point:

$$g(x) = b + 6x \quad \text{then at } (4, 5),$$

$$5 = b + 6(4)$$

$$b = -19$$

$$\text{Giving the line } g(x) = -19 + 6x$$

## Intersections of Lines

The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that  $f(x) = g(x)$ . In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

## Example 11

Find the intersection of the lines  $h(t) = 3t - 4$  and  $j(t) = 5 - t$

Setting  $h(t) = j(t)$ ,

$$3t - 4 = 5 - t$$

$$4t = 9$$

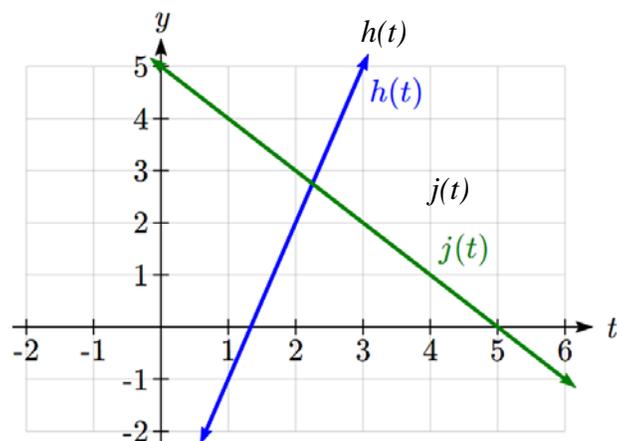
$$t = \frac{9}{4}$$

This tells us the lines intersect when the input is  $\frac{9}{4}$ .

We can then find the output value of the intersection point by evaluating either function at this input

$$j\left(\frac{9}{4}\right) = 5 - \frac{9}{4} = \frac{11}{4}$$

These lines intersect at the point  $\left(\frac{9}{4}, \frac{11}{4}\right)$ . Looking at the graph, this result seems reasonable.



Two parallel lines can also intersect if they happen to be the same line. In that case, they intersect at every point on the lines.

### Try it Now

3. Look at the graph in example 11 above and answer the following for the function  $j(t)$ :
  - a. Vertical intercept coordinates
  - b. Horizontal intercepts coordinates
  - c. Slope
  - d. Is  $j(t)$  parallel or perpendicular to  $h(t)$  (or neither)
  - e. Is  $j(t)$  an Increasing or Decreasing function (or neither)
  - f. Write a transformation description from the identity toolkit function  $f(x) = x$

Finding the intersection allows us to answer other questions as well, such as discovering when one function is larger than another.

## Example 12

Using the functions from the previous example, for what values of  $t$  is  $h(t) > j(t)$

To answer this question, it is helpful first to know where the functions are equal, since that is the point where  $h(t)$  could switch from being greater to smaller than  $j(t)$  or vice-versa. From the previous example, we know the functions are equal at  $t = \frac{9}{4}$ .

By examining the graph, we can see that  $h(t)$ , the function with positive slope, is going to be larger than the other function to the right of the intersection. So  $h(t) > j(t)$  when

$$t > \frac{9}{4}$$

## Important Topics of this Section

Methods for graphing linear functions

Another name for slope = rise/run

Horizontal intercepts  $(a,0)$

Horizontal lines

Vertical lines

Parallel lines

Perpendicular lines

Intersecting lines

## Try it Now Answers

1.  $(-3,7)$  found by starting at the vertical intercept, going up 2 units and 3 in the negative horizontal direction. You could have also answered,  $(-6, 9)$  or  $(-9, 11)$  etc...

2. Parallel  $f(t) = 2t$  ; Perpendicular  $g(t) = -\frac{1}{2}t$

3. Given  $j(t) = 5-t$

a.  $(0,5)$

b.  $(5,0)$

c. Slope -1

d. Neither parallel nor perpendicular

e. Decreasing function

f. Given the identity function, perform a vertical flip (over the  $t$  axis) and shift up 5 units.

## Section 1.6 Modeling with Linear Functions

When modeling scenarios with a linear function and solving problems involving quantities changing linearly, we typically follow the same problem solving strategies that we would use for any type of function:

### Problem solving strategy

- 1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
- 2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
- 3) Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
- 4) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
- 5) When needed, find a formula for the function.
- 6) Solve or evaluate using the formula you found for the desired quantities.
- 7) Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
- 8) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

### Example 1

A company purchased \$120,000 in new office equipment. Then expect the value to depreciate (decrease) by \$16,000 per year. Find a linear model for the value, then find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

In the problem, there are two changing quantities: time and value. The remaining value of the equipment depends on how long the company has had it. We can define our variables, including units.

Output:  $V$ , value remaining, in dollars

Input:  $t$ , time, in years

Reading the problem, we identify two important values. The first, \$120,000, is the initial value for  $V$ . The other value appears to be a rate of change – the units of dollars per year match the units of our output variable divided by our input variable. The value is depreciating, so you should recognize that the value remaining is decreasing each year and the slope is negative.

Using the intercept and slope provided in the problem, we can write the equation:

$$V(t) = 120,000 - 16,000t.$$

To find the horizontal intercept, we set the output to zero, and solve for the input:

$$0 = 120,000 - 16,000t$$

$$t = \frac{120,000}{16,000} = 7.5$$

The horizontal intercept is 7.5 years. Since this represents the input value where the output will be zero, interpreting this, we could say: The equipment will have no remaining value after 7.5 years.

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than zero. This model is also not valid after the horizontal intercept.

The domain represents the set of input values and so the reasonable domain for this function is  $0 \leq t \leq 7.5$ . The range represents the set of output values and the value starts at \$120,000 and ends with \$0 after 7.5 years so the corresponding range is

$$0 \leq V(t) \leq 120,000.$$

Most importantly remember that domain and range are tied together, and what ever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

### Example 2

Jamal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of \$20, then 59 cents a mile. The second, Budget, charges an up-front fee of \$16, then 63 cents a mile<sup>7</sup>. When will U-Haul be the better choice for Jamal?

The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

Input:  $m$ , miles driven

Outputs:

$Y(m)$ : cost, in dollars, for renting from U-Haul

$B(m)$ : cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: \$0.59 a mile for U-Haul, and \$0.63 a mile for Budget.

<sup>7</sup> Rates retrieved Aug 2, 2010 from <http://www.budgettruck.com> and <http://www.uhaul.com/>

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when  $Y(m) < B(m)$ . The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where the  $Y(m)$  function is smaller. Using the rates of change and initial charges, we can write the equations:

$$Y(m) = 20 + 0.59m$$

$$B(m) = 16 + 0.63m$$

These graphs are sketched to the right, with  $Y(m)$  drawn dashed.

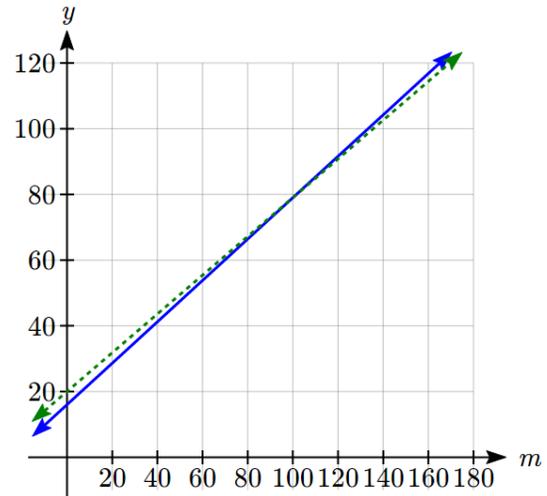
To find the intersection, we set the equations equal and solve:

$$Y(m) = B(m)$$

$$20 + 0.59m = 16 + 0.63m$$

$$4 = 0.04m$$

$$m = 100$$



This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that  $Y(m)$  is growing at a slower rate, we can conclude that U-Haul will be the cheaper price when more than 100 miles are driven.

### Example 3

A company's revenue has been growing linearly. In 2012 their revenue was \$1.45 million. By 2015 the revenue had grown to \$1.81 million. If this trend continues,

- Predict the revenue in 2020
- When will revenue reach \$3 million?

The two changing quantities are the revenue and time. While we could use the actual year value as the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!

To make things a little nicer, and to make our lives easier too, we will define our input as years since 2012:

Input:  $t$ , years since 2012

Output:  $R(t)$ , the company's revenue, in millions of dollars

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2012 would correspond to  $t = 0$ , giving the point  $(0, 1.45)$ . Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2015 would correspond to  $t = 3$ , giving the point  $(3, 1.81)$ .

To predict the population in 2020 ( $t = 8$ ), we would need an equation for the population. Likewise, to find when the revenue would reach \$3 million, we would need to solve for the input that would provide an output of 3. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{1.81 - 1.45}{3 - 0} = \frac{0.36}{3} = 0.12 \text{ million dollars per year}$$

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$R(t) = 1.45 + 0.12t$$

To predict the revenue in 2020, we evaluate our function at  $t = 8$

$$R(8) = 1.45 + 0.12(8) = 2.41$$

If the trend continues, our model predicts a revenue of \$2.41 million in 2020.

To find when the population will reach \$3 million, we can set  $R(t) = 3$  and solve for  $t$ .

$$15000 = 6200 + 380t$$

$$8800 = 380t$$

$$t \approx 23.158$$

$$3 = 1.45 + 0.12t$$

$$1.55 = 0.12t$$

$$t \approx 12.917$$

Our model predicts the revenue will reach \$3 million in a little under 13 years after 2012, or just before the year 2025.

In business, a very common application of functions is to model cost, revenue, and profit.

### Cost, Revenue, Profit

When a company produces  $q$  items, the **total cost** is the cost of total cost of producing those items. The total cost includes both **fixed costs**, which are startup costs, like equipment and buildings, and **variable costs**, which are costs that depend on the number of items produced, like materials and labor.

In the most simple case, **Total Cost = (Fixed Costs) + (Variable Costs)  $\cdot$   $q$**

**Revenue** is the amount of money a company brings in from sales.

In the most simple case, **Revenue = (Revenue per item)  $\cdot$   $q$**

**Profit** is the amount of money brings in, after expenses.

**Profit = Revenue – Costs**

We often talk about the **break-even** point. This is the level of production where Revenue equals Cost, or equivalently where Profit is zero. This is typically the minimum level of sales necessary for the company to make a profit.

## Example 4

A tech startup is looking at developing and launching a new mobile app. Initial development of the app will cost \$300,000, and they estimate marketing and support for each user will cost \$0.50. While the app will be free, they estimate they will be able to bring in \$2 per user on average from in-app purchases. How many users will the company need to break even?

We start by modeling the cost, revenue, and profit. Let  $q$  = number of users.

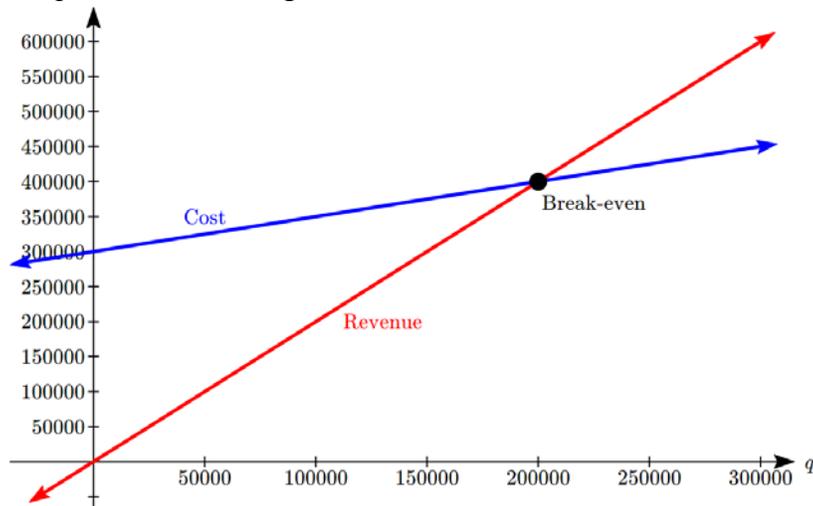
The fixed (initial) costs are \$300,000, and the variable (per-item) costs as \$0.50 per user. We can write the total cost equation:

$$TC(q) = 300,000 + 0.50q$$

The revenue will be \$2 per user, so the revenue equation will be:

$$R(q) = 2q$$

We could find the break-even point by setting the total cost equal to the revenue, which is equivalent to finding the intersection of the lines.



Alternatively, we could go ahead and find a profit equation first:

$$P(q) = R(q) - TC(q) = 2q - (300,000 + 0.50q) = 1.5q - 300,000$$

The break even point can be found by setting the profit equal to zero:

$$0 = 1.5q - 300,000$$

$$q = 200,000$$

The company will have to acquire 200,000 users to break even.

## Try it Now

1. A donut shop estimates their fixed daily expenses are \$600. If each donut costs about \$0.05 to make and sells for \$0.60, how many donuts do they need to sell to break even?

In economics, there is a model for how prices are determined in a free market which states that **supply and demand** for a product is related to the price. The demand relationship shows the quantity of a certain product that consumers are willing to buy at a certain price. Typically the quantity demanded will decrease for an item if the price increases. The supply relationship shows the quantity of a product that suppliers are willing to produce at a certain sales price. Typically the supply demanded will increase if the price increases. Economic theory says that supply and demand will interact, and the intersection will be the equilibrium price, or market price, where the quantity supplied and demanded will be equal.

### Supply and Demand

If  $p$  is the price of a product, then

$Q_d$  is the quantity demanded

$Q_s$  is the quantity supplied

The demand curve is a decreasing function, while the supply curve is an increasing function.

The intersection of the curves is the **equilibrium price and quantity**, also called the **market price and quantity**. This point is often notated as  $p^*$ ,  $Q^*$ .

In later chapters you will explore supply and demand curves that are non-linear, but in this chapter we will focus on linear supply and demand functions.

In most economic books, you will see the supply and demand curve written with price as the input and quantity as the output, like  $Q_d = 140 - 2p$ . However, supply and demand graphs are drawn with price on the vertical axis and quantity on the horizontal. In an effort to avoid confusion, most of the time in this text we will instead write supply and demand curves with price as the output, to match its placement on the vertical axis.

### Example 5

At a price of \$2.50 per gallon, there is a demand in a certain town for 42.5 thousand gallons of gas and a supply of 20 thousand gallons. At a price of \$3.50, there is demand for 25.5 thousand gallons and a supply of 28 thousand gallons. Assuming supply and demand are linear, find the equilibrium price and quantity.

We start by finding a linear equation for both supply and demand. We will use price,  $p$  in dollars, as the output and quantity,  $q$  in thousands of gallons, as the input.

For supply, we have the points (20, 2.50) and (28, 3.50).

Finding the slope:  $m = \frac{3.50 - 2.50}{28 - 20} = \frac{1}{8}$

We know the equation will look like  $p = \frac{1}{8}q + b$ , so substituting in (20, 2.50),

$$2.5 = \frac{1}{8}(20) + b$$

$$2.5 = 2.5 + b$$

$$b = 0$$

The supply equation is:  $p = \frac{1}{8}q$

For demand, we have the points (42.5, 2.50) and (25.5, 3.50). Using a similar approach, we can find the demand equation is:  $p = -\frac{1}{17}q + 5$

To find the equilibrium, we set the supply equal to the demand:

$$\frac{1}{8}q = -\frac{1}{17}q + 5 \quad \text{Multiplying through by } 8(17) = 136 \text{ to clear the fractions}$$

$$136\left(\frac{1}{8}q\right) = 136\left(-\frac{1}{17}q + 5\right)$$

$$17q = -8q + 680 \quad \text{Now we solve for } q$$

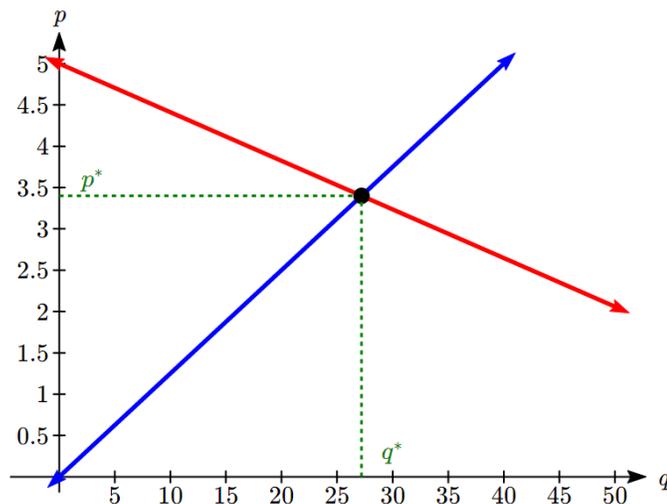
$$25q = 680$$

$$q = 27.2$$

To find the equilibrium price, we can substitute that value back into either equation:

$$p = \frac{1}{8}(27.2) = 3.4$$

The equilibrium quantity will be 27.2 thousand gallons of gas at a price of \$3.40.



## Try it Now

2. A company estimates that at a price of \$140 there will demand for 4000 items, and for each \$5 increase in price the demand will drop by 200 items. The supply curve is

$$p = \frac{1}{20}q.$$
 Find the equilibrium price and quantity.

## Important Topics of this Section

**The problem solving process**

- 1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
- 2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
- 3) Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
- 4) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
- 5) When needed, find a formula for the function.
- 6) Solve or evaluate using the formula you found for the desired quantities.
- 7) Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
- 8) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Cost, revenue, profit equations

Supply and demand

Equilibrium price and quantity

## Try it Now Answers

1. Revenue:  $R = 0.60q$

Break even when  $R = C$ , at a quantity of about 1091 donuts per day.

2. Demand:  $p = -\frac{1}{40}q + 240$

Supply = Demand when  $q = 3200$ ,  $p = \$160$

## Section 1.7 Fitting Linear Models to Data

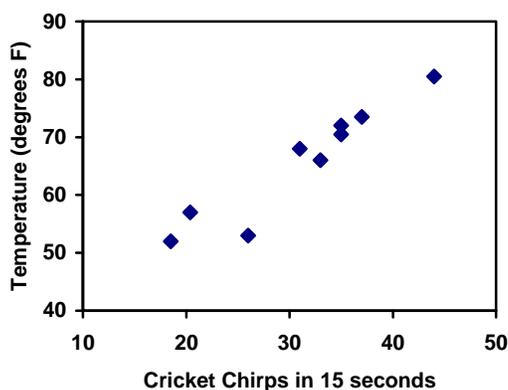
In the real world, rarely do things follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behavior of the data and allows us to use the linear model to make predictions about the data, inside and outside of the data range.

### Example 1

The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit<sup>8</sup>. Plot this data, and determine whether the data appears to be linearly related.

chirps	44	35	20.4	33	31	35	18.5	37	26
Temp	80.5	70.5	57	66	68	72	52	73.5	53

Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so.



The simplest way to find an equation to approximate this data is to try to “eyeball” a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

<sup>8</sup> Selected data from <http://classic.globe.gov/fsl/scientistsblog/2007/10/>. Retrieved Aug 3, 2010

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## Flashback

1. a. What descriptive variables would you choose to represent Temperature & Chirps?
- b. Which variable is the independent variable and which is the dependent variable?
- c. Based on this data and the graph, what is a reasonable domain & range?
- d. Based on the data alone, is this function one-to-one, explain?

## Example 2

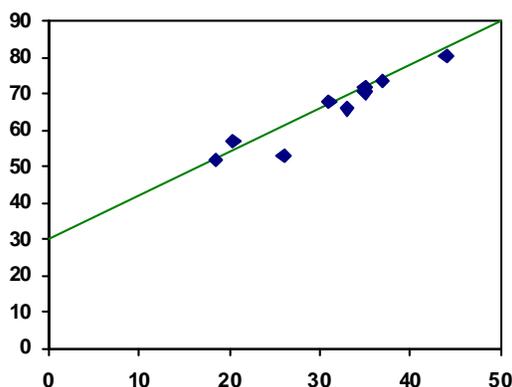
Using the table of values from the previous example, find a linear function that fits the data by “eyeballing” a line that seems to fit.

On a graph, we could try sketching in a line. Note the scale on the axes have been adjusted to start at zero to include the vertical axis and vertical intercept in the graph.

Using the starting and ending points of our “hand drawn” line, points  $(0, 30)$  and  $(50, 90)$ , this graph has a slope of  $m = \frac{60}{50} = 1.2$  and a vertical intercept at 30, giving an equation of

$$T(c) = 30 + 1.2c$$

where  $c$  is the number of chirps in 15 seconds, and  $T(c)$  is the temperature in degrees Fahrenheit.



This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. There is a difference, though, between making predictions inside the domain and range of values we have data for, and outside that domain and range.

## Interpolation and Extrapolation

**Interpolation:** When we predict a value inside the domain and range of the data

**Extrapolation:** When we predict a value outside the domain and range of the data

For the Temperature as a function of chirps in our hand drawn model above:

Interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44.

Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

**Example 3**

a) Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

b) Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

With our cricket data, our number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$T(30) = 30 + 1.2(30) = 66 \text{ degrees.}$$

Based on the data we have, this value seems reasonable.

The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation since 40 is outside the range of our data. Using our model:

$$40 = 30 + 1.2c$$

$$10 = 1.2c$$

$$c \approx 8.33$$

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

When our model no longer applies after some point, it is sometimes called **model breakdown**.

**Try it Now**

What temperature would you predict if you counted 20 chirps in 15 seconds?

**Fitting Lines with Technology**

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values<sup>9</sup>. This technique is called **least-square regression**, and can be computed by many graphing calculators, spreadsheet software like Excel or Google Docs, statistical software, and many web-based calculators<sup>10</sup>.

<sup>9</sup> Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.

<sup>10</sup> For example, <http://www.shodor.org/unchem/math/lls/leastsq.html>

### Example 4

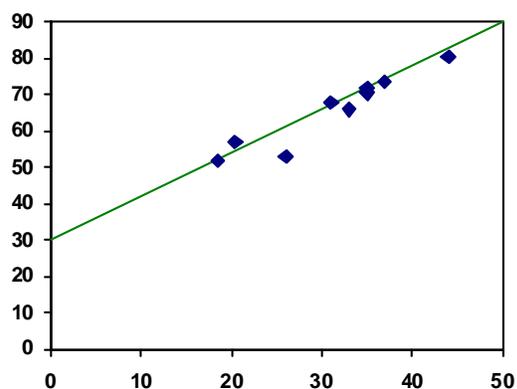
Find the least-squares regression line using the cricket chirp data from above.

Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c$$

Notice that this line is quite similar to the equation we “eyeballed”, but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$T(30) = 30.281 + 1.143(30) = 64.571 \approx 64.6 \text{ degrees.}$$



Most calculators and computer software will also provide you with the **correlation coefficient**, a measure of how closely the line fits the data.

### Correlation Coefficient

The **correlation coefficient** is a value,  $r$ , between -1 and 1.

$r > 0$  suggests a positive (increasing) relationship

$r < 0$  suggests a negative (decreasing) relationship

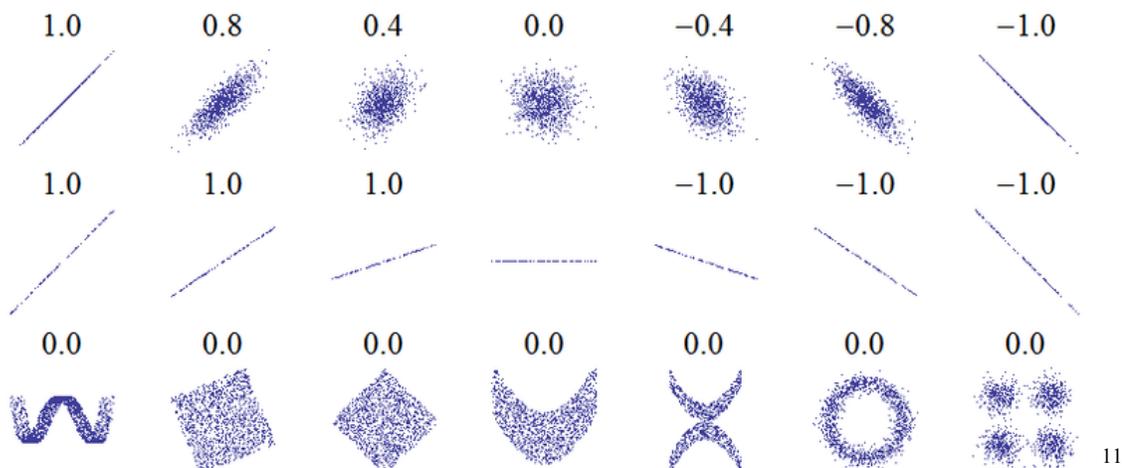
The closer the value is to 0, the more scattered the data

The closer the value is to 1 or -1, the less scattered the data is

The correlation coefficient provides an easy way to get some idea of how close to a line the data falls.

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of  $r$  and the graph of the data, here are some large data sets with their correlation coefficients:

## Examples of Correlation Coefficient Values



11

## Example 5

Calculate the correlation coefficient for our cricket data.

Because the data appears to follow a linear pattern, we can use technology to calculate  $r = 0.9509$ . Since this value is very close to 1, it suggests a strong increasing linear relationship.

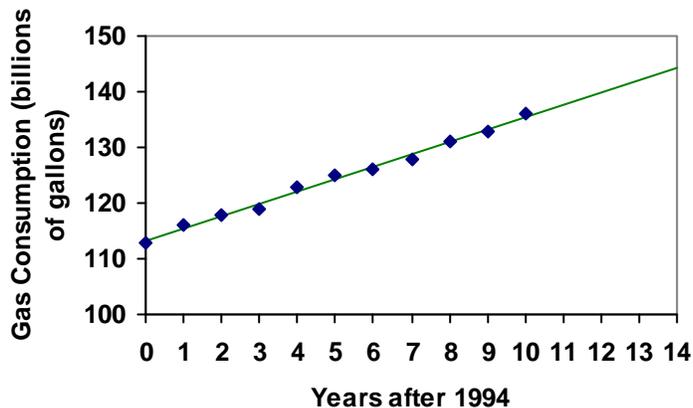
## Example 6

Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004 is shown below.<sup>12</sup> Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

To make things simpler, a new input variable is introduced,  $t$ , representing years since 1994.

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.



<sup>11</sup> [http://en.wikipedia.org/wiki/File:Correlation\\_examples.png](http://en.wikipedia.org/wiki/File:Correlation_examples.png)

<sup>12</sup> [http://www.bts.gov/publications/national\\_transportation\\_statistics/2005/html/table\\_04\\_10.html](http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html)

The least-squares regression equation is:

$$C(t) = 113.318 + 2.209t .$$

Using this to predict consumption in 2008 ( $t = 14$ ),

$$C(14) = 113.318 + 2.209(14) = 144.244 \text{ billions of gallons}$$

The model predicts 144.244 billion gallons of gasoline will be consumed in 2008.

### Try it Now

2. Use the model created by technology in example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

### Important Topics of this Section

Fitting linear models to data by hand  
Fitting linear models to data using technology  
Interpolation  
Extrapolation  
Correlation coefficient

### Flashback Answers

- a.  $T = \text{Temperature}$ ,  $C = \text{Chirps}$  (answers may vary)  
b. Independent (Chirps) , Dependent (Temperature)  
c. Reasonable Domain (18.5, 44) , Reasonable Range (52, 80.5) (answers may vary)  
d. NO, it is not one-to-one, there are two different output values for 35 chirps.

### Try it Now Answers

- 54 degrees Fahrenheit
- 150.871 billion gallons, extrapolation

## Chapter 2: Systems of Equations and Matrices

Section 2.1: Systems of Equations.....	59
Section 2.2: Solving Systems using Matrices.....	76
Section 2.3: Matrix Operations.....	90
Section 2.4: Solving Systems with Inverses.....	101

### Section 2.1: Systems of Equations

Back when studying linear equations, we found the intersection of two lines. Doing so allowed us to solve interesting problems by finding a pair of values that satisfied two different equations. While we didn't call it this at the time, we were solving a system of equations. To start out, we'll review an example of the type of problem we've solved before.

#### Example 1

A small business produces soap and lotion gift baskets. Labor, utilities, and other fixed expenses cost \$6,000 a month. Each basket costs \$8 to produce, and sells for \$20. How many baskets does the company need to sell each month to break even?

In business terms, "break even" means for revenue (money brought in) to equal costs. While this problem can be approached in several ways, we'll approach here by first creating two linear functions, one for the costs, and another for revenue.

Let's define  $n$  to be the number of gift baskets the company sells in a month. There are \$6,000 of fixed costs each month, and costs increase by \$8 for each basket, so we can write the linear function for costs,  $C$ , as:

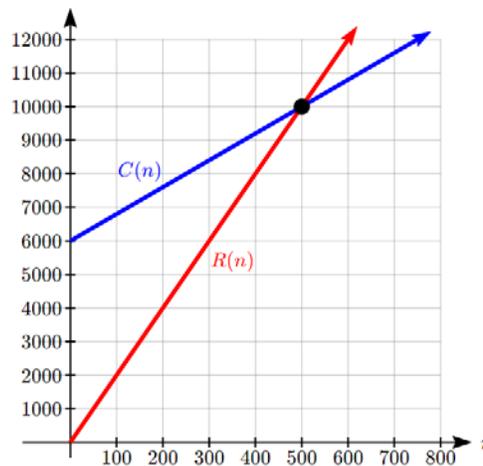
$$C(n) = 6000 + 8n$$

Each sale brings in \$20, so the revenue,  $R$ , after selling  $n$  baskets will be:

$$R(n) = 20n$$

To find the break-even point, we are looking for the number of baskets where the revenue will equal the costs. In other words, if we were to graph the two linear functions, we are looking for the point that lies on both lines; the solution is the point that satisfies both equations.

In this case we could probably solve the problem from the graph itself, but we can also solve it algebraically by setting the equations equal:



$$R(n) = C(n)$$

$$20n = 6000 + 8n$$

Subtract  $8n$  from both sides

$$12n = 6000$$

Divide

$$n = \frac{6000}{12} = 500$$

Evaluate either function at this input

$$R(500) = C(500) = 10,000$$

The break even point is at 500 baskets. The company must sell 500 baskets a month, at which point their revenue of \$10,000 will cover their total costs of \$10,000.

The example above illustrates one type of system of equations, one where both equations are given in functional form. When the equations are written this way, it is easy to solve the system using substitution, by setting the two outputs equal, and solving for the input. However, many system of equations problems aren't written this way.

### Example 2

A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. If the company wants to fully utilize all staffed hours, how many of each item should they produce?

Notice first that this problem has two variables, or two unknowns - the number of basic products to make, and the number of premium products to make. There are also two constraints - the hours of assembly and the hours of painting available. This is going to give us two equations in two unknowns, what we call a 2 by 2 system of equations.

We'll start by defining our variables:

$b$ : the number of basic products produced

$p$ : the number of premium products produced

Now we can create our equations based on the constraints. Each basic product requires 20 minutes of assembly, so producing  $b$  items will require  $20b$  minutes. Each premium product requires 30 minutes of assembly, so producing  $p$  items will require  $30p$  minutes. Together we have 3,900 minutes available, giving us the equation:

$$20b + 30p = 3900$$

Using the same approach for painting gives the equation

$$15b + 30p = 3300$$

Together, these form our system of equations. They are sometimes written as a pair with a curly bracket on the left to indicate that they should be considered as connected equations.

$$\begin{cases} 20b + 30p = 3900 \\ 15b + 30p = 3300 \end{cases}$$

As before, our goal is to find a pair of values,  $(b, p)$ , that satisfies both equations. We'll return to this problem and solve it shortly.

While it may not be clear, the equation  $20b + 30p = 3900$  we constructed above is a linear equation, like the linear equations from the first example, it's just written differently. We could, if desired, solve this equation for  $p$  to get it written in slope-intercept form:

$$30p = 3900 - 20b, \text{ so } p = 130 - \frac{2}{3}b$$

We typically don't do this, since it often makes the system harder to solve than when using other techniques. To dive into this further, let's first clarify what it means to find a solution to a system of linear equations.

### System of Linear Equations

A system of linear equations consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously.

A solution to a system is a set of numerical values for each variable in the system that will satisfy all equations in the system at the same time.

Not every system will have exactly one solution, but we'll look more closely at that later. To check to see if an ordered pair is a solution to a system of equations, you would:

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

### Example 3

Determine whether the ordered pair  $(5, 1)$  is a solution to the given system of equations.

$$\begin{cases} x + 3y = 8 \\ 2x - 9 = y \end{cases}$$

Substitute the ordered pair  $(5, 1)$  into both equations.

$$\begin{aligned}(5) + 3(1) &= 8 \\ 8 &= 8 \quad \text{True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \quad \text{True}\end{aligned}$$

The ordered pair  $(5, 1)$  satisfies both equations, so it is the solution to the system.

There are three common methods for solving systems of linear equations with two variables. The first is solving by graphing. In the first example above we graphed both equations, and the solution to the system was the intersection of the lines.

#### Example 4

Solve the following system of equations by graphing.

$$\begin{aligned}2x + y &= -8 \\ x - y &= -1\end{aligned}$$

Solve the first equation for  $y$ .

$$\begin{aligned}2x + y &= -8 \\ y &= -2x - 8\end{aligned}$$

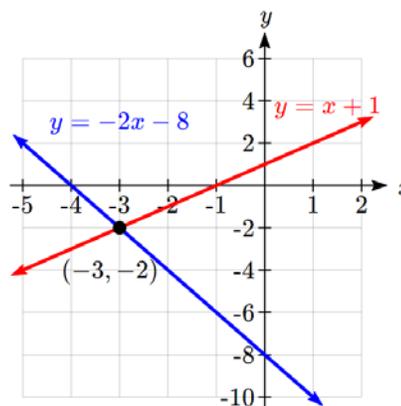
Solve the second equation for  $y$ .

$$\begin{aligned}x - y &= -1 \\ y &= x + 1\end{aligned}$$

Graph both equations on the same set of axes. The lines appear to intersect at the point  $(-3, -2)$ . We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned}2(-3) + (-2) &= -8 \\ -8 &= -8 \quad \text{True} \\ (-3) - (-2) &= -1 \\ -1 &= -1 \quad \text{True}\end{aligned}$$

The solution to the system is the ordered pair  $(-3, -2)$ .



#### Try it Now

1. Solve the following system of equations by graphing.

$$\begin{aligned}2x - 5y &= -25 \\ -4x + 5y &= 35\end{aligned}$$

While this method can work well enough when the solution values are both integers, it is not very useful when the intersection is not at a clear point. Additionally, it requires solving both equations for  $y$ , which adds extra steps. Because of these limitations, solving by graphing is rarely used, but can be useful for checking whether your algebraic answers are reasonable.

### Solving a System by Substitution

Another method for solving a system of equations is the substitution method, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable.

#### Solving a system using substitution

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

The problem we did in Example 1 was technically done by substitution, but it was made easier since both equations were already solved for one variable,  $y$ . An example of a more typical case is shown next.

#### Example 5

Solve the following system of equations by substitution.

$$-x + y = -5$$

$$2x - 5y = 1$$

First, we will solve the first equation for  $y$ .

$$-x + y = -5$$

$$y = x - 5$$

Now we can substitute the expression  $x - 5$  for  $y$  in the second equation.

$$2x - 5y = 1$$

$$2x - 5(x - 5) = 1$$

$$2x - 5x + 25 = 1$$

$$-3x = -24$$

$$x = 8$$

Now, we substitute  $x = 8$  into the first equation and solve for  $y$ .

$$-(8) + y = -5$$

$$y = 3$$

Our solution is  $(8, 3)$ .

We can check the solution by substituting  $(8, 3)$  into both equations.

$$-x + y = -5$$

$$-(8) + (3) = -5 \quad \text{True}$$

$$2x - 5y = 1$$

$$2(8) - 5(3) = 1 \quad \text{True}$$

### Try it Now

2. Solve the following system of equations by substitution.

$$x = y + 3$$

$$4 = 3x - 2y$$

Substitution can always be used, but is an especially good choice when one of the variables in one of the equations has a coefficient of 1 or -1, making it easy to solve for that variable without introducing fractions. This is fairly common in many applications.

### Example 6

Julia has just retired, and has \$600,000 in her retirement account that she needs to reallocate to produce income. She is looking at two investments: a very safe guaranteed annuity that will provide 3% interest, and a somewhat riskier bond fund that averages 7% interest. She would like to invest as little as possible in the riskier bond fund, but needs to produce \$40,000 a year in interest to live on. How much should she invest in each account?

Notice there are two unknowns in this problem: the amount she should invest in the annuity and the amount she should invest in the bond fund. We can start by defining variables for the unknowns:

- $a$ : The amount (in dollars) she invests in the annuity
- $b$ : The amount (in dollars) she invests in the bond fund.

Our first equation comes from noting that together she is going to invest \$600,000:

$$a + b = 600,000$$

Our second equation will come from the interest. She earns 3% on the annuity, so the interest earned in a year would be  $0.03a$ . Likewise, the interest earned on the bond fund in a year would be  $0.07b$ . Together, these need to total \$40,000, giving the equation:

$$0.03a + 0.07b = 40,000$$

Together, these two equations form our system. The first equation is an ideal candidate for the first step of substitution - we can easily solve the equation for  $a$  or  $b$ :

$$a = 600,000 - b$$

Then we can substitute this expression for  $a$  in the second equation and solve.

$$0.03(600,000 - b) + 0.07b = 30,000$$

$$18,000 - 0.03b + 0.07b = 30,000$$

$$0.04b = 12,000$$

$$b = 300,000$$

Now substitute this back into the equation  $a = 600,000 - b$  to find  $a$

$$a = 600,000 - 300,000$$

$$a = 300,000$$

In order to reach her goal, Julia will have to invest \$300,000 in the bond fund, and \$300,000 in the annuity.

### Solving a System by the Addition Method

A third method of solving systems of linear equations is the addition method, also called the elimination method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

#### Solving a System by the Addition Method

1. Write both equations with  $x$ - and  $y$ -variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Often using the addition method will require multiplying one or both equations by a constant so terms will eliminate. Using these approaches, we can revisit the equation from Example 2.

### Example 7

In Example 2, we set up the system below. Solve it.

$$\begin{cases} 20b + 30p = 3900 \\ 15b + 30p = 3300 \end{cases}$$

Adding the equations would not eliminate a variable, but we notice that the coefficients on  $p$  are the same, so multiplying one of the equations by  $-1$  will change the sign of the coefficients. Multiplying the second equation by  $-1$  gives the system

$$\begin{aligned} 20b + 30p &= 3900 \\ -15b - 30p &= -3300 \end{aligned}$$

Adding these equations gives

$$\begin{aligned} 5b &= 600 \\ b &= 120 \end{aligned}$$

Substituting  $b = 120$  into the first equation,

$$\begin{aligned} 20(120) + 30p &= 3900 \\ 2400 + 30p &= 3900 \\ 30p &= 1500 \\ p &= 50 \end{aligned}$$

The solution is  $b = 120$ ,  $p = 50$ , meaning the company should produce 120 basic products and 50 premium products to full utilize staffed hours.

Checking our answer in the second equation:

$$\begin{aligned} 15(120) + 30(50) &= 3300 \\ 1800 + 1500 &= 3300 \\ 3300 &= 3300 \end{aligned}$$

### Try it Now 3

3. Solve the system of equations by addition.

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

## Example 8

Solve the given system of equations in two variables by addition.

$$2x + 3y = -16$$

$$5x - 10y = 30$$

One equation has  $2x$  and the other has  $5x$ . The least common multiple is  $10x$  so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate  $x$  by multiplying the first equation by  $-5$  and the second equation by  $2$ .

$$-5(2x + 3y) = -5(-16)$$

$$-10x - 15y = 80$$

$$2(5x - 10y) = 2(30)$$

$$10x - 20y = 60$$

Then, we add the two equations together.

$$-10x - 15y = 80$$

$$10x - 20y = 60$$

$$\hline -35y = 140$$

$$y = -4$$

Substitute  $y = -4$  into the original first equation.

$$2x + 3(-4) = -16$$

$$2x - 12 = -16$$

$$2x = -4$$

$$x = -2$$

The solution is  $(-2, -4)$ . Check it in the other equation.

$$5x - 10y = 30$$

$$5(-2) - 10(-4) = 30$$

$$-10 + 40 = 30$$

$$30 = 30$$

## Try it Now

4. Solve the system of equations by addition.

$$2x + 3y = 8$$

$$3x + 5y = 10$$

### Dependent and Inconsistent Systems

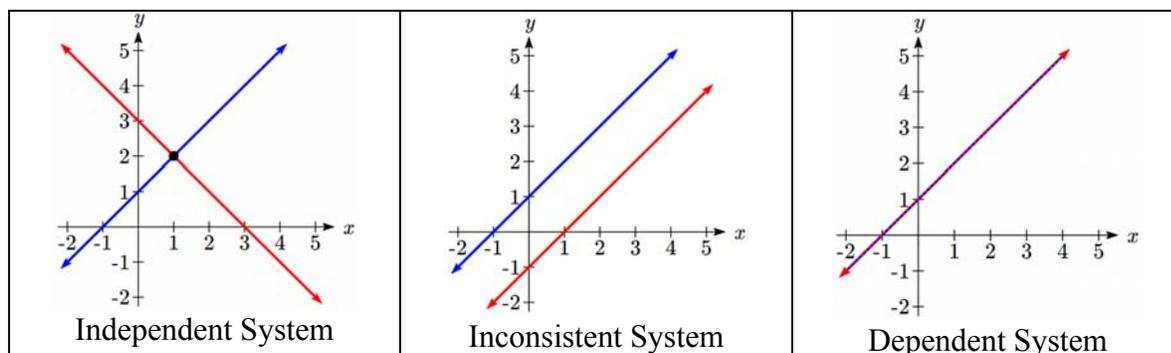
Up until now, we have only considered cases where there is exactly one solution to the system. We can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the examples we just explored. The two lines have different slopes and intersect at one point in the plane.

A consistent system is considered to be a **dependent system** if the equations have the same slope and the same  $y$ -intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different  $y$ -intercepts. There are no points common to both lines; hence, there is no solution to the system.

#### Types of Linear Systems

- An **independent system** has exactly one solution pair  $(x, y)$ . The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.



We can use substitution or addition to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different  $y$ -intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as  $12 = 0$ .

## Example 9

Solve the following system of equations.

$$x = 9 - 2y$$

$$x + 2y = 13$$

We can approach this problem in two ways. Because one equation is already solved for  $x$ , the most obvious step is to use substitution.

$$x + 2y = 13$$

$$(9 - 2y) + 2y = 13$$

$$9 + 0y = 13$$

$$9 = 13$$

Clearly, this statement is a contradiction (a false statement) because  $9 \neq 13$ . Therefore, the system has no solution, and the system is inconsistent.

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as  $0 = 0$ .

## Example 10

Find a solution to the system of equations using the addition method.

$$x + 3y = 2$$

$$3x + 9y = 6$$

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating  $x$ . If we multiply both sides of the first equation by  $-3$ , then we will be able to eliminate the  $x$ -variable.

$$x + 3y = 2$$

$$(-3)(x + 3y) = (-3)(2) \quad \text{Multiply both sides of the equation by } -3.$$

$$-3x - 9y = -6$$

$$-3x - 9y = -6$$

$$\underline{3x + 9y = 6}$$

$$0 = 0$$

We can see that there will be an infinite number of solutions that satisfy both equations. In some cases, realizing there are an infinite number of solutions is enough, and we can stop there. In other cases, we will want to describe the set of solutions.

One way is to simply say it's the set of points that satisfy  $x + 3y = 2$ , but often we would solve that equation for  $y$  and describe the solution as set of points

$$\left(x, -\frac{1}{3}x + \frac{2}{3}\right).$$

### Try it Now

5. Solve the systems:

$$\begin{array}{ll} 2y - 2x = 2 & y - 2x = 5 \\ \text{a. } 2y - 2x = 6 & \text{b. } -3y + 6x = -15 \end{array}$$

### Systems with 3 variables in 3 unknowns

In systems of two variables, a solution was an ordered pair  $(x, y)$  that satisfied both equations. The **solution set** to a three-by-three system is an ordered triple  $(x, y, z)$ .

Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

### Example 11

Determine whether the ordered triple  $(3, -2, 1)$  is a solution to the system.

$$\begin{array}{l} x + y + z = 2 \\ 6x - 4y + 5z = 31 \\ 5x + 2y + 2z = 13 \end{array}$$

We will check each equation by substituting in the values of the ordered triple for  $x$ ,  $y$ , and  $z$ .

$x + y + z = 2$	$6x - 4y + 5z = 31$	$5x + 2y + 2z = 13$
$(3) + (-2) + (1) = 2$	$6(3) - 4(-2) + 5(1) = 31$	$5(3) + 2(-2) + 2(1) = 13$
True	$18 + 8 + 5 = 31$	$15 - 4 + 2 = 13$
	True	True

The ordered triple  $(3, -2, 1)$  is indeed a solution to the system.

We can utilize the techniques we learned in the last section to solve 3-by-3 systems of equations by reducing the problem to one we already know how to solve.

Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

### Example 12

Find a solution to the following system. The equations are numbered so we can refer to them more easily.

$$x - 2y + 3z = 9 \quad (1)$$

$$-x + 3y - z = -6 \quad (2)$$

$$2x - 5y + 5z = 17 \quad (3)$$

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate  $x$  by adding equations (1) and (2).

$$x - 2y + 3z = 9 \quad (1)$$

$$\underline{-x + 3y - z = -6 \quad (2)}$$

$$y + 2z = 3 \quad (4)$$

The second step is multiplying equation (1) by  $-2$  and adding the result to equation (3). These two steps will eliminate the variable  $x$ .

$$-2x + 4y - 6z = -18 \quad (1) \text{ multiplied by } -2$$

$$\underline{2x - 5y + 5z = 17 \quad (3)}$$

$$-y - z = -1 \quad (5)$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for  $z$  by adding the two equations.

$$y + 2z = 3 \quad (4)$$

$$\underline{-y - z = -1 \quad (5)}$$

$$z = 2 \quad (6)$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$x - 2y + 3z = 9 \quad (1)$$

$$y + 2z = 3 \quad (4)$$

$$z = 2 \quad (6)$$

Next, we back-substitute  $z = 2$  into equation (4) and solve for  $y$ .

$$y + 2(2) = 3$$

$$y + 4 = 3$$

$$y = -1$$

Finally, we can back-substitute  $z = 2$  and  $y = -1$  into equation (1). This will yield the solution for  $x$ .

$$x - 2(-1) + 3(2) = 9$$

$$x + 2 + 6 = 9$$

$$x = 1$$

The solution is the ordered triple  $(1, -1, 2)$ .

### Try it Now

6. Solve the system of equations in three variables.

$$2x + y - 2z = -1$$

$$3x - 3y - z = 5$$

$$x - 2y + 3z = 6$$

Many problems in real life depend on more than two unknowns.

### Example 13

Chad is trying to plan a meal to meet specific nutritional goals. He wants to construct a dish containing rice, tofu, and peanuts that will provide 30g of protein, 14g of fat, and 50g of carbohydrates. How much of each ingredient should he use?

First off, we're assuming any other ingredients used in the recipe aren't contributing significantly enough to the nutrition to be considered. To answer this question, we'll first need to know the nutritional content for the ingredients. Looking these up:

White rice: 1 cup provides: 0g fat, 44g carbohydrates, 4g protein

Tofu: 1 cup provides: 10g fat, 5g carbohydrates, 20g protein

Peanuts: 1 cup provides: 72g fat, 31g carbohydrates, 35g protein

Now we can define our variables. We are interested in the amount of each ingredient to use, so we'll define our variables as the quantity of each ingredient:

$r$ : cups of rice,  $t$ : cups of tofu,  $p$ : cups of peanuts.

Now for each nutrient, we can create an equation. Since 1 cup of rice provides 44g carbohydrates,  $r$  cups will provide  $44r$  grams of carbohydrates. Likewise  $t$  cups of tofu will provide  $5t$  grams, and  $p$  cups of peanuts will provide  $31p$  grams. Together we want our recipe to provide 50g of carbohydrates, giving the equation:

$$44r + 5t + 31p = 50$$

Doing the same for fat and protein gives the full system:

$$44r + 5t + 31p = 50$$

$$10t + 72p = 14$$

$$4r + 20t + 35p = 30$$

Now we can solve the system.

Step 1. Notice that the section equation already does not involve the variable  $r$ . To make things simpler, a first step might be to interchange the last two equations so the two equations with three variables will line up.

$$44r + 5t + 31p = 50 \quad (1)$$

$$4r + 20t + 35p = 30 \quad (2)$$

$$10t + 72p = 14 \quad (3)$$

Step 2. Since  $r$  is already eliminated in the last equation, we'll eliminate  $r$  from the first two equations. Multiply equation (2) by  $-11$

$$44r + 5t + 31p = 50 \quad (1)$$

$$-44r - 220t - 385p = -330 \quad (2)$$

$$10t + 72p = 14 \quad (3)$$

Step 3. Add equations (1) and (2), writing the result as row 2.

$$44r + 5t + 31p = 50 \quad (1)$$

$$-215t - 384p = -280 \quad (2)$$

$$10t + 72p = 14 \quad (3)$$

Step 4. Multiply equation (2) by 2 and equation (3) by 43

$$44r + 5t + 31p = 50 \quad (1)$$

$$-430t - 708p = -560 \quad (2)$$

$$430t + 3096p = 602 \quad (3)$$

Step 5. Add equations (2) and (3), writing the result in row 3

$$44r + 5t + 31p = 50 \quad (1)$$

$$-430t - 708p = -560 \quad (2)$$

$$2388p = 42 \quad (3)$$

Step 6. Solve for  $p$  in equation 3. For a real life problem like this, decimal approximations are probably fine.

$$2388p = 42$$

$$p = \frac{42}{2388} \approx 0.0176$$

Step 7. Back substitute the value for  $p$  into equation (2) to solve for  $t$ .

$$-430t - 708(0.0176) = -560$$

$$-430t - 12.4608 = -560$$

$$-430t = -547.5392$$

$$t = \frac{-547.5392}{-430} \approx 1.273$$

Step 8. Back substitute the values for  $p$  and  $t$  into equation (1) and solve for  $r$ .

$$44r + 5(1.273) + 31(0.0176) = 50$$

$$44r = 43.0894$$

$$r \approx 0.979$$

To meet his nutritional goals, Chad should use 0.979 cups of rice, 1.273 cups of tofu, and 0.0176 cups of peanuts.

This particular system was rather unpleasant to solve. In the rest of the chapter we will learn some other techniques for solving complex systems.

### Important Topics of this Section

Solving systems

Graphing

Elimination

Addition

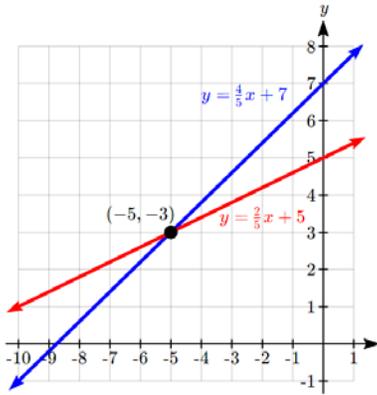
Setting up systems of equations

Inconsistent and dependent systems

Solving 3 by 3 systems

## Try it Now Answers

1. The solution to the system is the ordered pair  $(-5, 3)$ .



2.  $(-2, -5)$   
 3.  $(-6, -2)$   
 4.  $(10, -4)$   
 5a. No solution. The system is inconsistent.  
 5b. The system is dependent so there are infinite solutions of the form  $(x, 2x + 5)$ .  
 6.  $(1, -1, 1)$

## Section 2.2: Solving Systems using Matrices

While the techniques we learned in the last section can be used to solve any 2-by-2 or 3-by-3 system of linear equations, mathematicians often look for ways to do problems while writing less (which is why we use single letters for variables instead of full words), and to make solving problems more procedural. For systems of linear equations, matrices are the tool we use. In addition to making solving small systems more straightforward, the techniques can be extended to solve 4-by-4, 100-by-100, or even larger systems of linear equations that commonly arise in science.

### Matrix

A matrix is a rectangular array of numbers arranged into rows and columns.

### Writing the Augmented Matrix of a System of Equations

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

Given a system of equations, write an augmented matrix.

1. Write the coefficients of the  $x$ -terms as the numbers down the first column.
2. Write the coefficients of the  $y$ -terms as the numbers down the second column.
3. If there are  $z$ -terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

### Example 1

Consider the following  $2 \times 2$  system of equations. Write this system as an augmented matrix:

$$3x + 4y = 7$$

$$4x - 2y = 5$$

Extracting the coefficients from the system and write them in a rectangular array. This is called the **coefficient matrix**.

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$$

We then draw a vertical line and extract the constants from the right hand side of the system equations. This is the augmented matrix for the given system.

$$\left[ \begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array} \right]$$

### Example 2

Write the augmented matrix for the three-by-three system of equations

$$3x - y - z = 0$$

$$x + y = 5$$

$$2x - 3z = 2$$

The coefficient matrix is

$$\left[ \begin{array}{ccc} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3 \end{array} \right]$$

And the system is represented by the augmented matrix

$$\left[ \begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array} \right]$$

Notice that the matrix is written so that the variables line up in their own columns:  $x$ -terms go in the first column,  $y$ -terms in the second column, and  $z$ -terms in the third column. It is very important that each equation is written in standard form  $ax + by + cz = d$  so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0.

### Try it Now

1. Write the augmented matrix of the given system of equations.

$$4x - 3y = 11$$

$$3x + 2y = 4$$

### Writing a System of Equations from an Augmented Matrix

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

#### Example 3

Find the system of equations from the augmented matrix.

$$\left[ \begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array} \right]$$

When the columns represent the variables  $x$ ,  $y$ , and  $z$ ,

$$\left[ \begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array} \right] \rightarrow \begin{array}{l} x - 3y - 5z = -2, \\ 2x - 5y - 4z = 5, \\ -3x + 5y + 4z = 6 \end{array}$$

#### Try it Now

2. Write the system of equations from the augmented matrix.

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{array} \right]$$

### Performing Row Operations on a Matrix

Now that we can write systems of equations in augmented matrix form, we will examine the various **row operations** that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows. These are all operations that can be performed without changing the solution to the corresponding system.

Performing row operations on a matrix is the method we use for solving a system of equations, since these operations allow us to change a complex system into an equivalent simpler one with the same solution. In order to solve the system of equations, we want to convert the matrix to **row-echelon form**, in which there are ones down the **main diagonal** from the upper left corner to the lower right corner, and zeros in every position below the main diagonal as shown.

Row-echelon form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

We use row operations corresponding to equation operations to obtain a new matrix that is **row-equivalent** in a simpler form.

#### Guidelines for row-echelon form.

1. In any nonzero row, the first nonzero number is a 1. It is called a *leading 1*.
2. Any all-zero rows are placed at the bottom on the matrix.
3. Any leading 1 is below and to the right of a previous leading 1.
4. Any column containing a leading 1 has zeros in all other positions in the column.

#### Solving a system of equations using row operations

To solve a system of equations we can perform the following row operations to convert the coefficient matrix to row-echelon form and do back-substitution to find the solution.

- Interchange rows. (Notation:  $R_i \leftrightarrow R_j$ )
- Multiply a row by a constant. (Notation:  $cR_i$ )
- Add the product of a row multiplied by a constant to another row. (Notation:  $R_i + cR_j$ )

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a 1 as the first entry so that row 1 can be used to convert the remaining rows.

#### Gaussian Elimination

The **Gaussian elimination** method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix  $A$  with the number 1 as the entry down the main diagonal and have all zeros below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{After Gaussian elimination}} A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

### Using row operations on an augmented matrix to achieve row-echelon form.

1. The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of 1.
3. Use row operations to obtain a 1 in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a 1 in row 3, column 3.
6. Continue this process for all rows until there is a 1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

### Example 4

Solve the given system by Gaussian elimination.

$$2x + 3y = 6$$

$$x - y = \frac{1}{2}$$

First, we write this as an augmented matrix.

$$\left[ \begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array} \right]$$

We want a 1 in row 1, column 1. This can be accomplished by interchanging row 1 and row 2.

$$R_1 \leftrightarrow R_2 \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array} \right]$$

We now have a 1 as the first entry in row 1, column 1. Now let's obtain a 0 in row 2, column 1. This can be accomplished by multiplying row 1 by  $-2$ , and then adding the result to row 2.

$$-2R_1 + R_2 = R_2 \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array} \right]$$

We only have one more step, to multiply row 2 by  $\frac{1}{5}$ .

$$\frac{1}{5}R_2 = R_2 \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array} \right]$$

Use back-substitution. The second row of the matrix represents  $y = 1$ . Back-substitute  $y = 1$  into the first equation.

$$x - (1) = \frac{1}{2}$$

$$x = \frac{3}{2}$$

The solution is the point  $\left(\frac{3}{2}, 1\right)$ .

### Try it Now

3. Solve the given system by Gaussian elimination.

$$4x + 3y = 11$$

$$x - 3y = -1$$

### Example 5

Use Gaussian elimination to solve the given  $2 \times 2$  system of equations.

$$2x + y = 1$$

$$4x + 2y = 6$$

Write the system as an augmented matrix.

$$\left[ \begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

Obtain a 1 in row 1, column 1. This can be accomplished by multiplying the first row by  $\frac{1}{2}$ .

$$\frac{1}{2}R_1 = R_1 \rightarrow \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 2 & 6 \end{array} \right]$$

Next, we want a 0 in row 2, column 1. Multiply row 1 by  $-4$  and add row 1 to row 2.

$$-4R_1 + R_2 = R_2 \rightarrow \left[ \begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{array} \right]$$

The second row represents the equation  $0 = 4$ . Therefore, the system is inconsistent and has no solution.

## Example 6

Solve the system of equations.

$$3x + 4y = 12$$

$$6x + 8y = 24$$

Perform row operations on the augmented matrix to try and achieve row-echelon form.

$$A = \left[ \begin{array}{cc|c} 3 & 4 & 12 \\ 6 & 8 & 24 \end{array} \right]$$

$$-\frac{1}{2}R_2 + R_1 = R_1 \rightarrow \left[ \begin{array}{cc|c} 0 & 0 & 0 \\ 6 & 8 & 24 \end{array} \right]$$

$$R_1 \leftrightarrow R_2 \rightarrow \left[ \begin{array}{cc|c} 6 & 8 & 24 \\ 0 & 0 & 0 \end{array} \right]$$

The matrix ends up with all zeros in the last row:  $0y = 0$ . Thus, there are an infinite number of solutions and the system is classified as dependent. To find the generic solution, return to one of the original equations and solve for  $y$ .

$$3x + 4y = 12$$

$$4y = 12 - 3x$$

$$y = 3 - \frac{3}{4}x$$

So the solution to this system is  $\left(x, 3 - \frac{3}{4}x\right)$ .

Row reduction can be applied to any size system.

## Example 7

Perform row operations on the given matrix to obtain row-echelon form.

$$\left[ \begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{array} \right]$$

The first row already has a 1 in row 1, column 1. The next step is to multiply row 1 by  $-2$  and add it to row 2. Then replace row 2 with the result.

$$-2R_1 + R_2 = R_2 \rightarrow \left[ \begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ -3 & 3 & 4 & 6 \end{array} \right]$$

Next, obtain a zero in row 3, column 1.

$$3R_1 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{array} \right]$$

Next, obtain a zero in row 3, column 2.

$$6R_2 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 15 \end{array} \right]$$

The last step is to obtain a 1 in row 3, column 3.

$$\frac{1}{2}R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & \frac{21}{2} \end{array} \right]$$

### Try it Now

4. Write the system of equations as an augmented matrix, then perform row operations to obtain row-echelon form.

$$x - 2y + 3z = 9$$

$$-x + 3y = -4$$

$$2x - 5y + 5z = 17$$

### Solving a 3-by-3 System of Linear Equations Using Matrices

We have seen how to write a system of equations with an augmented matrix, and then how to use row operations and back-substitution to obtain row-echelon form. Now, we will take row-echelon form a step farther to solve a 3 by 3 system of linear equations. The general idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

### Example 8

Solve the system of linear equations using matrices.

$$x - y + z = 8$$

$$2x + 3y - z = -2$$

$$3x - 2y - 9z = 9$$

First, we write the augmented matrix.

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array} \right]$$

Next, we perform row operations to obtain row-echelon form.

$$-2R_1 + R_2 = R_2 \rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 3 & -2 & -9 & 9 \end{array} \right]$$

$$-3R_1 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{array} \right]$$

The easiest way to obtain a 1 in row 2 of column 1 is to interchange  $R_2$  and  $R_3$ .

$$\text{Interchange } R_2 \text{ and } R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array} \right]$$

Then

$$-5R_2 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array} \right]$$

$$\frac{1}{57}R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

The last matrix represents the equivalent system.

$$x - y + z = 8$$

$$y - 12z = -15$$

$$z = 1$$

Using back-substitution, we obtain the solution as  $(4, -3, 1)$ .

## Try it Now

5. Solve the system using matrices.

$$x + 4y - z = 4$$

$$2x + 5y + 8z = 15$$

$$x + 3y - 3z = 1$$

Using these approaches, a system of linear equations of any size can be solved by Gaussian elimination. One of the real benefits of Gaussian elimination is that it is very procedural, so it is something we can teach a computer or calculator to do.

## Example 9

Peter is planning on investing in what's called a three fund portfolio, consisting of a U.S. stock mutual fund, an international stock mutual fund, and a bond mutual fund. He has a total of \$200,000 to invest, and wants four times as much invested in stocks as in bonds. The bond fund has a historical return of 4.4%, the U.S. stock fund has a historical return of 8.3%, and the international stock fund has a historical return of 5.4%. If Peter is hoping for a \$13,300 return, how should he allocate his investment?

Of course, historical returns cannot predict future earnings, but they're the best information we have, so we'll use them.

We have a system of three equations in three variables.

let  $x$  be the amount invested in U.S. stocks at 8.3% return, and

let  $y$  be the amount invested in international stock at 5.4% return.

Let  $z$  be the amount invested in the bond fund at 4.4% return,

We have a total of \$200,000 to invest, so

$$x + y + z = 200,000$$

He wants four times as much invested in stocks as in bonds, so

$$x + y = 4z$$

$$x + y - 4z = 0$$

And using the return rates and the desired earnings,

$$0.083x + 0.054y + 0.044z = 13,300$$

We can take this system and put it into an augmented matrix.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 200,000 \\ 1 & 1 & -4 & 0 \\ 0.083 & 0.054 & 0.044 & 13,300 \end{array} \right].$$

Now, we perform Gaussian elimination to achieve row-echelon form.

$$-R_1 + R_2 = R_2 \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 200,000 \\ 0 & 0 & -5 & -200,000 \\ 0.083 & 0.054 & 0.044 & 13,300 \end{array} \right]$$

$$-0.083R_1 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 200,000 \\ 0 & 0 & -5 & -200,000 \\ 0 & -0.029 & -0.039 & -3,300 \end{array} \right]$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 200,000 \\ 0 & -0.029 & -0.039 & -3,300 \\ 0 & 0 & -5 & -200,000 \end{array} \right]$$

$$\frac{1}{-0.029} R_2 = R_2 \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 200,000 \\ 0 & 1 & \frac{39}{29} & \frac{3,300,000}{29} \\ 0 & 0 & -5 & -200,000 \end{array} \right]$$

$$\frac{1}{-5} R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 200,000 \\ 0 & 1 & \frac{39}{29} & \frac{3,300,000}{29} \\ 0 & 0 & 1 & 40,000 \end{array} \right]$$

The third row tells us  $z = 40,000$ .

The second row tells us  $y + \frac{39}{29}z = \frac{3,300,000}{29}$ , or  $29y + 39z = 3,300,000$ . Substituting

$z = 40,000$  we get

$$29y + 39(40,000) = 3,300,000$$

$$29y + 1,560,000 = 3,300,000$$

$$29y = 1,740,000$$

$$y = 60,000$$

The first row tells us  $x + y + z = 200,000$ . Substituting  $y = 60,000$  and  $z = 40,000$  we get

$$x + 60,000 + 40,000 = 200,000$$

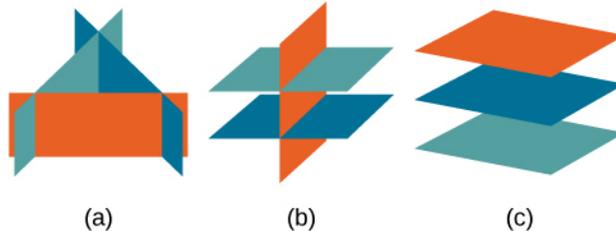
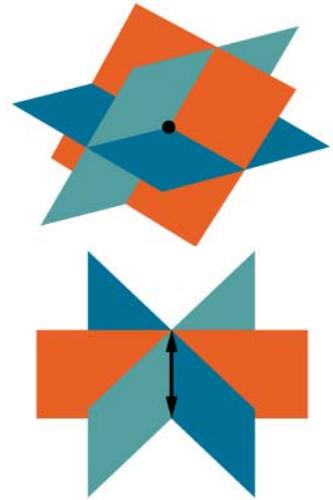
$$x = 100,000$$

The answer is \$100,000 invested in U.S. stock, \$60,000 in international stock, and \$40,000 in bonds.

**Number of Possible Solutions: Dependent and Inconsistent Systems**

With three-by-three systems, the possibilities for number of solutions are:

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple  $(x, y, z)$ . Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after Gaussian elimination, have a row of all zeros, equivalent to the statement  $0 = 0$ . Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after Gaussian elimination, have a row of all zeros in the coefficient matrix and a nonzero-entry on the right hand side, equivalent to a statement such as  $0 = 3$ . Graphically, a system with no solution is represented by three planes with no point in common. There are several ways this can happen, as shown below.



All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection

**Example 10**

Solve the following system of linear equations using matrices.

$$-x - 2y + z = -1$$

$$2x + 3y = 2$$

$$y - 2z = 0$$

Write the augmented matrix.

$$\left[ \begin{array}{ccc|c} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array} \right]$$

First, multiply row 1 by  $-1$  to get a 1 in row 1, column 1. Then, perform row operations to obtain row-echelon form.

$$-R_1 \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array} \right]$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & 2 \end{array} \right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right]$$

$$R_2 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The last matrix represents the following system.

$$x + 2y - z = 1$$

$$y - 2z = 0$$

$$0 = 0$$

We see by the identity  $0 = 0$  that this is a dependent system with an infinite number of solutions. We then find the generic solution. By solving the second equation for  $y$  and substituting it into the first equation we can solve for  $z$  in terms of  $x$ .

$$x + 2y - z = 1$$

$$y = 2z$$

$$x + 2(2z) - z = 1$$

$$x + 3z = 1$$

$$z = \frac{1-x}{3}$$

Now we substitute the expression for  $z$  into the second equation to solve for  $y$  in terms of  $x$ .

$$y - 2z = 0$$

$$z = \frac{1-x}{3}$$

$$y - 2\left(\frac{1-x}{3}\right) = 0$$

$$y = \frac{2-2x}{3}$$

The generic solution is  $\left(x, \frac{2-2x}{3}, \frac{1-x}{3}\right)$ .

### Important Topics of this Section

- Writing a system as an augmented matrix
- Row reduction (Gaussian elimination)
- Reading the solutions from an augmented matrix
- Recognizing dependent and inconsistent systems

### Try it Now Answers

1.  $\left[ \begin{array}{cc|c} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array} \right]$

$$x - y + z = 5$$

2.  $2x - y + 3z = 1$

$$y + z = -9$$

3. (2,1)

4.  $\left[ \begin{array}{ccc|c} 1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{array} \right]$

5. (1, 1, 1)

## Section 2.3: Matrix Operations

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. The table below shows the needs of both teams.

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

A goal costs \$300; a ball costs \$10; and a jersey costs \$30. How can we find the total cost for the equipment needed for each team?

In the last section we learned how matrices can be used to solve systems of linear equations. In this section, we will explore other uses of matrices, and discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment in a way that can be easily translated to a computer or calculator.

### Finding the Sum and Difference of Two Matrices

To solve a problem like the one in the section opener, we can use a **matrix**, which is a rectangular array of numbers. A **row** in a matrix is a set of numbers that are aligned horizontally. A **column** in a matrix is a set of numbers that are aligned vertically. Each number is an **entry**, sometimes called an element, of the matrix. Matrices (plural) are enclosed in  $[ ]$  or  $( )$ , and are usually named with capital letters. For example, three matrices named  $A$ ,  $B$ , and  $C$  are shown below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$$

### Describing Matrices

A matrix is often referred to by its size or dimensions:  $m \times n$  indicating  $m$  rows and  $n$  columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix  $A$  identified as  $a_{ij}$ , we look for the entry in row  $i$ , column  $j$ .

In matrix  $A$ , shown below, the entry in row 2, column 3 is  $a_{23}$ .

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A square matrix is a matrix with dimensions  $n \times n$ , meaning that it has the same number of rows as columns. The  $3 \times 3$  matrix above is an example of a square matrix.

A row matrix is a matrix consisting of one row with dimensions  $1 \times n$ .

$$[a_{11} \quad a_{12} \quad a_{13}]$$

A column matrix is a matrix consisting of one column with dimensions  $m \times 1$ .

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic matrix operations.

### Matrices

A **matrix** is a rectangular array of numbers that is usually named by a capital letter:  $A$ ,  $B$ ,  $C$ , and so on. Each **entry** in a matrix is referred to as  $a_{ij}$ , such that  $i$  represents the **row** and  $j$  represents the **column**. Matrices are often referred to by their dimensions:  $m \times n$  indicating  $m$  rows and  $n$  columns.

### Example 1

Given matrix  $A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$

- What are the dimensions of matrix  $A$ ?
- What are the entries at  $a_{31}$  and  $a_{22}$ ?

a) The dimensions are  $3 \times 2$  because there are three rows and two columns.

b) Entry  $a_{31}$  is the number at row 3, column 1, which is 3. The entry  $a_{22}$  is the number at row 2, column 2, which is 4. Remember, the row comes first, then the column.

### Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries.

In order to do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a  $3 \times 3$  matrix and another  $3 \times 3$  matrix, but we cannot add or subtract a  $2 \times 3$  matrix and a  $3 \times 3$  matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

### Adding and Subtracting Matrices

Given matrices  $A$  and  $B$  of like dimensions, addition and subtraction of  $A$  and  $B$  will produce matrix  $C$  or matrix  $D$  of the same dimension.

$$A + B = C \text{ such that } a_{ij} + b_{ij} = c_{ij}$$

$$A - B = D \text{ such that } a_{ij} - b_{ij} = d_{ij}$$

Matrix addition is commutative:  $A + B = B + A$

It is also associative:  $(A + B) + C = A + (B + C)$

### Example 2

Given  $A$  and  $B$ :

$$A = \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix}$$

- Find the sum.
- Find the difference.

a) Add the corresponding entries.

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2+6 & -10+10 & -2-2 \\ 14+0 & 12-12 & 10-4 \\ 4-5 & -2+2 & 2-2 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 0 & -4 \\ 14 & 0 & 6 \\ -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

b) Subtract the corresponding entries.

$$\begin{aligned}
 A - B &= \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix} \\
 &= \begin{bmatrix} 2-6 & -10-10 & -2+2 \\ 14-0 & 12+12 & 10+4 \\ 4+5 & -2-2 & 2+2 \end{bmatrix} \\
 &= \begin{bmatrix} -4 & -20 & 0 \\ 14 & 24 & 14 \\ 9 & -4 & 4 \end{bmatrix}
 \end{aligned}$$

### Try it Now

1. Add matrix  $A$  and matrix  $B$ .

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$$

### Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A **scalar multiple** is any entry of a matrix that results from scalar multiplication.

### Scalar Multiplication

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple  $cA$  is

$$\begin{aligned}
 cA &= c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
 &= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}
 \end{aligned}$$

Scalar multiplication is distributive. For the matrices  $A$ ,  $B$ , and  $C$  with scalars  $a$  and  $b$ ,

$$a(A + B) = aA + aB$$

$$(a + b)A = aA + bA$$

### Example 3

Multiply matrix  $A$  by the scalar 3.

$$A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Multiply each entry in  $A$  by the scalar 3.

$$3A = 3 \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 8 & 3 \cdot 1 \\ 3 \cdot 5 & 3 \cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 3 \\ 15 & 12 \end{bmatrix}$$

### Try it Now

2. Given matrix  $B$ , find  $-2B$  where

$$B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

### Example 4

Find the sum  $3A + 2B$ .

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$$

First, find  $3A$ , then  $2B$ .

$$\begin{aligned}
 3A &= \begin{bmatrix} 3 \cdot 1 & 3(-2) & 3 \cdot 0 \\ 3 \cdot 0 & 3(-1) & 3 \cdot 2 \\ 3 \cdot 4 & 3 \cdot 3 & 3(-6) \end{bmatrix} \\
 &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 2B &= \begin{bmatrix} 2(-1) & 2 \cdot 2 & 2 \cdot 1 \\ 2 \cdot 0 & 2(-3) & 2 \cdot 2 \\ 2 \cdot 0 & 2 \cdot 1 & 2(-4) \end{bmatrix} \\
 &= \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix}
 \end{aligned}$$

Now, add  $3A + 2B$ .

$$\begin{aligned}
 3A + 2B &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} \\
 &= \begin{bmatrix} 3-2 & -6+4 & 0+2 \\ 0+0 & -3-6 & 6+4 \\ 12+0 & 9+2 & -18-8 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & -2 & 2 \\ 0 & -9 & 10 \\ 12 & 11 & -26 \end{bmatrix}
 \end{aligned}$$

### Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If  $A$  is an  $m \times r$  matrix and  $B$  is an  $r \times n$  matrix, then the product matrix  $AB$  is an  $m \times n$  matrix. For example, the product  $AB$  is possible because the number of columns in  $A$  is the same as the number of rows in  $B$ . If the inner dimensions do not match, the product is not defined.

$$\begin{array}{ccc}
 A & \cdot & B \\
 2 \times 3 & & 3 \times 3 \\
 & \underbrace{\hspace{2cm}} & \\
 & \text{same} &
 \end{array}$$

We multiply entries of  $A$  with entries of  $B$  according to a specific pattern as outlined below. The process of matrix multiplication becomes clearer when working a problem with real numbers.

To obtain the entries in row  $i$  of  $AB$ , we multiply the entries in row  $i$  of  $A$  by column  $j$  in  $B$  and add. For example, given matrices  $A$  and  $B$ , where the dimensions of  $A$  are  $2 \times 3$  and the dimensions of  $B$  are  $3 \times 3$ , the product of  $AB$  will be a  $2 \times 3$  matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

### Multiplying Matrices

To obtain the entry in row 1, column 1 of  $AB$ , multiply the first row in  $A$  by the first column in  $B$ , and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$$

To obtain the entry in row 1, column 2 of  $AB$ , multiply the first row of  $A$  by the second column in  $B$ , and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$$

To obtain the entry in row 1, column 3 of  $AB$ , multiply the first row of  $A$  by the third column in  $B$ , and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$$

We proceed the same way to obtain the second row of  $AB$ . In other words, row 2 of  $A$  times column 1 of  $B$ ; row 2 of  $A$  times column 2 of  $B$ ; row 2 of  $A$  times column 3 of  $B$ . When complete, the product matrix will be

$$AB = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \end{bmatrix}$$

### Properties of Matrix Multiplication

For the matrices  $A$ ,  $B$ , and  $C$  the following properties hold.

- Matrix multiplication is associative:  $(AB)C = A(BC)$ .
- Matrix multiplication is distributive:  $C(A+B) = CA + CB$ ,  
 $(A+B)C = AC + BC$ .

Note that matrix multiplication is **not** commutative.

### Example 5

Multiply matrix  $A$  and matrix  $B$ .

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

First, we check the dimensions of the matrices. Matrix  $A$  has dimensions  $2 \times 2$  and matrix  $B$  has dimensions  $2 \times 2$ . The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions  $2 \times 2$ .

We perform the operations outlined previously.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

### Example 6

Given  $A$  and  $B$ :

a) Find  $AB$ .

b) Find  $BA$ .

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$

a) As the dimensions of  $A$  are  $2 \times 3$  and the dimensions of  $B$  are  $3 \times 2$ , these matrices can be multiplied together because the number of columns in  $A$  matches the number of rows in  $B$ . The resulting product will be a  $2 \times 2$  matrix, the number of rows in  $A$  by the number of columns in  $B$ .

$$\begin{aligned}
 AB &= \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} -1(5) + 2(-4) + 3(2) & -1(-1) + 2(0) + 3(3) \\ 4(5) + 0(-4) + 5(2) & 4(-1) + 0(0) + 5(3) \end{bmatrix} \\
 &= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix}
 \end{aligned}$$

b) The dimensions of  $B$  are  $3 \times 2$  and the dimensions of  $A$  are  $2 \times 3$ . The inner dimensions match so the product is defined and will be a  $3 \times 3$  matrix.

$$\begin{aligned}
 BA &= \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 5(-1) + (-1)(4) & 5(2) + (-1)(0) & 5(3) + (-1)(5) \\ -4(-1) + 0(4) & -4(2) + 0(0) & -4(3) + 0(5) \\ 2(-1) + 3(4) & 2(2) + 3(0) & 2(3) + 3(5) \end{bmatrix} \\
 &= \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix}
 \end{aligned}$$

Notice in the example above that the products  $AB$  and  $BA$  are not equal.

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA$$

This illustrates the fact that matrix multiplication is not commutative.

Also, this means that it is possible for  $AB$  to be defined but not  $BA$ . Consider a matrix  $A$  with dimension  $3 \times 4$  and matrix  $B$  with dimension  $4 \times 2$ . For the product  $AB$  the inner dimensions are 4 and the product is defined, but for the product  $BA$  the inner dimensions are 2 and 3 so the product is undefined.

### Try it Now

$$3. \text{ Multiply: } \begin{bmatrix} 1 & -2 \\ 4 & 0 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

**Example 7**

Let's return to the problem presented at the opening of this section. We have the table below representing the equipment needs of two soccer teams.

	<b>Wildcats</b>	<b>Mud Cats</b>
<b>Goals</b>	6	10
<b>Balls</b>	30	24
<b>Jerseys</b>	14	20

We are also given the prices of the equipment, as shown in below.

<b>Goal</b>	\$300
<b>Ball</b>	\$10
<b>Jersey</b>	\$30

We will convert the data to matrices. Thus, the equipment need matrix is written as

$$E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}$$

The cost matrix is written as

$$C = [300 \quad 10 \quad 30]$$

We perform matrix multiplication to obtain costs for the equipment.

$$\begin{aligned} CE &= [300 \quad 10 \quad 30] \cdot \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= [300(6) + 10(30) + 30(14) \quad 300(10) + 10(24) + 30(20)] \\ &= [2,520 \quad 3,840] \end{aligned}$$

The total cost for equipment for the Wildcats is \$2,520, and the total cost for equipment for the Mud Cats is \$3,840.

The calculation in the last example could easily be calculated using a calculator and computer, which would also easily handle a version of the problem with dozens of teams and hundreds of expenses. Representing the problem as matrix operations allows us to utilize technology to help solve this type of problem.

### Important Topics of this Section

Size of a matrix  
 Sum and difference of matrices  
 Scalar multiple of a matrix  
 Product of matrices

### Try it Now Answers

$$1. A + B = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 2+3 & 6+(-2) \\ 1+1 & 0+5 \\ 1+(-4) & -3+3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & 5 \\ -3 & 0 \end{bmatrix}$$

$$2. -2B = \begin{bmatrix} -8 & -2 \\ -6 & -4 \end{bmatrix}$$

$$3. \begin{bmatrix} 1 & -2 \\ 4 & 0 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1(2)+(-2)(1) & 1(3)+(-2)(-1) \\ 4(2)+0(1) & 4(3)+0(-1) \\ (-1)(2)+(3)(1) & (-1)(3)+(3)(-1) \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 8 & 12 \\ 1 & -6 \end{bmatrix}$$

## Section 2.4: Solving Systems with Inverses

Nancy plans to invest \$10,500 into two different bonds to spread out her risk. The first bond has an annual return of 10%, and the second bond has an annual return of 6%. In order to receive an 8.5% return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem?

There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

### Finding the Inverse of a Matrix

The multiplicative inverse of a real number  $a$  is  $a^{-1}$ , where the product is 1:

$aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1$ . For example,  $2^{-1} = \frac{1}{2}$  and  $\left(\frac{1}{2}\right)2 = 1$ . The **multiplicative inverse**

**of a matrix** is similar in concept, except that the product of matrix  $A$  and its inverse  $A^{-1}$  equals the **identity matrix**. The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by  $I_n$  where  $n$  represents the dimension of the matrix. Shown below are the identity matrices for a  $2 \times 2$  matrix and a  $3 \times 3$  matrix, respectively.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix acts as a 1 in matrix algebra. For example,  $AI = IA = A$ . A matrix that has a multiplicative inverse has the properties

$$AA^{-1} = I$$

$$A^{-1}A = I$$

A matrix that has a multiplicative inverse is called an invertible matrix. Only a square matrix may have a multiplicative inverse, as the reversibility,  $AA^{-1} = A^{-1}A = I$ , is a requirement. Not all square matrices have an inverse, but if  $A$  is invertible, then  $A^{-1}$  is unique. We will look at two methods for finding the inverse of a  $2 \times 2$  matrix and a third method that can be used on both  $2 \times 2$  and  $3 \times 3$  matrices.

### The Identity Matrix and Multiplicative Inverse

The **identity matrix**,  $I_n$ , is a square matrix containing ones down the main diagonal and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$2 \times 2$                        $3 \times 3$

If  $A$  is an  $n \times n$  matrix and  $B$  is an  $n \times n$  matrix such that  $AB = BA = I_n$ , then  $B = A^{-1}$ , the **multiplicative inverse of a matrix**  $A$ .

#### Example 1

Given matrix  $A$ , show that  $AI = IA = A$ .

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Use matrix multiplication to show that the product of  $A$  and the identity is equal to the product of the identity and  $A$ .

$$AI = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Given two matrices, show that one is the multiplicative inverse of the other.

1. Given matrix  $A$  of order  $n \times n$  and matrix  $B$  of order  $n \times n$ , multiply  $AB$ .
2. If  $AB = I$ , then find the product  $BA$ . If  $BA = I$ , then  $B = A^{-1}$  and  $A = B^{-1}$ .

#### Example 2

Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

Multiply  $AB$  and  $BA$ . If both products equal the identity, then the two matrices are inverses of each other.

$$\begin{aligned}
 AB &= \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \cdot \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1(-9) + 5(2) & 1(-5) + 5(1) \\ -2(-9) - 9(2) & -2(-5) - 9(1) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \\
 &= \begin{bmatrix} -9(1) - 5(-2) & -9(5) - 5(-9) \\ 2(1) + 1(-2) & 2(-5) + 1(-9) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

$A$  and  $B$  are inverses of each other.

### Try it Now

1. Show that the following two matrices are inverses of each other.

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

### Finding the Multiplicative Inverse by Augmenting with the Identity

One way to find the multiplicative inverse is by augmenting with the identity. When matrix  $A$  is transformed into  $I$ , the augmented matrix  $I$  transforms into  $A^{-1}$ .

For example, given

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

augment  $A$  with the identity

$$\left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array} \right]$$

Perform row operations with the goal of turning  $A$  into the identity.

1. Switch row 1 and row 2.

$$\left[ \begin{array}{cc|cc} 5 & 3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array} \right]$$

2. Multiply row 2 by  $-2$  and add to row 1.

$$\left[ \begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 2 & 1 & 1 & 0 \end{array} \right]$$

3. Multiply row 1 by  $-2$  and add to row 2.

$$\left[ \begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 0 & -1 & 5 & -2 \end{array} \right]$$

4. Add row 2 to row 1.

$$\left[ \begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & -1 & 5 & -2 \end{array} \right]$$

5. Multiply row 2 by  $-1$ .

$$\left[ \begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array} \right]$$

The matrix we have found is  $A^{-1}$ .

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

### Finding the Multiplicative Inverse of $2 \times 2$ Matrices Using a Formula

When we need to find the multiplicative inverse of a  $2 \times 2$  matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

#### Formula for the Inverse of a $2$ by $2$ Matrix

If  $A$  is a  $2 \times 2$  matrix, such as  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

the multiplicative inverse of  $A$  is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where  $ad - bc \neq 0$ . If  $ad - bc = 0$ , then  $A$  has no inverse.

## Example 3

Use the formula to find the multiplicative inverse of

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

Using the formula, we have

$$\begin{aligned} A^{-1} &= \frac{1}{(2)(3) - (5)(1)} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \\ &= \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \end{aligned}$$

Notice this matches the inverse we found above by augmenting with the identity matrix.

## Try it Now

2. Use the formula to find the inverse of matrix  $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Verify your answer by augmenting with the identity matrix.

## Example 4

Find the inverse, if it exists, of the given matrix:  $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

We will use the method of augmenting with the identity.

$$\left[ \begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right]$$

Switch row 1 and row 2.

$$\left[ \begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{array} \right]$$

Multiply row 1 by  $-3$  and add it to row 2.

$$\left[ \begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 0 & -3 & 1 \end{array} \right]$$

There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

**Finding the Multiplicative Inverse of  $3 \times 3$  Matrices**

Unfortunately, we do not have a formula similar to the one for a  $2 \times 2$  matrix to find the inverse of a  $3 \times 3$  matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a  $3 \times 3$  matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

augment  $A$  with the identity matrix

$$A|I = \left[ \begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array} \right]$$

To begin, we write the augmented matrix with the identity on the right and  $A$  on the left. Performing elementary row operations so that the identity matrix appears on the left, we will obtain the inverse matrix on the right. We will find the inverse of this matrix in the next example.

Given a 3 by 3 matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that  $AA^{-1} = I$  and  $A^{-1}A = I$ .

**Example 5**

Given the  $3 \times 3$  matrix  $A$ , find the inverse.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

Augment  $A$  with the identity matrix, and then begin row operations until the identity matrix replaces  $A$ . The matrix on the right will be the inverse of  $A$ .

$$\left[ \begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{Interchange } R_2 \text{ and } R_1} \left[ \begin{array}{ccc|ccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$-R_2 + R_1 = R_1 \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$-R_2 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array} \right]$$

$$R_3 \leftrightarrow R_2 \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array} \right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array} \right]$$

$$-3R_2 + R_3 = R_3 \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array} \right]$$

Thus,

$$A^{-1} = B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}.$$

To prove that  $B = A^{-1}$ , let's multiply the two matrices together to see if the product equals the identity, if  $AA^{-1} = I$  and  $A^{-1}A = I$ .

$$\begin{aligned}
 AA^{-1} &= \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \\
 &= \begin{bmatrix} 2(-1)+3(-1)+1(6) & 2(1)+3(0)+1(-2) & 2(0)+3(1)+1(-3) \\ 3(-1)+3(-1)+1(6) & 3(1)+3(0)+1(-2) & 3(0)+3(1)+1(-3) \\ 2(-1)+4(-1)+1(6) & 2(1)+4(0)+1(-2) & 2(0)+4(1)+1(-3) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 A^{-1}A &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -1(2)+1(3)+0(2) & -1(3)+1(3)+0(4) & -1(1)+1(1)+0(1) \\ -1(2)+0(3)+1(2) & -1(3)+0(3)+1(4) & -1(1)+0(1)+1(1) \\ 6(2)+-2(3)+-3(2) & 6(3)+-2(3)+-3(4) & 6(1)+-2(1)+-3(1) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

### Try it Now

3. Find the inverse of the  $3 \times 3$  matrix.

$$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

### Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices:  $X$  is the matrix representing the variables of the system, and  $B$  is the matrix representing the constants. Using matrix multiplication, we may define a system of equations with the same number of equations as variables as

$$AX = B$$

To solve a system of linear equations using an inverse matrix, let  $A$  be the coefficient matrix, let  $X$  be the variable matrix, and let  $B$  be the constant matrix. Thus, we want to solve a system  $AX = B$ . For example, look at the following system of equations.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

From this system, the coefficient matrix is

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

And the constant matrix is

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then  $AX = B$  looks like

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solving this system of equations written in matrix form is similar to solving a linear equation, except there is no matrix “division” so instead we need to use the inverse. The goal is the same—to isolate the variable. We will investigate this idea in detail, but it is helpful to begin with a  $2 \times 2$  system and then move on to a  $3 \times 3$  system.

### Solving a System of Equations Using the Inverse of a Matrix

Given a system of equations, write the coefficient matrix  $A$ , the variable matrix  $X$ , and the constant matrix  $B$ . Then

$$AX = B$$

Multiply both sides by the inverse of  $A$  to obtain the solution.

$$(A^{-1})AX = (A^{-1})B$$

$$[(A^{-1})A]X = (A^{-1})B$$

$$IX = (A^{-1})B$$

$$X = (A^{-1})B$$

**Important:** If the coefficient matrix does not have an inverse, the system could be inconsistent and have no solution, **or** be dependent and have infinitely many solutions.

### Example 6

Solve the given system of equations using the inverse of a matrix.

$$3x + 8y = 5$$

$$4x + 11y = 7$$

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Using the formula to calculate  $A^{-1}$ , we have:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{3(11) - 8(4)} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by  $A^{-1}$ .

$$(A^{-1})AX = (A^{-1})B$$

$$\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11(5) + (-8)7 \\ -4(5) + 3(7) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is  $(-1, 1)$ .

Note that you cannot solve for  $X$  by finding the product  $BA^{-1}$ . Recall that matrix multiplication is not commutative, so  $A^{-1}B \neq BA^{-1}$ . Consider our steps for solving the matrix equation.

$$\begin{aligned}(A^{-1})AX &= (A^{-1})B \\ [(A^{-1})A]X &= (A^{-1})B \\ IX &= (A^{-1})B \\ X &= (A^{-1})B\end{aligned}$$

Notice in the first step we multiplied both sides of the equation by  $A^{-1}$ , but the  $A^{-1}$  was to the left of  $A$  on the left side and to the left of  $B$  on the right side. Because matrix multiplication is not commutative, order matters.

### Example 7

Solve the following system using the inverse of a matrix.

$$5x + 15y + 56z = 35$$

$$-4x - 11y - 41z = -26$$

$$-x - 3y - 11z = -7$$

Write the equation  $AX = B$ .

$$\begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

First, we will find the inverse of  $A$  by augmenting with the identity.

$$\left[ \begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array} \right]$$

Multiply row 1 by  $\frac{1}{5}$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array} \right]$$

Multiply row 1 by 4 and add to row 2.

$$\left[ \begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array} \right]$$

Add row 1 to row 3.

$$\left[ \begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array} \right]$$

Multiply row 2 by  $-3$  and add to row 1.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array} \right]$$

Multiply row 3 by 5.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array} \right]$$

Multiply row 3 by  $\frac{1}{5}$  and add to row 1.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array} \right]$$

Multiply row 3 by  $-\frac{19}{5}$  and add to row 2.

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array} \right]$$

So,

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix}.$$

Multiply both sides of the equation by  $A^{-1}$ . We want  $A^{-1}AX = A^{-1}B$ :

$$\begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}.$$

Thus,

$$A^{-1}B = \begin{bmatrix} -70+78-7 \\ -105-26+133 \\ 35+0-35 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

The solution is  $(1, 2, 0)$ .

### Try it Now

4. Solve the system using the inverse of the coefficient matrix.

$$2x - 17y + 11z = 0$$

$$-x + 11y - 7z = 8$$

$$3y - 2z = -2$$

Systems of equations in many variables can get very complex, but solving using matrices can easily be handled by calculators or computers.

Given a system of equations, solve with matrix inverses using a calculator.

1. Save the coefficient matrix and the constant matrix as matrix variables  $[A]$  and  $[B]$ .
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

### Important Topics of this Section

Finding the inverse of a matrix

Using the formula for 2 by 2 matrices

By augmenting with the identity and row reducing

Using the inverse to solve a system of equations

### Try it Now Answers

$$1. \quad AB = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(-3)+4(1) & 1(-4)+4(1) \\ -1(-3)+-3(1) & -1(-4)+-3(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -3(1)+-4(-1) & -3(4)+-4(-3) \\ 1(1)+1(-1) & 1(4)+1(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2. \quad A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$3. \quad A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

$$4. \quad X = \begin{bmatrix} 4 \\ 38 \\ 58 \end{bmatrix}, \text{ so } x = 4, y = 38, z = 58$$

## Chapter 3: Linear Programming

Section 3.1: Inequalities in One Variable .....	115
Section 3.2: Linear Inequalities .....	125
Section 3.3: Graphical Solutions.....	131
Section 3.4: Simplex Method.....	140
Section 3.5: Applications of Linear Programming .....	146

### Section 3.1: Inequalities in One Variable

When learning about domain and range, you learned about inequalities and using set-builder and interval notation to represent them. In this section, we will explore how to solve linear and absolute value inequalities in one variable. The process is very similar to solve equations, but instead of the solution being a single value, the solution will be an inequality.

Notice that if an inequality is true, like  $2 < 5$ , then these operations result in a true statement as well, just like with equations:

Adding a number to both sides:

$$2 + 4 < 5 + 4 \qquad 6 < 9 \qquad \text{True}$$

Subtracting a number on both sides:

$$2 - 3 < 5 - 3 \qquad -1 < 2 \qquad \text{True}$$

Multiplying a positive number on both sides:

$$2(3) < 5(3) \qquad 6 < 15 \qquad \text{True}$$

Dividing by a positive number on both sides:

$$2/2 < 5/2 \qquad 1 < 2.5 \qquad \text{True}$$

We can use these operations just like when solving equations.

#### Example 1

Solve  $3x + 7 \geq 1$

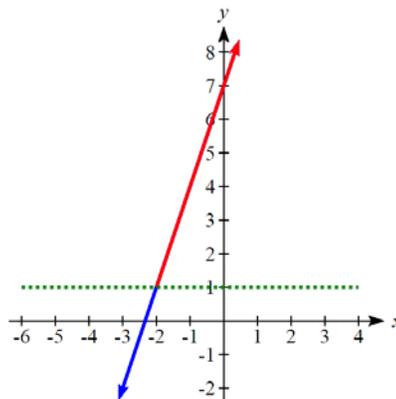
$$3x + 7 \geq 1 \qquad \text{Subtract 7 from both sides}$$

$$3x \geq -6 \qquad \text{Divide both sides by 3}$$

$$x \geq -2$$

This inequality represents the solution set. It tells us that all numbers greater than or equal to -2 will satisfy the original inequality. We could also write this solution in interval notation, as  $[-2, \infty)$ .

To understand what is happening, we could also consider the problem graphically. If we were to graph the equation  $y = 3x + 7$ , then solving  $3x + 7 \geq 1$  would correspond with asking “for what values of  $x$  is  $y \geq 1$ ”. Notice that the part of the graph where this is true corresponds to where  $x \geq -2$ .



While most operations in solving inequalities are the same as in solving equations, we run into a problem when multiplying or dividing both sides by a negative number. Notice, for example:

$$2(-3) < 5(-3) \quad -6 < -15 \quad \text{Not True}$$

To account for this, when multiplying or dividing by a negative number, we must reverse the sign of the inequality.

### Rules for Solving Linear Inequalities

- 1) You may add or subtract a positive or negative number to both sides of the inequality.
- 2) You can multiply or divide both sides of the inequality by a positive number.
- 3) You can multiply or divide both sides of the inequality by a negative number, but you must reverse the direction of the inequality.

### Example 2

Solve  $12 - 4x < 6$

$$12 - 4x < 6$$

Subtract 12 from both sides

$$-4x < -6$$

Divide both sides by  $-4$ , and reverse the inequality direction

$$x > \frac{-6}{-4}$$

Simplify

$$x > \frac{3}{2}$$

### Try it Now

1. Solve:  $6 + 2x \leq 18 + 5x$

**Example 3**

A company spends \$1200 per day on overhead and labor, and each item they produce costs \$5 for materials. If they sell the items for \$15 each, how many items will they need to sell each day for their profits to be positive?

While we could solve this problem using equations, it also lends itself to inequalities, since we want the profit to be positive:  $P > 0$ .

Costs:  $C(q) = 1200 + 5q$

Revenue:  $R(q) = 10q$

Profit:  $P(q) = 10q - (1200 + 5q) = 5q - 1200$

Solving  $P(q) > 0$ :

$$5q - 1200 > 0$$

$$5q > 1200$$

$$q > 240$$

The company will need to sell at least 240 items a day to make a profit.

**Compound Inequalities**

Compound inequalities are inequalities that consist of more than one part. The most common type is called a tripartite inequality. The basic version looks like this:

$$-1 < 3x + 5 < 14.$$

When we write these it is important that both inequalities point in the same direction and that the “outside” inequality is also true – in this case  $-1 < 14$  is true, so this is valid. Expressions like  $10 < x < 2$  and  $1 < x > 5$  are *not* valid notation.

The most universal way to solve a tripartite inequality is to:

- 1) Break it into two separate inequalities
- 2) Solve each inequality separately
- 3) Combine the solutions if possible.

**Example 4**

Solve  $-1 < -3x + 5 < 14$

First we separate this into two inequalities:

$$-1 < -3x + 5 \quad \text{and} \quad -3x + 5 < 14$$

Now we solve each:

$$-6 < -3x \quad \text{and} \quad -3x < 9$$

$$2 > x \quad \text{and} \quad x > -3$$

Now we can combine these solution sets. The numbers where both  $2 > x$  and  $x > -3$  are true is the set:

$$2 > x > -3$$

While this solution is valid and correct, it is more common to write the solution to tripartite inequalities with the smaller number on the left. We could rewrite the solution as:

$$-3 < x < 2$$

This also has the advantage of corresponding better with the answer in interval notation:  $(-3, 2)$

With this particular inequality, it would also be possible to skip the step of breaking it apart, and instead just subtract 5 from all three “parts” of the inequality. This works for simple problems like this, but may fail if the inequality has variables in more than one “part” of the inequality.

### Try it Now

2. Solve:  $4 \leq 2x + 6 < 16$

## Absolute Value

So far in this section we have been looking at inequalities that are linear. We will now turn to absolute value inequalities. The absolute value function is a piecewise-defined function made up of two linear functions.

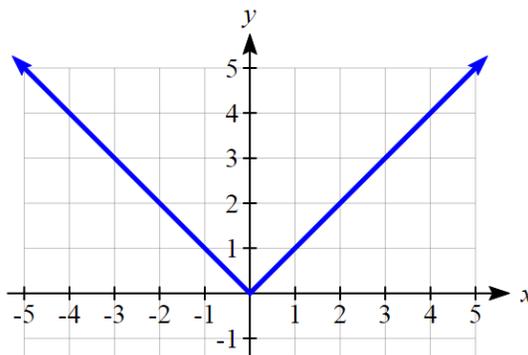
### Absolute Value Function

The absolute value function can be defined as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph of the absolute value looks like a V:

The absolute value function is commonly used to determine the distance between two numbers on the number line. Given two values  $a$  and  $b$ , then  $|a - b|$  will give the distance, a positive quantity, between these values, regardless of which value is larger.



**Example 5**

Describe all values,  $x$ , within a distance of 4 from the number 5.

We want the distance between  $x$  and 5 to be less than or equal to 4. The distance can be represented using the absolute value, giving the expression

$$|x - 5| \leq 4$$

**Example 6**

A 2010 poll reported 78% of Americans believe that people who are gay should be able to serve in the US military, with a reported margin of error of 3%<sup>1</sup>. The margin of error tells us how far off the actual value could be from the survey value<sup>2</sup>. Express the set of possible values using absolute values.

Since we want the size of the difference between the actual percentage,  $p$ , and the reported percentage to be less than 3%,

$$|p - 78| \leq 3$$

**Try it Now**

3. Students who score within 20 points of 80 will pass the test. Write this as a distance from 80 using the absolute value notation.

**Solving Absolute Value Equations**

To solve an equation like  $8 = |2x - 6|$ , we can notice that the absolute value will be equal to eight if the quantity *inside* the absolute value were 8 or -8. This leads to two different equations we can solve independently:

$$\begin{array}{l} 2x - 6 = 8 \quad \text{or} \quad 2x - 6 = -8 \\ 2x = 14 \quad \quad \quad 2x = -2 \\ x = 7 \quad \quad \quad \quad x = -1 \end{array}$$

**Solutions to Absolute Value Equations**

An equation of the form  $|A| = B$ , with  $B \geq 0$ , will have solutions when

$$A = B \text{ or } A = -B$$

<sup>1</sup> <http://www.pollingreport.com/civil.htm>, retrieved August 4, 2010

<sup>2</sup> Technically, margin of error usually means that the surveyors are 95% confident that actual value falls within this range.

## Example 7

Solve:  $0 = |4x + 1| - 7$

$0 = |4x + 1| - 7$       Isolate the absolute value on one side of the equation

$7 = |4x + 1|$       Now we can break this into two separate equations:

$7 = 4x + 1$        $-7 = 4x + 1$

$6 = 4x$       or       $-8 = 4x$

$x = \frac{6}{4} = \frac{3}{2}$        $x = \frac{-8}{4} = -2$

There are two solutions:  $x = \frac{3}{2}$  and  $x = -2$

## Example 8

Solve  $1 = 4|x - 2| + 2$

Isolating the absolute value on one side the equation,

$1 = 4|x - 2| + 2$

$-1 = 4|x - 2|$

$-\frac{1}{4} = |x - 2|$

At this point, we notice that this equation has no solutions – the absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value.

## Try it Now

4. Find the horizontal & vertical intercepts for the function  $f(x) = -|x + 2| + 3$

**Solving Absolute Value Inequalities**

When absolute value inequalities are written to describe a set of values, like the inequality  $|x - 5| \leq 4$  we wrote earlier, it is sometimes desirable to express this set of values without the absolute value, either using inequalities, or using interval notation.

We will explore two approaches to solving absolute value inequalities:

- 1) Using a graph
- 2) Using test values

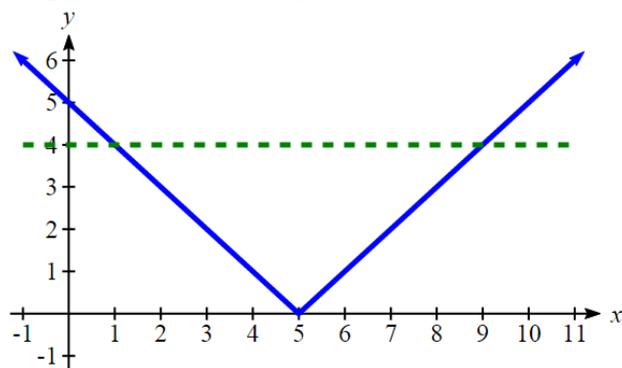
## Example 9

Solve  $|x - 5| \leq 4$ 

With both approaches, we will need to know first where the corresponding *equality* is true. In this case we first will find where  $|x - 5| = 4$ . We do this because the absolute value is a nice friendly function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve  $|x - 5| = 4$ ,

$$\begin{array}{l} x - 5 = 4 \quad \text{or} \quad x - 5 = -4 \\ x = 9 \quad \quad \quad x = 1 \end{array}$$

To use a graph, we can sketch the function  $f(x) = |x - 5|$ . To help us see where the outputs are 4, the line  $g(x) = 4$  could also be sketched.



On the graph, we can see that indeed the output values of the absolute value are equal to 4 at  $x = 1$  and  $x = 9$ . Based on the shape of the graph, we can determine the absolute value is less than or equal to 4 between these two points, when  $1 \leq x \leq 9$ . In interval notation, this would be the interval  $[1, 9]$ .

As an alternative to graphing, after determining that the absolute value is equal to 4 at  $x = 1$  and  $x = 9$ , we know the graph can only change from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:  $x < 1$ ,  $1 < x < 9$ , and  $x > 9$ . To determine when the function is less than 4, we could pick a value in each interval and see if the output is less than or greater than 4.

Interval	Test $x$	$f(x)$	$<4$ or $>4$ ?
$x < 1$	0	$ 0 - 5  = 5$	greater
$1 < x < 9$	6	$ 6 - 5  = 1$	less
$x > 9$	11	$ 11 - 5  = 6$	greater

Since  $1 \leq x \leq 9$  is the only interval in which the output at the test value is less than 4, we can conclude the solution to  $|x - 5| \leq 4$  is  $1 \leq x \leq 9$ .

## Example 10

Given the function  $f(x) = -\frac{1}{2}|4x - 5| + 3$ , determine for what  $x$  values the function values are negative.

We are trying to determine where  $f(x) < 0$ , which is when  $-\frac{1}{2}|4x - 5| + 3 < 0$ . We begin by isolating the absolute value:

$$-\frac{1}{2}|4x - 5| < -3 \quad \text{when we multiply both sides by } -2, \text{ it reverses the inequality}$$

$$|4x - 5| > 6$$

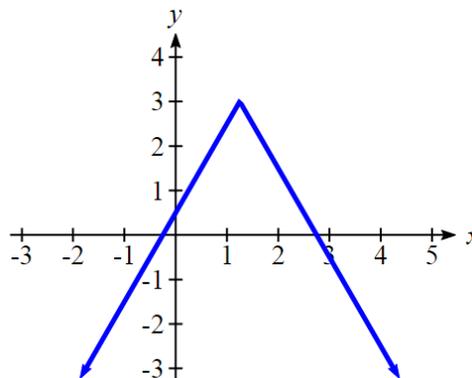
Next we solve for the equality  $|4x - 5| = 6$

$$4x - 5 = 6 \quad 4x - 5 = -6$$

$$4x = 11 \quad \text{or} \quad 4x = -1$$

$$x = \frac{11}{4} \quad x = \frac{-1}{4}$$

We can now either pick test values or sketch a graph of the function to determine on which intervals the original function values are negative. Notice that it is not even really important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at  $x = \frac{-1}{4}$  and  $x = \frac{11}{4}$ , and that the graph has been flipped.



From the graph of the function, we can see the function values are negative to the left of the first horizontal intercept at  $x = \frac{-1}{4}$ , and negative to the right of the second

intercept at  $x = \frac{11}{4}$ . This gives us the solution to the inequality:

$$x < \frac{-1}{4} \quad \text{or} \quad x > \frac{11}{4}$$

In interval notation, this would be  $\left(-\infty, \frac{-1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$

## Try it Now

6. Solve  $-2|k - 4| \leq -6$

There is a third approach to solving absolute value inequalities that is formulaic. While it works, and you are welcome to use it, it is much more likely that you will remember the other approaches.

## Solutions to Absolute Value Inequalities

To solve  $|A| < B$ , solve:  $-B < A < B$

To solve  $|A| > B$ , solve:  $A > B$  or  $A < -B$

## Example 11

Solve  $3|x + 4| - 2 \geq 7$

We need to start by isolating the absolute value:

$$3|x + 4| - 2 \geq 7 \quad \text{Add 2 to both sides}$$

$$3|x + 4| \geq 9 \quad \text{Divide both sides by 3}$$

$$|x + 4| \geq 3$$

Now we can break this apart and solve each piece separately:

$$x + 4 \geq 3 \quad \text{or} \quad x + 4 \leq -3$$

$$x \geq -1 \quad \text{or} \quad x \leq -7$$

In interval notation, this would be  $(-\infty, -7] \cup [-1, \infty)$ .

## Important Topics of this Section

The properties of the absolute value function

Solving absolute value equations

Finding intercepts

Solving absolute value inequalities

## Try it Now Answers

1.  $x \geq -4$
2.  $4 \leq 2x + 6 < 16$
3. Using the variable  $p$ , for passing,  $|p - 80| \leq 20$
4.  $f(x) = -|x + 2| + 3$
5.  $f(0) = 1$ , so the vertical intercept is at  $(0, 1)$ .  $f(x) = 0$  when  $x = -5$  and  $x = 1$  so the horizontal intercepts are at  $(-5, 0)$  &  $(1, 0)$
6.  $k < 1$  or  $k > 7$ ; in interval notation this would be  $(-\infty, 1) \cup (7, \infty)$

## Section 3.2: Linear Inequalities

In the systems of equations chapter, we looked at a company that produces a basic and premium version of its product, and we determined how many of each item they should produce to fully utilize all staffed hours. In some cases, though, it might not make the most sense for the company to utilize all the staffed hours; if the premium product has high demand and a high price, it might make more sense for the company to make as many of those, even if some staff hours go unused. In this case, the company might be most interested in what combinations of basic and premium product production are possible. For that, we need linear inequalities.

A linear **equation** in two variables is an equation like  $f(x) = 2x + 1$ , which is sometimes written without function notation as  $y = 2x + 1$ . Recall that the graph of this equation is a line, formed by all the points  $(x, y)$  that satisfy the equation. A linear **inequality** in two variables is similar, but involves an inequality. Some examples:

$$y < 2x + 1 \quad y > 4x - 1 \quad y \leq \frac{2}{3}x + 4 \quad y \geq 5 - 2x$$

Linear inequalities can also be written with both variables on the same side of the equation, like  $2x - 3y < 4$ .

The solution set to a linear inequality will be all the points  $(x, y)$  that satisfy the inequality. Notice that the line  $y = 2x + 1$  divides the coordinate plane into two halves; on one half  $y < 2x + 1$ , and on the other  $y > 2x + 1$ . The solution set to a linear inequality will be a half plane, and to show the solution set we shade the part of the coordinate plane where the points lie in the solution set.

### To graph the solution to a linear inequality

1. If necessary, rewrite the linear inequality into a form convenient for graphing, like slope-intercept form.
2. Graph the corresponding linear equation.
  - a. For a strict inequality ( $<$  or  $>$ ), draw a dashed line to show that the points on the line are not part of the solution
  - b. For an inequality that includes the equal sign ( $\leq$  or  $\geq$ ), draw a solid line to show that the points on the line are part of the solution.
3. Choose a test point, not on the line.
  - a. Substitute the test point into the inequality
  - b. If the inequality is true at the test point, shade the half plane on the side including the test point
  - c. If the inequality is not true at the test point, shade the half plane on the side that doesn't include the test point

## Example 1

Graph the solution to  $y < 2x + 1$

Since this is a strict inequality, we will graph  $y = 2x + 1$  as a dashed line.

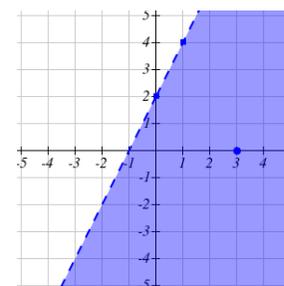
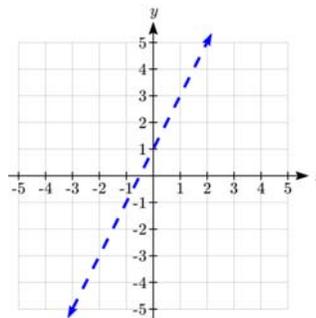
Now we choose a test point not on the line. Usually picking a point where one coordinate is zero will make things easier. Let's use  $(3,0)$ .

Substituting  $(3,0)$  into the inequality, we get

$$0 < 2(3) + 1$$

$$0 < 7$$

This is a true statement, so we will shade the side of plane that includes  $(3,0)$



## Example 2

Graph the solution to  $x \geq 2y + 4$

This is not written in a form we are used to use for graphing, so we might solve it for  $y$  first.

$$x \geq 2y + 4 \quad \text{Subtract } 2y \text{ from both sides}$$

$$x - 2y \geq 4 \quad \text{Subtract } x \text{ from both sides}$$

$$-2y \geq 4 - x \quad \text{Divide by } -2, \text{ reversing the order of the inequality}$$

$$y \leq -2 + \frac{1}{2}x$$

Since this inequality includes the equals sign, we will graph

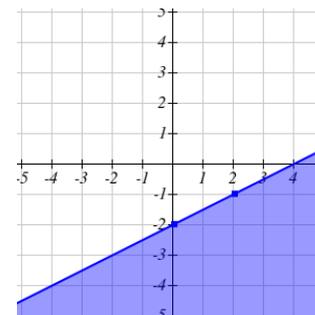
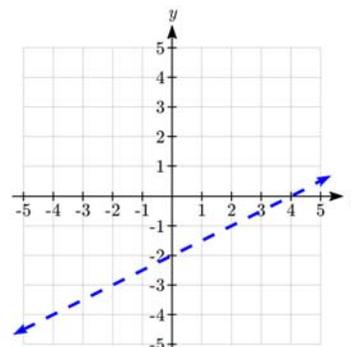
$$y = -2 + \frac{1}{2}x \text{ as a solid line.}$$

Now we choose a test point not on the line.  $(0,0)$  is a convenient choice. Substituting  $(0,0)$  into the inequality, we get

$$0 \leq -2 + \frac{1}{2}(0)$$

$$0 \leq -2$$

This is a false statement, so we shade the half of the plane that does not include  $(0,0)$ .



## Try it Now

- Graph the solution to  $y \geq -\frac{1}{2}x + 1$

## Example 3

A store sells peanuts for \$4/pound and cashews for \$8/pound, and plans to sell a new mixed-nut blend in a jar. What combinations of peanuts and cashews are possible, if they want the mix to cost \$6 or less?

We start by defining our variables:

$p$ : The number of pounds of peanuts in 1 pound of mix

$c$ : The number of pounds of cashews in 1 pound of mix

The cost of one pound of mix will be  $4p + 8c$ , so all the mixes costing \$6 or less will satisfy the inequality  $4p + 8c \leq 6$ .

We can graph the equation  $4p + 8c = 6$  fairly easily by finding the intercepts:

When  $p = 0$ ,

$$4(0) + 8c = 6$$

$$8c = 6$$

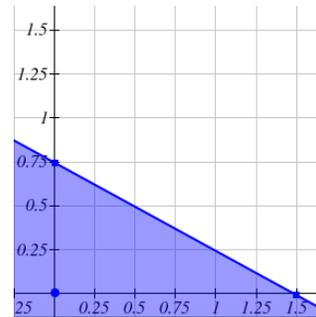
$$c = \frac{6}{8} = \frac{3}{4}. \text{ So the point } \left(0, \frac{3}{4}\right) \text{ is on the line.}$$

When  $c = 0$ ,  $4p = 6$ , so  $p = \frac{6}{4} = \frac{3}{2}$ , giving the point

$$\left(\frac{3}{2}, 0\right).$$

Notice the test point  $(0,0)$  will satisfy the inequality, so we will shade the side of the line including the origin.

Because of the context, only values in the first quadrant are reasonable. The graph shows all the possible combinations the store could use, including 1 pound of peanuts with  $1/4$  pound of cashews, or  $1/2$  a pound of each.



## Systems of Linear Inequalities

In the systems of equations chapter, we looked for solutions to a system of linear equations - a point that would satisfy all the equations in the system. Likewise, we can consider a system of linear inequalities. The solution to a system of linear inequalities is the set of points that satisfy all the inequalities in the system.

With a single linear inequality, we can show the solution set graphically. Likewise, with a system of linear inequalities we show the solution set graphically. We find it by looking for where the regions indicated by the individual linear inequalities overlap.

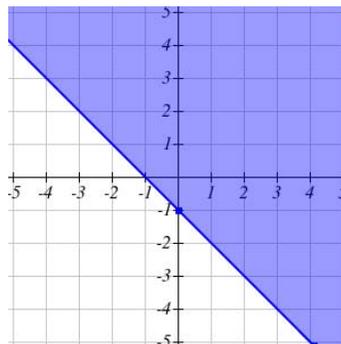
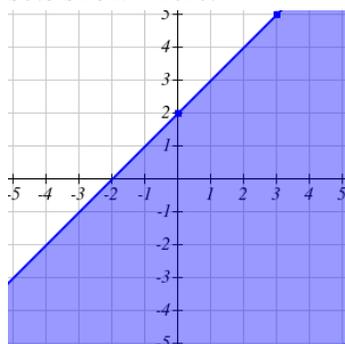
#### Example 4

Graph the solution to the system of linear inequalities

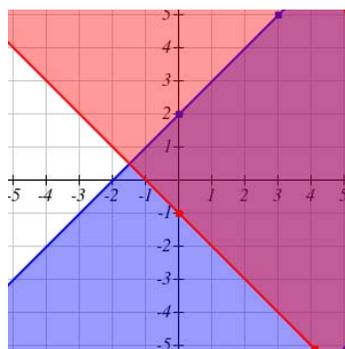
$$y \leq x + 2$$

$$y \geq 1 - x$$

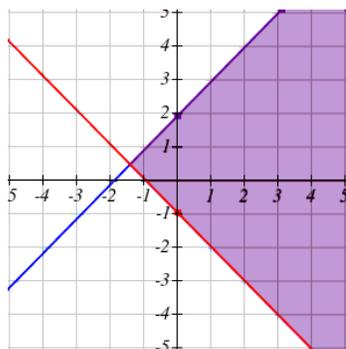
If we graph the solution set to each inequality individually, we get the two solutions sets shown here.



Graphing these solution sets on the same axes reveals the solution to the system of inequalities as the region where the two overlap.



The solution set, where the regions overlap



The solution set, drawn alone

#### Try it Now

2. Graph the solution to the system of linear inequalities

$$y \leq 3 - 2x$$

$$y \geq \frac{1}{2}x + 1$$

The following question is similar to a problem we solved using systems in the systems chapter.

### Example 5

A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. How many items can then produce within the limits of their staffing?

Notice this problem is different than the question we asked in the first section, since we are no longer concerned about fully utilizing staffing, we are only interested in what is possible. As before, we'll define

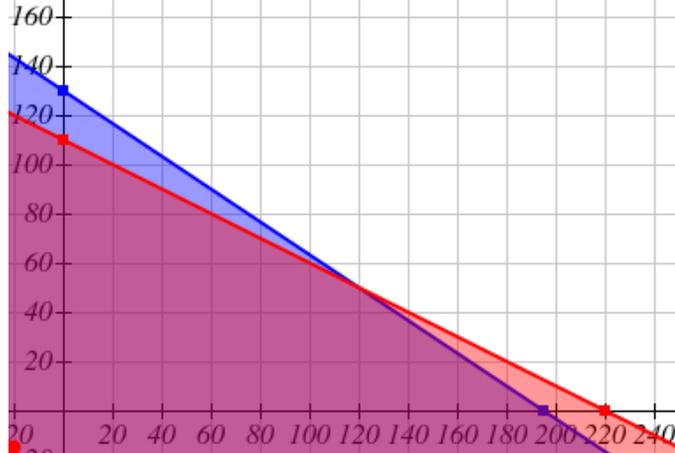
$b$ : The number of basic products made,  $p$ : The number of premium products made.

Just as we created equations in the first section, we can now create inequalities, since we know the hours used in production needs to be less than or equal to the hours available. This leads to two inequalities:

$$20b + 30p \leq 3900$$

$$15b + 30p \leq 3300$$

Graphing these inequalities gives us the solution set.



Since it isn't reasonable to consider negative numbers of items, we can further restrict the solutions to the first quadrant. This could also be represented by adding the inequalities  $b \geq 0$ ,  $p \geq 0$ .

In the solution set we can see the solution to the system of equations we solved in the previous section: 120 basic products and 50 premium products. The solution set shows that if the company is willing to not fully utilize the staffing, there are many other possible combinations of products they could produce.

The techniques we used above are key to a branch of mathematics called linear programming, which is used extensively in business. We will explore linear programming further in the next section.

### Important Topics of this Section

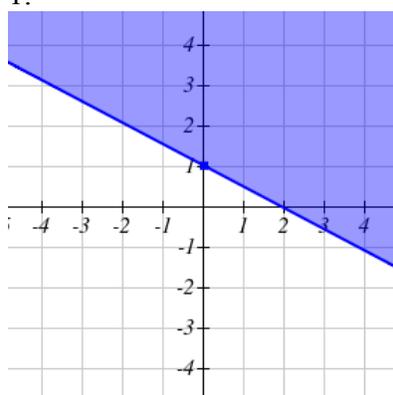
Graphing linear inequalities

Dashed line for strict inequalities, solid for  $\leq$  or  $\geq$

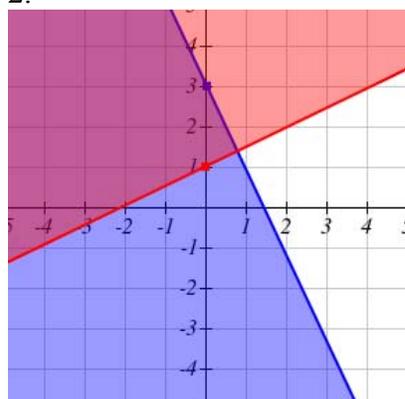
Graphing systems of linear inequalities

### Try it Now Answers

1.



2.



### Section 3.3: Graphical Solutions

Wouldn't it be nice if we could simply produce and sell infinitely many units of a product and thus make a never-ending amount of money? In business (and in day-to-day living) we know that some things are just unreasonable or impossible. Instead, our hope is to maximize or minimize some quantity, given a set of constraints.

In order to have a linear programming problem, we must have:

- Constraints, represented as inequalities
- An **objective function**, that is, a function whose value we either want to be as large as possible (want to maximize it) or as small as possible (want to minimize it).

Consider this extension of the example from the end of the last section.

#### Example 1

A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. They sell the basic products for a profit of \$30 and the premium products for a profit of \$40. How many of each version should be produced to maximize profit?

Let  $b$  = the number of basic products made, and  $p$  = the number of premium products made. Our objective function is what we're trying to maximize or minimize. In this case, we're trying to maximize profit. The total profit,  $P$ , is

$$P = 30b + 40p$$

In the last section, the example developed our constraints. Together, these define our linear programming problem:

Objective function:  $P = 30b + 40p$

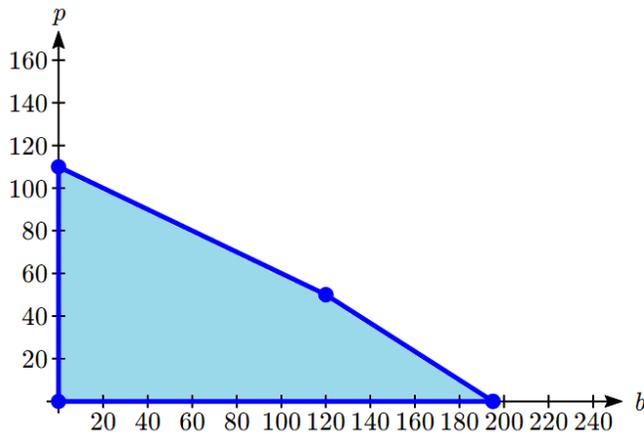
Constraints:

$$20b + 30p \leq 3900$$

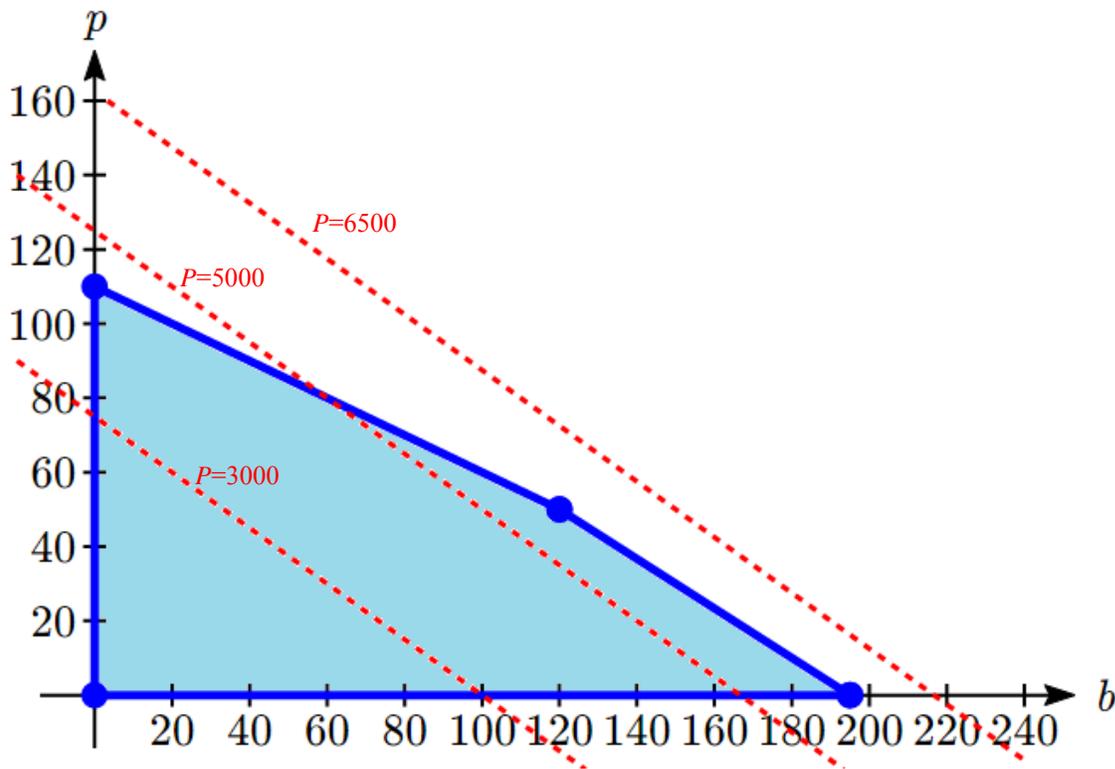
$$15b + 30p \leq 3300$$

$$b \geq 0, p \geq 0$$

In this section, we will approach this type of problem graphically. We start by graphing the constraints to determine the **feasible region** – the set of possible solutions. Just showing the solution set where the four inequalities overlap, we see a clear region.

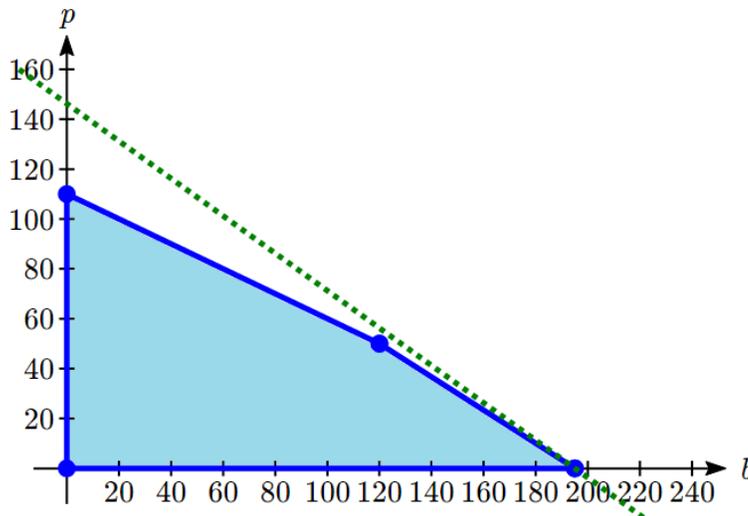


To consider how the objective function connects, suppose we considered all the possible production combinations that gave a profit of  $P = \$3000$ , so that  $3000 = 30b + 40p$ . That set of combinations would form a line in the graph. Doing the same for a profit of \$5000 and \$6500 would give additional lines. Graphing those on top of our feasible region, we see a pattern:



Notice that all the constant-profit lines are parallel, and that in general the profit increases as we move up to upper right. Notice also that for a profit of \$5000 there are some production levels inside the feasible region for that profit level, but some are outside. That means we could feasibly make \$5000 profit by producing, for example, 167 basic items and no premium items, but we can't make \$5000 by producing 125 premium items and no basic items because that falls outside our constraints.

The solution to our linear programming problem will be the largest possible profit that is still feasible. Graphically, that means the line furthest to the upper-right that still touches the feasible region on at least point. That solution is the one below:



This profit line touches the feasible region where  $b = 195$  and  $p = 0$ , giving a profit of  $P = 30(195) + 40(0) = \$5850$ .

Notice that this is slightly larger than the profit that would be made by completely utilizing all staffing at  $b = 120$ ,  $p = 50$ , where the profit would be \$5600.

The objective function along with the four corner points above forms a **bounded** linear programming problem. That is, imagine you are looking at three fence posts connected by fencing (black point and lines, respectively). If you were to put your dog in the middle, you could be sure it would not escape (assuming the fence is tall enough). If this is the case, then you have a bounded linear programming problem. If the dog could walk infinitely in any one direction, then the problem is unbounded.

In the past example, you can see that the line of maximum profit will always touch the boundary of the feasible region. That observation inspires the fundamental theorem of linear programming.

#### Fundamental Theorem of Linear Programming

- If a solution exists to a bounded linear programming problem, then it occurs at one of the corner points.
- If a feasible region is unbounded, then a maximum value for the objective function does not exist.
- If a feasible region is unbounded, and the objective function has only positive coefficients, then a minimum value exists.

In the last example we solve the problem somewhat intuitively by “sliding” the profit line up. Typically we use a more procedural approach.

### Solving a Linear Programming Problem Graphically

1. Define the variables to be optimized. The question asked is a good indicator as to what these will be.
2. Write the objective function, first in words, then convert to a mathematical equation
3. Write the constraints, first in words, then convert to mathematical inequalities
4. Graph the constraints inequalities, and shade the feasible region
5. Identify the corner points by solving systems of linear equations whose intersection represents a corner point.
6. Test all corner points in the objective function. The “winning” point is the point that optimizes the objective function (biggest if maximizing, smallest if minimizing)

### Try it Now

1. Maximize  $P = 14x + 9y$  subject to the constraints:

$$x + y \leq 9$$

$$3x + y \leq 15$$

$$x \geq 0, y \geq 0$$

### Example 2

A health-food business would like to create a high-potassium blend of dried fruit in the form of a box of 10 fruit bars. It decides to use dried apricots, which have 407 mg of potassium per serving, and dried dates, which have 271 mg of potassium per serving. The company can purchase its fruit through in bulk for a reasonable price. Dried apricots cost \$9.99/lb. (about 3 servings) and dried dates cost \$7.99/lb. (about 4 servings). The company would like the box of bars to have at least the recommended daily potassium intake of about 4700 mg, and contain at least 1 serving of each fruit. In order to minimize cost, how many servings of each dried fruit should go into the box of bars?

We begin by defining the variables. Let  
 $x$  = number of servings of dried apricots  
 $y$  = number of servings of dried dates

We next work on the objective function.

For apricots, there are 3 servings in one pound. This means that the cost per serving is  $\$9.99/3 = \$3.33$ . The cost for  $x$  servings would thus be  $3.33x$ .

For dates, there are 4 servings per pound. This means that the cost per serving is  $\$7.99/4 = \$2.00$ . The cost for  $y$  servings would thus be  $2.00y$ .

The total cost,  $C$ , for apricots and dates would be  
 $C = 3.33x + 2.00y$

Normally we would have constraints  $x \geq 0$  and  $y \geq 0$  since negative servings can't be used. But in this case, we're further restricted. In words:

- There must be at least 1 serving of each fruit
- The product must contain at least 4700 mg of potassium

Mathematically,

- Since there must be at least 1 serving of each fruit,  $x \geq 1$  and  $y \geq 1$
- There are  $407x$  mg of potassium in  $x$  servings of apricots and  $271y$  mg of potassium in  $y$  servings of dates. The sum should be greater than or equal to 4700 mg of potassium, or  $407x + 271y \geq 4700$

Thus we have,

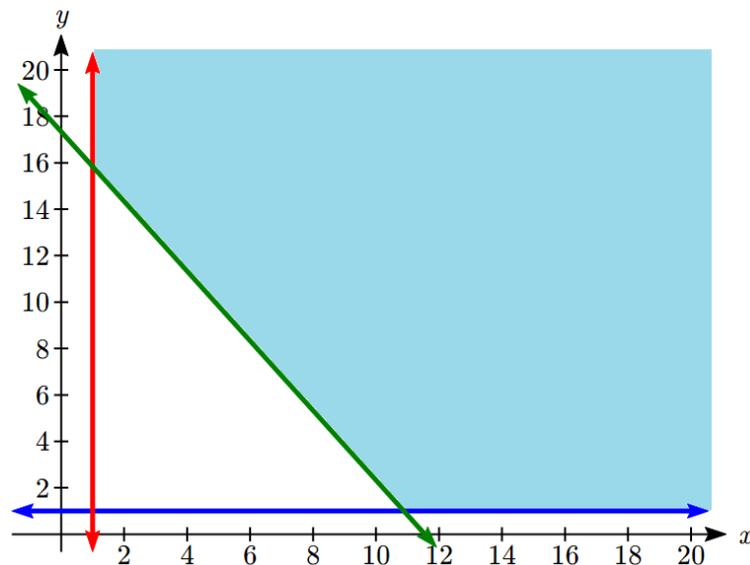
Objective function:  $C = 3.33x + 2.00y$

Constraints:

$$407x + 271y \geq 4700$$

$$x \geq 1, y \geq 1$$

We graph the constraints and shade the feasible region:



The region is unbounded, but we will be able to find a minimum still. We can see there are two corner points.

The one in the upper left is the intersection of the lines  $407x + 271y = 4700$  and  $x = 1$ .

Solving for the intersection using substitution:

$$407(1) + 271y = 4700$$

$$y \approx 15.8$$

Point: (1, 15.8)

The one in the lower right is the intersection of the lines  $407x + 271y = 4700$  and  $y = 1$ .

$$407x + 271(1) = 4700$$

$$x \approx 10.9$$

Point: (10.9, 1)

Testing the objective function at each of these corner points:

Point	Cost, $C = 3.33x + 2.00y$
(10.9, 1)	$C = 3.33(10.9) + 2.00(1) = \$38.30$
(1, 15.8)	$C = 3.33(1) + 2.00(15.8) = \$34.96$

The company can minimize cost by using 1 serving of apricots and 15.8 servings of dates.

### Try it Now

2. A company makes two products. Product A requires 3 hours of manufacturing and 1 hour of assembly. Product B requires 4 hours of manufacturing and 2 hours of assembly. There are a total of 84 hours of manufacturing and 32 hours of assembly available. Determine the production to maximize profit if the profit on product A is \$50 and the profit on product B is \$60.

### Example 3

A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per unit for a chair and \$30 for a table, how many units of each should be manufactured to maximize profit?

We begin by defining the variables. Let

$c$  = number of chairs made

$t$  = number of tables made

The profit,  $P$ , will be  $P = 20c + 30t$ .

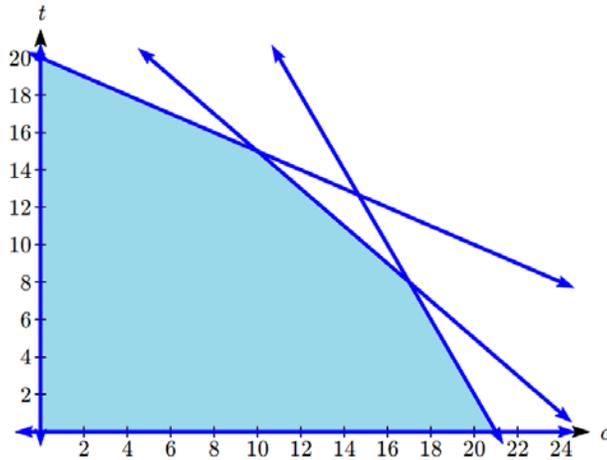
For cutting,  $c$  chairs will require  $1c$  hours and  $t$  tables will require  $2t$  hours. We can use at most 40 hours, so  $c + 2t \leq 40$ .

For assembly,  $c$  chairs will require  $2c$  hours and  $t$  tables will require  $1t$  hours. We can use at most 42 hours, so  $2c + t \leq 42$ .

For finishing,  $c$  chairs will require  $1c$  hours and  $t$  tables will require  $1t$  hours. We can use at most 25 hours, so  $c + t \leq 25$ .

Since we can't produce negative items,  $c \geq 0, t \geq 0$ .

Graphing the constraints, we can see the feasible region.



There are five corner points for this region.

Point 1: In the lower left, where  $t = 0$  crosses  $c = 0$ . Point:  $(0, 0)$

Point 2: In the upper left, where  $c = 0$  crosses  $c + 2t = 40$ .

Using substitution,  $0 + 2t = 40$ , so  $t = 20$ .

Point:  $(0, 20)$

Point 3: In the lower right, where  $t = 0$  crosses  $2c + t = 42$ .

Using substitution,  $2c + 0 = 42$ , so  $c = 21$ .

Point:  $(21, 0)$

Point 4: Where  $c + 2t = 40$  crosses  $c + t = 25$ .

We can solve this as a system using any techniques we know. We could solve the second equation for  $c$ , giving  $c = 25 - t$ , then substitute into the first equation:

$$(25 - t) + 2t = 40$$

$$25 + t = 40$$

$$t = 15$$

$$\text{Then } c = 25 - 15 = 10.$$

Point:  $(10, 15)$

Point 5: Where  $2c + t = 42$  crosses  $c + t = 25$ .

We can solve this as a system using any techniques we know. Using a different technique this time, we could multiply the bottom equation by  $-1$  then add it to the first:

$$\begin{array}{r} 2c + t = 42 \\ -c - t = -25 \\ \hline c = 17 \end{array}$$

Then using  $c + t = 25$ , we have  $17 + t = 25$ , so  $t = 8$ .

Point: (17, 8)

Testing the objective function at each of these corner points:

Point	Profit, $P = 20c + 30t$
(0, 0)	$P = 20(0) + 30(0) = \$0$
(0, 20)	$P = 20(0) + 30(20) = \$600$
(21, 0)	$P = 20(21) + 30(0) = \$420$
(10, 15)	$P = 20(10) + 30(15) = \$650$
(17, 8)	$P = 20(17) + 30(8) = \$580$

The profit will be maximized by producing 10 chairs and 15 tables.

### Important Topics of this Section

- Objective function
- Constraint equations
- Feasible region
- Corner points
- Fundamental Theorem of Linear Programming
- Solving a linear programming problem using a graph

### Try it Now Answers

1. Maximize  $P = 14x + 9y$

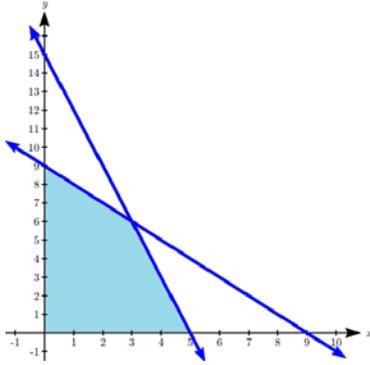
subject to:

$$x + y \leq 9$$

$$3x + y \leq 15$$

$$x \geq 0, y \geq 0$$

Graphing the feasible region, we see there are three corner points of potential interest: (0, 9), (5, 0), and the intersection of the two lines at (3, 6). We then evaluate the objective function at each corner point.



Point	$P = 14x + 9y$
(0, 9)	81
(5, 0)	70
(3, 6)	96

$P$  is maximized when  $x = 3, y = 6$ .

2. Let  $a$  be the number of product A produced, and  $b$  be the number of product B.

Our goal is to maximize profit:  $P = 50a + 60b$ .

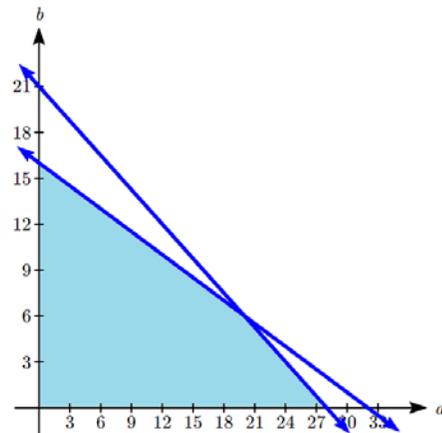
From manufacturing we get the constraint:  
 $3a + 4b \leq 84$

From assembly we get the constraint:  
 $1a + 2b \leq 32$

We have corner points at (0, 16), (28, 0). The third is at the intersection of the two lines. To find that we could multiply the second equation by -2 and add:

$$\begin{array}{r} 3a + 4b = 84 \\ -2a - 4b = -64 \\ \hline a = 20 \end{array}$$

The third corner point is at (20, 6).



Evaluating the objective function at each of these points:

Point	$P$
(0, 16)	$P = 50(0) + 60(16) = 960$
(28, 0)	$P = 50(28) + 60(0) = 1400$
(20, 6)	$P = 50(20) + 60(6) = 1360$

Profit will be maximized by producing 28 units of product A and 0 units of product B.

## Section 3.4: Simplex Method

The graphical approach to linear programming problems we learned in the last section works well for problems involving only two variables, but does not extend easily to problems involving three or more unknowns. To tackle those more complex problems, we have two options:

1. Use by-hand solution methods that have been developed to solve these types of problems in a compact, procedural way.
2. Use technology that has automated those by-hand methods.

In this section we will explore the traditional by-hand method for solving linear programming problems.

To handle linear programming problems that contain upwards of two variables, mathematicians developed what is now known as the **simplex method**. It is an efficient algorithm (set of mechanical steps) that “toggles” through corner points until it has located the one that maximizes the objective function. In order to be able to find a solution, we need problems in the form of a standard maximization problem.

### Standard maximization problem

A standard maximization problem will include

- an objective function, and
- one or more constraints of the form,

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots a_nx_n \leq V$$

All of the  $a_{\text{number}}$  represent real-numbered coefficients and the  $x_{\text{number}}$  represent the corresponding variables.

$V$  is a non-negative (0 or larger) real number.

Having constraints that have upper limits should make sense, since when maximizing a quantity, we probably have caps on what we can do. If we had no caps, then we could continue to increase, say profit, infinitely! This contradicts what we know about the real world.

In order to use the simplex method, either by technology or by hand, we must set up an **initial simplex tableau**, which is a matrix containing information about the linear programming problem we wish to solve.

### Setting Up the Initial Simplex Tableau

First off, matrices don't do well with inequalities. For one, a matrix does not have a simple way of keeping track of the direction of an inequality. This alone discourages the use of inequalities in matrices. How, then, do we avoid this?

Consider the following linear programming problem

**Maximize:**

$$P = 7x + 12y$$

**Subject to:**

$$2x + 3y \leq 6$$

$$3x + 7y \leq 12$$

Because we know that the left sides of both inequalities will be quantities that are smaller than the corresponding values on the right, we can be sure that adding "something" to the left-hand side will make them exactly equal. That is:

$$2x + 3y + s_1 = 6$$

$$3x + 7y + s_2 = 12$$

For instance, suppose that  $x = 1, y = 1$ , Then

$$2\overset{x}{(1)} + 3\overset{y}{(1)} + \overset{s_1}{1} = 6$$

$$3\overset{x}{(1)} + 7\overset{y}{(1)} + \overset{s_2}{2} = 12$$

It is important to note that these two variables,  $s_1$  and  $s_2$ , are not necessarily the same. They simply act on the inequality by picking up the "slack" that keeps the left side from looking like the right side. Hence, we call them **slack variables**. This takes care of the inequalities for us. Since augmented matrices contain all variables on the left and constants on the right, we will rewrite the objective function to match this format:

$$-7x - 12y + P = 0$$

This will require us to have a matrix that can handle  $x, y, s_1, s_2$ , and  $P$ . We will put it in this order. Finally, the simplex method requires that the objective function be listed as the bottom line in the matrix so that we have:

$$\left[ \begin{array}{ccccc|c} x & y & s_1 & s_2 & P & \\ \hline 2 & 3 & 1 & 0 & 0 & 6 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -7 & -12 & 0 & 0 & 1 & 0 \end{array} \right]$$

We have established the initial simplex tableau. Note that the horizontal and vertical lines are used simply to separate constraint coefficients from constants and objective function coefficients. Also notice that the slack variable columns, along with the objective function output, form the identity matrix.

### Solving the Linear Programming Problem by Using the Initial Tableau

We will present the algorithm for solving, however, note that it is not entirely intuitive.

#### 1. Select a pivot column

We first select a pivot column, which will be the column that contains the largest negative coefficient in the row containing the objective function. Note that the largest negative number belongs to the term that contributes most to the objective function. This is intentional since we want to focus on values that make the output as large as possible.

Our pivot is thus the  $y$  column.

$$\begin{array}{c}
 x \quad y \quad s_1 \quad s_2 \quad P \\
 \left[ \begin{array}{cc|ccc|c}
 2 & 3 & 1 & 0 & 0 & 6 \\
 3 & 7 & 0 & 1 & 0 & 12 \\
 \hline
 -7 & -12 & 0 & 0 & 1 & 0
 \end{array} \right]
 \end{array}$$

#### 2. Select a pivot row.

Do this by computing the ratio of each constraint constant to its respective coefficient in the pivot column – this is called the **test ratio**. Select the row with the smallest test ratio.

We first calculate the test ratios:

$$\begin{array}{c}
 x \quad y \quad s_1 \quad s_2 \quad P \\
 \left[ \begin{array}{cc|ccc|c}
 2 & 3 & 1 & 0 & 0 & 6 \\
 3 & 7 & 0 & 1 & 0 & 12 \\
 \hline
 -7 & -12 & 0 & 0 & 1 & 0
 \end{array} \right]
 \begin{array}{l}
 6/3 = 2 \\
 12/7 \approx 1.7
 \end{array}
 \end{array}$$

Since the test ratio is smaller for row 2, we select it as the pivot row. The boxed value is now called our **pivot**. To justify why we do this, observe that 2 and 1.7 are simply the vertical intercepts of the two inequalities. We select the smaller one to ensure we have a corner point that is in our feasible region.

3. Using Gaussian elimination, we next eliminate rows 1 and 3. We want to take  $-3/7$  multiplied by row 2 and add it to row 1, so that we eliminate the 3 in the second column. We also want next to eliminate the  $-12$  in row 3. To do this, we must multiply 7 by  $12/7$

and add it to row 3 (recall that placing the value you wish to cancel out in the denominator of a multiple and the value you wish to achieve in the numerator of the multiple, you obtain the new value).

We can see that we have effectively zeroed out the second column non-pivot values. We get the following matrix:

$$\left[ \begin{array}{ccccc|c} .71 & 0 & 1 & -.43 & 0 & .86 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -1.86 & 0 & 0 & 1.71 & 1 & 20.57 \end{array} \right]$$

What have we done? For one, we have maxed out the contribution of the 2-2 entry  $y$ -value coefficient to the objective function. Have we optimized the function? Not quite, as we still see that there is a negative value in the first column. This tells us that  $x$  can still contribute to the objective function. To eliminate this, we first find the pivot row by obtaining test ratios:

$$\left[ \begin{array}{ccccc|c} \boxed{.71} & 0 & 1 & -.43 & 0 & .86 \\ 3 & 7 & 0 & 1 & 0 & 12 \\ \hline -1.86 & 0 & 0 & 1.71 & 1 & 20.57 \end{array} \right] \quad \begin{array}{l} .86/.71 \approx 1.2 \\ 12/3 = 4 \end{array}$$

We proceed to eliminate all non-pivot values by multiplying the top row by  $-3/0.71$  and adding it to the second row, and adding  $1.86/0.71$  times the first row to the third row.

There remain no additional negative entries in the objective function row. We thus have the following matrix:

$$\left[ \begin{array}{ccccc|c} .71 & 0 & 1 & -.43 & 0 & .86 \\ 0 & 7 & -4.23 & 2.81 & 0 & 8.38 \\ \hline 0 & 0 & 2.62 & .59 & 1 & 22.82 \end{array} \right]$$

We are thus prepared to read the solutions.

4. To identify the solution set, focus we focus only on the columns with exactly one non-zero entry – these are called **active variables** (columns with more than one non-zero entry are thus called **inactive variables**).

We notice that both the  $x$  and  $y$  columns are active variables. We really don't care about the slack variables, much like we ignore inequalities when we are finding intersections. We now see that,

$$.71x + s_1 - .43s_2 = .86$$

$$7y - 4.23s_1 + 2.81s_2 = 8.38$$

$$2.62s_1 + .59s_2 + P = 22.82$$

Setting the slack variables to 0, gives:

$$.71x = .86 \rightarrow x \approx 1.21$$

$$7y = 8.38 \rightarrow y \approx 1.20$$

$$P = 22.82$$

Thus, the triplet,  $(x, y, z) \sim (1.21, 1.20, 22.82)$  is the solution to the linear programming problem. That is, inputs of 1.21 and 1.20 will yield a maximum objective function value of 22.82.

The entire process of solving using simplex method is:

### Simplex Method

1. Set up the problem. That is, write the objective function and the constraints.
2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.
3. Construct the initial simplex tableau. Write the objective function as the bottom row.
4. The most negative entry in the bottom row identifies a column.
5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element. The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
6. Perform pivoting to make all other entries in this column zero. This is done the same way as we did with the Gauss-Jordan method for matrices.
7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.
8. Read off your answers. Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.

### Try it Now

1. Use simplex method to solve:

$$\text{Maximize } P = 5x + 7y + 9z$$

subject to:

$$x + 4y + 2z \leq 8$$

$$3x + 5y + z \leq 6$$

$$x \geq 0, y \geq 0, z \geq 0$$

## Important Topics of this Section

Initial tableau  
 Pivot row and column  
 Simplex method  
 Reading the answer from a reduced tableau

## Try it Now Answers

1. We set up the initial tableau. The most negative entry in the bottom row is in the third column, so we select that column. Calculating the quotients we have  $8/2 = 4$  in the first row, and  $6/1 = 6$  in the second row. The smaller value is in row one, so we choose that row. Our pivot is in row 1 column 3.

$$\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P \\ \hline 1 & 4 & 2 & 1 & 0 & 0 & 8 \\ 3 & 5 & 1 & 0 & 1 & 0 & 6 \\ \hline -5 & -7 & -9 & 0 & 0 & 1 & 0 \end{array}$$

Now we perform the pivot. We might start by scaling the top row by  $1/2$  to get a 1 in the pivot position. Then we can add  $-1$  times the top row to the second row, and  $9$  times the top row to the third row.

Now we are prepared to pivot again. The most negative entry in the bottom row is in column 1, so we select that column. Looking at the ratios,  $\frac{4}{1/2} = 8$  and  $\frac{2}{5/2} = 0.8$ .

Choosing the smaller, we have our pivot in row 2 column 1.

$$\begin{array}{cccccc|c} 1/2 & 2 & 1 & 1/2 & 0 & 0 & 4 \\ \hline 5/2 & 3 & 0 & -1/2 & 1 & 0 & 2 \\ \hline -1/2 & 11 & 0 & 9/2 & 0 & 1 & 36 \end{array}$$

We can multiply the second row by  $2/5$  to get a 1 in the pivot position, then add  $-1/2$  times the second row to the first row and  $1/2$  times the second row to the third row to reduce.

$$\begin{array}{cccccc|c} 0 & 7/5 & 1 & 3/5 & -1/5 & 0 & 18/5 \\ \hline 1 & 6/5 & 0 & -1/5 & 2/5 & 0 & 4/5 \\ \hline 0 & 58/5 & 0 & 22/5 & 1/5 & 1 & 182/5 \end{array}$$

Only the first and third columns contain only one non-zero value and are active variables. We set the remaining variables equal to zero and find our solution:

$$x = \frac{4}{5}, y = 0, z = \frac{18}{5}$$

## Section 3.5: Applications of Linear Programming

In the previous section we looked at the Simplex method, a procedure for solving linear programming problems with many variables. While this method can be done by-hand, it can easily be automated by a computer. In the remainder of this chapter, we will focus on setting up the objective function and constraints and interpreting the solution, and omit the details of solving.

### Example 1

A catering company is to make lunch for a business meeting. It will serve ham sandwiches, light ham sandwiches, and vegetarian sandwiches. A ham sandwich has 1 serving of vegetables, 4 slices of ham, 1 slice of cheese, and 2 slices of bread. A light ham sandwich has 2 serving of vegetables, 2 slices of ham, 1 slice of cheese and 2 slices of bread. A vegetarian sandwich has 3 servings of vegetables, 2 slices of cheese, and 2 slices of bread. A total of 10 bags of ham are available, each of which has 40 slices; 18 loaves of bread are available, each with 14 slices; 200 servings of vegetables are available, and 15 bags of cheese, each with 60 slices, are available. Given the resources, how many of each sandwich can be produced if the goal is to maximize the number of sandwiches?

We wish to maximize the number of sandwiches, so let:

$x$  = number of ham sandwiches

$y$  = number of light ham sandwiches

$z$  = number of vegetarian sandwiches

The total number of sandwiches is given by:  $S = x + y + z$

The constraints will be given by considering the total amount of ingredients available. That is, the company has a limited amount of ham, vegetables, cheese, and bread.

In total, the company has  $10(40) = 400$  slices of ham,  $18(14) = 252$  slices of bread, 200 servings of vegetables, and  $15(60) = 900$  slices of cheese available. At most, the company can use the above amounts.

There are two sandwiches that use ham – the first requires 4 slices of ham and the second requires only 2, per sandwich, and the total number of slices of ham cannot exceed 400:

$$4x + 2y \leq 400$$

Each sandwich requires 2 slices of bread so:

$$2x + 2y + 2z \leq 252$$

The ham sandwiches have 1 and 2 servings of vegetables, respectively, and the vegetarian sandwich has 3 servings of vegetables. So,

$$1x + 2y + 3z \leq 200$$

Both ham sandwiches require one slice of cheese, and the vegetarian sandwich requires two slices of cheese, so,

$$1x + 1y + 2z \leq 900$$

Our final setup is:

$$\text{Maximize: } S = x + y + z$$

Subject to:

$$4x + 2y \leq 400$$

$$2x + 2y + 2z \leq 252$$

$$2x + 2y + 2z \leq 252$$

Solving this, we get

$$\text{Optimal Solution: } S = 126; x = 100, y = 0, z = 26$$

We find that 100 ham sandwiches, 26 vegetarian sandwiches, and 0 light ham sandwiches should be made to maximize the total number of sandwiches made.

Notice that this will effectively use up all of the bread, which is the first to go.

### Example 2

A factory manufactures three products, A, B, and C. Each product requires the use of two machines, Machine I and Machine II. The total hours available, respectively, on Machine I and Machine II per month are 180 and 300. The time requirements and profit per unit for each product are listed below.

	A	B	C
Machine I	1	2	2
Machine II	2	2	4
Profit	20	30	40

How many units of each product should be manufactured to maximize profit, and what is the maximum profit?

As usual, we start by defining our variables:

$A$  = number of units of product A manufactured

$B$  = number of units of product B manufactured

$C$  = number of units of product B manufactured

We are trying to maximize profit. Producing  $A$  units of item A will result in a profit of  $20A$ , producing  $B$  units of item B will profit  $30B$ , and  $C$  units of item C will profit  $40C$ , giving our objective function:

$$P = 20A + 30B + 40C$$

Next we consider the time available on each machine to establish constraints.

Producing  $A$  units of item A will require  $1A$  hours on Machine 1, producing  $B$  units of item B will require  $2B$  hours, and producing  $C$  units of item C will require  $2C$  hours.

Together these need to not exceed the 180 hours available. This leads to the constraint:  
 $1A + 2B + 2C \leq 180$

We can construct a similar constraint using the hours on Machine 2:

$$2A + 2B + 4C \leq 300$$

Our final setup is:

$$\text{Maximize } P = 20A + 30B + 40C$$

Subject to:

$$1A + 2B + 2C \leq 180$$

$$2A + 2B + 4C \leq 300$$

$$A \geq 0, B \geq 0, C \geq 0$$

Solving this:

$$\text{Optimal Solution: } P = 3300; A = 120, B = 30, C = 0$$

We will maximize profit at \$3300 by producing 120 units of item A, 30 units of item B, and no units of item C.

In addition to maximization problems, linear programming can also be used to solve minimization problems. When done by-hand, these would require a modification of the Simplex method shown in the last section, but these problems can be solved by most technologic methods.

### Example 3

A company is creating a meal replacement bar. They plan to incorporate peanut butter, oats, and dried cranberries as the primary ingredients. The nutritional content of 10 grams of each is listed below, along with the cost, in cents, of each ingredient. Find the amount of each ingredient they should use to minimize the cost of producing a bar containing a minimum of 15g of each ingredient, at least 10g of protein and at most 14g of fat.

	Peanut Butter, 10g	Oats, 10g	Cranberries, 10g
Protein (grams)	2.5	1.7	0
Fat (grams)	5	0.7	0.1
Cost (cents)	6	1	2

We start by introducing variables:

$p$  = number of 10g servings of peanut butter

$a$  = number of 10g servings of oats

$c$  = number of 10g servings of cranberries

The total cost,  $C$ , of producing the bar, in cents, will be  $C = 6p + 1a + 2c$ .

Our first constraints come from the requirement for 15g of each ingredient, which is 1.5 servings (1.5 servings at 10g per serving = 15g). Constructing those constraints:

$$p \geq 1.5, a \geq 1.5, c \geq 1.5$$

Next we look at the nutritional components. For protein,  $p$  servings of peanut butter will contain  $2.5p$  grams of protein. Likewise,  $a$  servings of oats will have  $1.7a$  grams of protein, and  $c$  servings of cranberries will have  $0c$  grams of protein. Together, these need to be at least 10 grams, giving the constraint

$$2.5p + 1.7a + 0c \geq 10$$

We can construct a similar constraint for fat, in this case noting we want the fat to be at most 14g:

$$5p + 0.7a + 0.1c \leq 14$$

We can now have our complete problem:

$$\text{Minimize } C = 6p + 1a + 2c$$

Subject to:

$$2.5p + 1.7a + 0c \geq 10$$

$$5p + 0.7a + 0.1c \leq 14$$

$$p \geq 1.5, a \geq 1.5, c \geq 1.5$$

Turning to technology, we get a solution:

$$\text{Optimal Solution: } C = 15.6765; p = 1.5, a = 3.67647, c = 1.5$$

Interpreting that result, the minimum cost of to produce the bar will be about 15.7 cents, by using 15 grams of peanut butter, 36.8 grams of oats, and 15 grams of dried cranberries.

Verifying our conditions, we can see that our recipe contains at least 1.5 servings of each ingredient. The protein content will be  $2.5(1.5) + 1.7(3.68) + 0(1.5) \approx 10$  grams. The fat content will be  $5(1.5) + 0.7(3.68) + 0.1(1.5) \approx 10.2$  grams.

### Try it Now

1. A factory manufactures three products, A, B, and C. Each product requires a certain number of hours of manufacturing, assembly and finishing time, shown below, along with the total time available and profit. Find production levels to maximize profit.

	A	B	C	Total Hours
Manufacturing	4	5	2	300
Assembly	7	4	1	250
Finishing	4	5	1	200
Profit	175	130	40	

In some cases we have to be clever with how we create our constraints to maintain the correct form of a linear programming problem while still meeting the requirements of the actual application.

#### Example 4

A distribution company needs to ship products from its two warehouses to three retailers. Warehouse A has 1000 products in stock, and Warehouse B has 1200 products. Retailer 1 needs 700 products, Retailer 2 needs 500 products, and Retailer 3 needs 600 products. The cost to ship a product from each warehouse to each retailer is shown below. Find the number of products the company should ship from each warehouse to each retailer to minimize shipping costs.

	Retailer 1	Retailer 2	Retailer 3
Warehouse A	3	5	6
Warehouse B	4	7	5

To start this problem, we first need to define our variables. Since there are six different routes, we will need to define six variables:

$A_1$  = the number of products shipped from Warehouse A to Retailer 1

$B_1$  = the number of products shipped from Warehouse B to Retailer 1

$A_2$  = the number of products shipped from Warehouse A to Retailer 2

We can similarly define variables  $B_2$ ,  $A_3$ , and  $B_3$ .

Our objective function is the total shipping cost. Shipping  $A_1$  items from Warehouse A to Retailer 1 will cost \$3 per item, so a total cost of  $3A_1$ . Doing the same for the other variables gives our total cost equation:

$$C = 3A_1 + 4B_1 + 5A_2 + 7B_2 + 6A_3 + 5B_3$$

We know that Warehouse A has 1000 products in stock, so the total number of items shipped out of Warehouse A needs to be no more than 1000. Likewise Warehouse B can't ship more than 1200 items. These give the constraints:

$$A_1 + A_2 + A_3 \leq 1000$$

$$B_1 + B_2 + B_3 \leq 1200$$

Retailer 1 needs 700 products. While is technically a strict equality, we can set it up as an inequality, indicating the total number of product arriving at retailer 1 needs to be at least 700 products. Since we're minimizing cost, there's no way we'd end up shipping more than 700 items to the retailer. Setting up this constraint, and similar ones for the other three retailers:

$$A_1 + B_1 \geq 700$$

$$A_2 + B_2 \geq 500$$

$$A_3 + B_3 \geq 600$$

Our final problem setup is:

$$\text{Minimize } C = 3A_1 + 4B_1 + 5A_2 + 7B_2 + 8A_3 + 5B_3$$

Subject to:

$$A_1 + A_2 + A_3 \leq 1000$$

$$B_1 + B_2 + B_3 \leq 1200$$

$$A_1 + B_1 \geq 700$$

$$A_2 + B_2 \geq 500$$

$$A_3 + B_3 \geq 600$$

$$A_1 \geq 0, B_1 \geq 0, A_2 \geq 0, B_2 \geq 0, A_3 \geq 0, B_3 \geq 0$$

Solving this, we get the solution:

$$\text{Optimal Solution: } C = 7800; A_1 = 500, B_1 = 200, A_2 = 500, B_2 = 0, A_3 = 0, B_3 = 600$$

### Important Topics of this Section

Setting up the objective and constraint equations for an application

Interpreting the solution to a linear programming question

### Try it Now Answers

1.

Our problem setup is:

$$\text{Maximize } P = 175A + 130B + 40C$$

Subject to:

$$4A + 5B + 2C \leq 300$$

$$7A + 4B + 1C \leq 250$$

$$4A + 5B + 1C \leq 200$$

Solving this, we get the solution:

$$\text{Optimal Solution: } P = 7907.89; A = 18.4211, B = 5.26316, C = 100$$

Rounding down, producing  $A = 18$ ,  $B = 5$ , and  $C = 100$  would use 297 hours of manufacturing, 246 hours of assembly, and 197 hours of finishing and have profit of \$7800. We could fiddle with a few other nearby integer solutions:

$$A = 18, B = 5, \text{ and } C = 101: \text{ Profit } \$7840$$

$$A = 18, B = 6, \text{ and } C = 98: \text{ Profit } \$7850$$

$$A = 19, B = 4, \text{ and } C = 101: \text{ Profit } \$7885$$

As you can see, finding optimal integer solutions is a harder problem.



## Chapter 4: Polynomial and Rational Functions

Section 4.1 Quadratic Functions .....	153
Section 4.2 Polynomial Functions .....	164
Section 4.3 Rational Functions .....	175

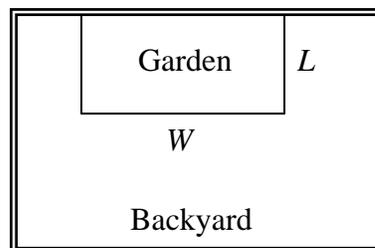
### Section 4.1 Quadratic Functions

In this section, we will explore the quadratic functions, a type of polynomial function. Quadratics commonly arise from problems involving areas, as well as revenue and profit, providing some interesting applications.

#### Example 1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4<sup>th</sup> side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length  $L$ .

In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable,  $W$ , to represent the side of fencing parallel to the 4<sup>th</sup> side or backyard fence.



Since we know we only have 80 feet of fence available, we know that  $L + W + L = 80$ , or more simply,  $2L + W = 80$ . This allows us to represent the width,  $W$ , in terms of  $L$ :  $W = 80 - 2L$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L)$$

$$A(L) = 80L - 2L^2$$

This formula represents the area of the fence in terms of the variable length  $L$ .

#### Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**.

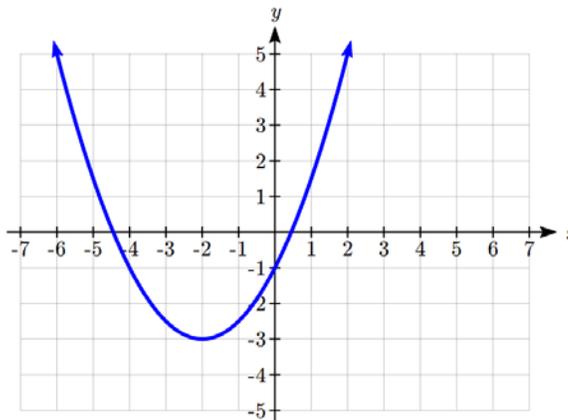
The **standard form** for a quadratic is  $f(x) = ax^2 + bx + c$ , but you will often see them written in the form  $f(x) = a(x - h)^2 + k$ . To see why, consider this example.

## Example 2

Sketch a graph of  $g(x) = \frac{1}{2}(x+2)^2 - 3$

We can create a table of values, which we can use to plot several points and connect them with a smooth curve.

$x$	$g(x)$
-5	1.5
-4	-1
-3	-2.5
-2	-3
-1	-2.5
0	-1
1	1.5



Notice that the turning point of the graph, where it changes from decreasing to increasing, is at the point  $(-2, -3)$ . We call this point the **vertex** of the quadratic. Notice that

$g(x) = \frac{1}{2}(x+2)^2 - 3$  can also be written as  $g(x) = \frac{1}{2}(x - (-2))^2 - 3$ . Comparing that to the form  $f(x) = a(x-h)^2 + k$ , you can see that the vertex of the graph,  $(-2, -3)$ , corresponds with the point  $(h, k)$ .

## Forms of Quadratic Functions

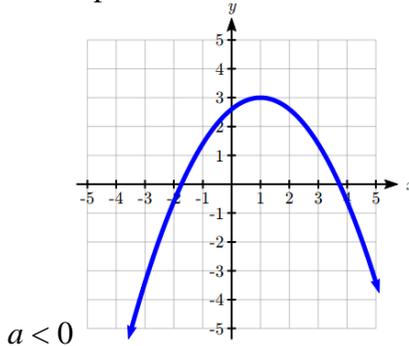
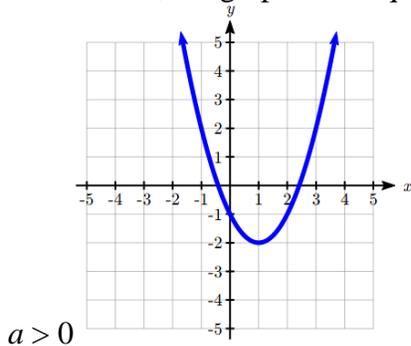
The **standard form** of a quadratic function is  $f(x) = ax^2 + bx + c$

The **vertex form** of a quadratic function is  $f(x) = a(x-h)^2 + k$

The **vertex** of the quadratic function is located at  $(h, k)$ , where  $h$  and  $k$  are the numbers in the vertex form of the function.

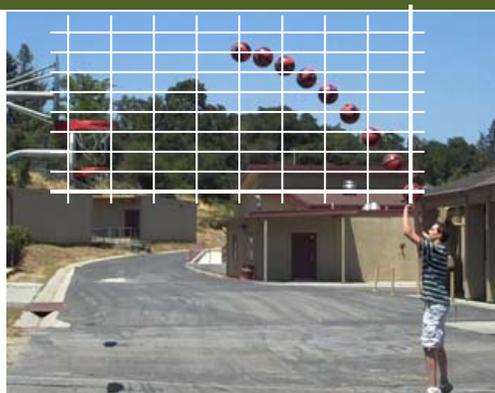
When  $a > 0$ , the graph of the quadratic will open upwards.

When  $a < 0$ , the graph of the quadratic will open downwards.



## Try it Now

1. A coordinate grid has been superimposed over the quadratic path of a basketball<sup>1</sup>. Find an equation for the path of the ball. Does he make the basket?



## Example 3

Write  $g(x) = \frac{1}{2}(x + 2)^2 - 3$  in standard form.

To write this in standard polynomial form, we could expand the formula and simplify terms:

$$g(x) = \frac{1}{2}(x + 2)^2 - 3$$

$$g(x) = \frac{1}{2}(x + 2)(x + 2) - 3$$

$$g(x) = \frac{1}{2}(x^2 + 4x + 4) - 3$$

$$g(x) = \frac{1}{2}x^2 + 2x + 2 - 3$$

$$g(x) = \frac{1}{2}x^2 + 2x - 1$$

In the previous example, we saw that it is possible to rewrite a quadratic function given in vertex form and rewrite it in standard form by expanding the formula. It would be useful to reverse this process, since the transformation form reveals the vertex.

Expanding out the general transformation form of a quadratic gives:

$$f(x) = a(x - h)^2 + k = a(x - h)(x - h) + k$$

$$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$$

This should be equal to the standard form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

<sup>1</sup> From <http://blog.mrmeyer.com/?p=4778>, © Dan Meyer, CC-BY

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$$-2ah = b, \text{ so } h = -\frac{b}{2a}$$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$$ah^2 + k = c, \text{ so } k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that  $k$  is the output value of the function when the input is  $h$ , so  $k = f(h)$ .

### Finding the Vertex of a Quadratic

For a quadratic given in standard form, the vertex  $(h, k)$  is located at:

$$h = -\frac{b}{2a}, \quad k = f(h) = f\left(-\frac{b}{2a}\right)$$

### Example 4

Find the vertex of the quadratic  $f(x) = 2x^2 - 6x + 7$ . Rewrite the quadratic into vertex form.

The horizontal coordinate of the vertex will be at  $h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$

The vertical coordinate of the vertex will be at  $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}$

Rewriting into vertex form, the value of  $a$  will remain the same as in the original quadratic.

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

### Try it Now

2. Given the equation  $g(x) = 13 + x^2 - 6x$  write the equation in standard form and then in vertex form.

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

### Example 5

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation  $A(L) = 80L - 2L^2$ . Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that  $a$  is the coefficient on the squared term, so  $a = -2$ ,  $b = 80$ , and  $c = 0$ .

Finding the vertex:

$$h = -\frac{80}{2(-2)} = 20, \quad k = A(20) = 80(20) - 2(20)^2 = 800$$

The maximum value of the function is an area of 800 square feet, which occurs when  $L = 20$  feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

### Example 6

A local newspaper currently has 84,000 subscribers, at a quarterly charge of \$30. Market research has suggested that if they raised the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables,  $C$  for charge per subscription and  $S$  for the number subscribers, giving us the equation:

$$\text{Revenue} = CS$$

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently  $S = 84,000$  and  $C = 30$ , and that if they raise the price to \$32 they would lose 5,000 subscribers, giving a second pair of values,  $C = 32$  and  $S = 79,000$ . From this we can find a linear equation relating the two quantities. Treating  $C$  as the input and  $S$  as the output, the equation will have form

$$S = mC + b. \quad \text{The slope will be}$$

$$m = \frac{79,000 - 84,000}{32 - 30} = \frac{-5,000}{2} = -2,500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$$\begin{aligned} S &= -2500C + b && \text{Plug in the point } S = 84,000 \text{ and } C = 30 \\ 84,000 &= -2500(30) + b && \text{Solve for } b \\ b &= 159,000 \end{aligned}$$

This gives us the linear equation  $S = -2,500C + 159,000$  relating cost and subscribers. We now return to our revenue equation.

$$\begin{aligned} \text{Revenue} &= CS && \text{Substituting the equation for } S \text{ from above} \\ \text{Revenue} &= C(-2,500C + 159,000) && \text{Expanding} \\ \text{Revenue} &= -2,500C^2 + 159,000C \end{aligned}$$

We now have a quadratic equation for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex:

$$h = -\frac{159,000}{2(-2,500)} = 31.8$$

The model tells us that the maximum revenue will occur if the newspaper charges \$31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:

$$\text{Maximum Revenue} = -2,500(31.8)^2 + 159,000(31.8) = \$2,528,100$$

Notice that the equation  $S = -2,500C + 159,000$  we found in the last example is essentially a demand function – a relationship between the price ( $C$ ) and the demand ( $S$ ). This is a common type of economic application.

### Maximizing Revenue or Profit

To solve a problem involving maximizing revenue or profit given data on demand at different price levels,

1. Use the data provided to create a demand equation relating price and quantity demanded. This will usually be a linear function.
2. Create a Revenue equation. Start with Revenue = price times quantity, then substitute the demand equation to create a revenue equation just in terms of quantity.
3. If the problem is to maximize profit, create a Profit equation. Profit = revenue minus cost.
4. Find the vertex of the Revenue or Profit function to find the quantity that maximizes revenue or profit.
5. If needed, use that quantity with the demand function to find the price that maximizes revenue or profit.

## Example 7

A company is planning to sell a new smart fitness device. Developing the product will cost \$700,000, and each product will cost \$30 to manufacture. Market research suggests that if they sell the device for \$100, they will be able to sell 30,000 items. For each \$10 they lower the price, they estimate they will sell 5,000 more items. Assuming quantity demanded is linearly related to price, determine the price that will maximize profit

Let  $p$  be the price per item, and  $q$  be the quantity the company can sell. We start by creating a linear demand function, of the form  $p = mq + b$ .

If they sell the device for \$100, they will be able to sell 30,000 items, giving the point (30000, 100).

We are told that for each \$10 they lower the price, they estimate they will sell 5,000 more items. We can either directly interpret this as a slope, or use it to create a second point. Lowering the price to \$90 would raise the quantity sold to 35,000, giving a second point (35000, 90).

Finding the slope:

$$m = \frac{90 - 100}{35000 - 30000} = \frac{-10}{5000} = -\frac{1}{500} = -0.002$$

The demand equation will look like  $p = -0.002q + b$ . Substituting in (30000, 100) to solve for  $b$ :

$$100 = -0.002(30000) + b$$

$$100 = -60 + b$$

$$b = 160$$

We now have our demand function:  $p = -0.002q + 160$ .

Now we construct our revenue function, using our demand curve

$$R = pq \quad \text{Substituting in the demand function for } p$$

$$R = (-0.002q + 160)q = -0.002q^2 + 160q$$

Now looking at costs, we know the fixed costs are \$700,000, and the per-item costs are \$30, leading to the cost equation

$$C = 700,000 + 30q$$

Finally we can construct the profit equation:

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$P = (-0.002q^2 + 160q) - (700,000 + 30q)$$

$$P = -0.002q^2 + 160q - 700,000 - 30q$$

$$P = -0.002q^2 + 130q - 700,000$$

Now to find the maximum we find the vertex of the quadratic:

$$h = -\frac{130}{2(-0.002)} = 32,500$$

To find the price that will produce a demand of 32,500 items, we use the demand function:

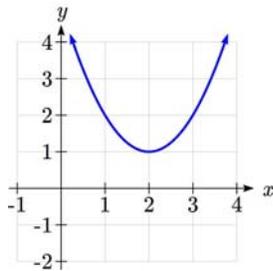
$$p = -0.002(32500) + 160 = 95$$

To maximize profit, the company should price the device at \$95. At that price they should expect to sell 35,000 items, with a total profit of:

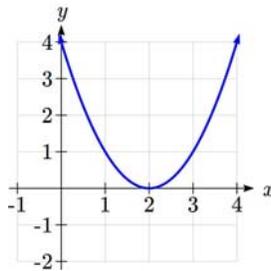
$$P = -0.002(32500)^2 + 130(32500) - 700,000 = \$1,412,500$$

### Short run Behavior: Intercepts

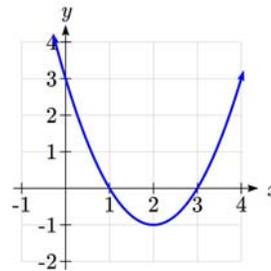
As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.



zero horizontal intercepts



one horizontal intercept



two horizontal intercepts

### Example 8

Find the vertical and horizontal intercepts of the quadratic  $f(x) = 3x^2 + 5x - 2$

We can find the vertical intercept by evaluating the function at an input of zero:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2 \quad \text{Vertical intercept at } (0, -2)$$

For the horizontal intercepts, we solve for when the output will be zero

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution

$$0 = (3x - 1)(x + 2)$$

$$0 = 3x - 1$$

$$x = \frac{1}{3}$$

or

$$0 = x + 2$$

$$x = -2$$

Horizontal intercepts at  $\left(\frac{1}{3}, 0\right)$  and  $(-2, 0)$

Notice that in the standard form of a quadratic, the constant term  $c$  reveals the vertical intercept of the graph.

For quadratics that can't be factored, we need another technique. Based on our previous work we showed that any quadratic in standard form can be written into vertex form as:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Solving for the horizontal intercepts using this general equation gives:

$$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \quad \text{start to solve for } x \text{ by moving the constants to the other side}$$

$$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2 \quad \text{divide both sides by } a$$

$$\frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 \quad \text{find a common denominator to combine fractions}$$

$$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2 \quad \text{combine the fractions on the left side of the equation}$$

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2 \quad \text{take the square root of both sides}$$

$$\pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a} \quad \text{subtract } b/2a \text{ from both sides}$$

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x \quad \text{combining the fractions}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Notice that this can yield two different answers for } x$$

### Quadratic Formula

For a quadratic function given in standard form  $f(x) = ax^2 + bx + c$ , the **quadratic formula** gives the horizontal intercepts of the graph of this function.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### Example 9

A ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation

$$H(t) = -16t^2 + 80t + 40.$$

What is the maximum height of the ball?

When does the ball hit the ground?

To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}, \quad k = H\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds.

To find when the ball hits the ground, we need to determine when the height is zero – when  $H(t) = 0$ . While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

### Example 10

The supply for a certain product can be modeled by  $p = 3q^2$  and the demand can be modeled by  $p = 1620 - 2q^2$ , where  $p$  is the price in dollars, and  $q$  is the quantity in thousands of items. Find the equilibrium price and quantity.

Recall that the equilibrium price and quantity is found by finding where the supply and demand curve intersect. We can find that by setting the equations equal:

$$3q^2 = 1620 - 2q^2 \quad \text{Add } 2q^2 \text{ to both sides}$$

$$5q^2 = 1620 \quad \text{Divide by 5 on both sides}$$

$$q^2 = 324 \quad \text{Take the square root of both sides}$$

$$q = \pm\sqrt{324} = \pm 18$$

Since it doesn't make sense to talk about negative quantities, the equilibrium quantity is  $q = 18$ . To find the equilibrium price, we evaluate either function at the equilibrium quantity.

$$p = 3(18)^2 = 972$$

The equilibrium is 18 thousand items, at a price of \$972.

### Try it Now

4. For these two equations determine if the vertex will be a maximum value or a minimum value.

- $g(x) = -8x + x^2 + 7$
- $g(x) = -3(3 - x)^2 + 2$

### Important Topics of this Section

- Quadratic functions
  - Standard form
  - Transformation form/Vertex form
  - Vertex as a maximum / Vertex as a minimum
- Short run behavior
  - Vertex / Horizontal & Vertical intercepts
- Quadratic formula

### Try it Now Answers

1. The path passes through the origin with vertex at  $(-4, 7)$ .

$h(x) = -\frac{7}{16}(x+4)^2 + 7$ . To make the shot,  $h(-7.5)$  would need to be about 4.  $h(-7.5) \approx 1.64$ ; he doesn't make it.



2.  $g(x) = x^2 - 6x + 13$  in Standard form;  $g(x) = (x - 3)^2 + 4$  in Transformation form

3. Vertical intercept at  $(0, 13)$ , NO horizontal intercepts.

- Vertex is a minimum value
- Vertex is a maximum value

## Section 4.2 Polynomial Functions

In the previous section we explored the short run behavior of quadratics, a special case of polynomials. In this section we will explore the behavior of polynomials in general. The basic building blocks of polynomials are power functions.

### Power Function

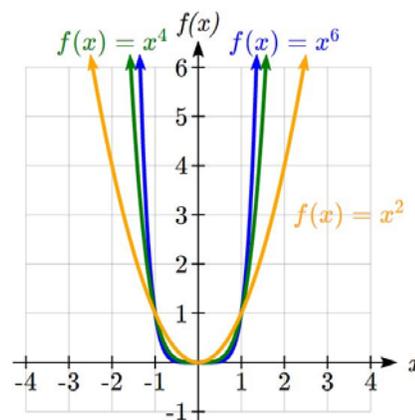
A **power function** is a function that can be represented in the form

$$f(x) = x^p$$

Where the base is a variable and the exponent,  $p$ , is a number.

### Characteristics of Power Functions

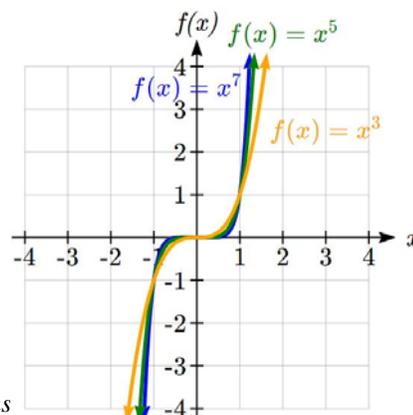
Shown to the right are the graphs of  $f(x) = x^2$ ,  $f(x) = x^4$ , and  $f(x) = x^6$ , all even whole number powers. Notice that all these graphs have a fairly similar shape, very similar to a quadratic, but as the power increases the graphs flatten somewhat near the origin, and become steeper away from the origin.



To describe the behavior as numbers become larger and larger, we use the idea of infinity. The symbol for positive infinity is  $\infty$ , and  $-\infty$  for negative infinity. When we say that “ $x$  approaches infinity”, which can be symbolically written as  $x \rightarrow \infty$ , we are describing a behavior – we are saying that  $x$  is getting large in the positive direction.

With the even power function, as the input becomes large in either the positive or negative direction, the output values become very large positive numbers. Equivalently, we could describe this by saying that as  $x$  approaches positive or negative infinity, the  $f(x)$  values approach positive infinity. In symbolic form, we could write: as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow \infty$ .

Shown here are the graphs of  $f(x) = x^3$ ,  $f(x) = x^5$ , and  $f(x) = x^7$ , all odd whole number powers. Notice all these graphs look similar, but again as the power increases the graphs flatten near the origin and become steeper away from the origin.



For these odd power functions, as  $x$  approaches negative infinity,  $f(x)$  approaches negative infinity. As  $x$  approaches positive infinity,  $f(x)$  approaches positive infinity. In symbolic form we write: as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$  and as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$ .

### Long Run Behavior

The behavior of the graph of a function as the input takes on large negative values ( $x \rightarrow -\infty$ ) and large positive values ( $x \rightarrow \infty$ ) as is referred to as the **long run behavior** of the function.

## Polynomials

### Terminology of Polynomial Functions

A **polynomial** is function that can be written as  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Each of the  $a_i$  constants are called **coefficients** and can be positive, negative, or zero, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, that is any  $a_ix^i$ . Each individual term is a transformed power function.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the “leading” term we often rearrange polynomials so that the powers are descending.

$$f(x) = a_nx^n + \dots + a_2x^2 + a_1x + a_0$$

### Example 1

Identify the degree, leading term, and leading coefficient of these polynomials:

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

For the function  $f(x)$ , the degree is 3, the highest power on  $x$ . The leading term is the term containing that power,  $-4x^3$ . The leading coefficient is the coefficient of that term, -4.

For  $g(t)$ , the degree is 5, the leading term is  $5t^5$ , and the leading coefficient is 5.

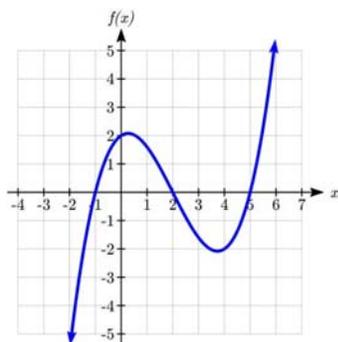
For  $h(p)$ , the degree is 3, the leading term is  $-p^3$ , so the leading coefficient is -1.

### Long Run Behavior of Polynomials

For any polynomial, the **long run behavior** of the polynomial will match the long run behavior of the leading term.

#### Example 2

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?



Since the output grows large and positive as the inputs grow large and positive, we describe the long run behavior symbolically by writing: as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$ .

Similarly, as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$ .

In words, we could say that as  $x$  values approach infinity, the function values approach infinity, and as  $x$  values approach negative infinity the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function which has not been reflected, so the degree of the polynomial creating this graph must be odd, and the leading coefficient would be positive.

#### Short Run Behavior: Intercepts

Characteristics of the graph such as vertical and horizontal intercepts and the places the graph changes direction are part of the short run behavior of the polynomial.

Like with all functions, the vertical intercept is where the graph crosses the vertical axis, and occurs when the input value is zero. Since a polynomial is a function, there can only be one vertical intercept, which occurs at the point  $(0, a_0)$ . The horizontal intercepts occur at the input values that correspond with an output value of zero. It is possible to have more than one horizontal intercept.

Horizontal intercepts are also called **zeros**, or **roots** of the function.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and 4<sup>th</sup> degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

- 1) The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
- 2) The polynomial is given in factored form.
- 3) Technology is used to determine the intercepts.

### Example 3

Find the horizontal intercepts of  $f(x) = x^6 - 3x^4 + 2x^2$ .

We can attempt to factor this polynomial to find solutions for  $f(x) = 0$ .

$$\begin{array}{ll}
 x^6 - 3x^4 + 2x^2 = 0 & \text{Factoring out the greatest common factor} \\
 x^2(x^4 - 3x^2 + 2) = 0 & \text{Factoring the inside as a quadratic in } x^2 \\
 x^2(x^2 - 1)(x^2 - 2) = 0 & \text{Then break apart to find solutions} \\
 \begin{array}{lll}
 x^2 = 0 & \text{or} & (x^2 - 1) = 0 \\
 x = 0 & & x^2 = 1 \\
 & & \text{or} \\
 & & (x^2 - 2) = 0 \\
 & & x^2 = 2 \\
 & & x = \pm\sqrt{2}
 \end{array}
 \end{array}$$

This gives us 5 horizontal intercepts.

### Example 4

Find the vertical and horizontal intercepts of  $g(t) = (t - 2)^2(2t + 3)$

The vertical intercept can be found by evaluating  $g(0)$ .

$$g(0) = (0 - 2)^2(2(0) + 3) = 12$$

The horizontal intercepts can be found by solving  $g(t) = 0$

$$\begin{array}{ll}
 (t - 2)^2(2t + 3) = 0 & \text{Since this is already factored, we can break it apart:} \\
 (t - 2)^2 = 0 & (2t + 3) = 0 \\
 t - 2 = 0 & \text{or} \\
 t = 2 & t = \frac{-3}{2}
 \end{array}$$

We can always check our answers are reasonable by graphing the polynomial.

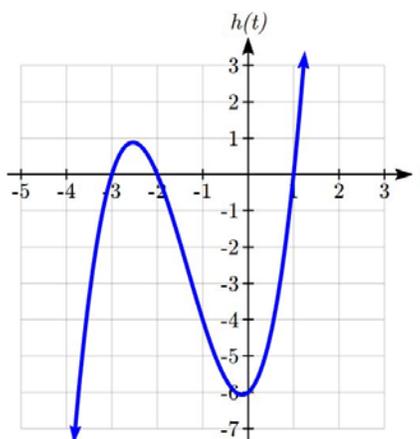
## Example 5

Find the horizontal intercepts of  $h(t) = t^3 + 4t^2 + t - 6$

Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

Graphing this function, it appears there are horizontal intercepts at  $t = -3, -2,$  and  $1$ .

We could check these are correct by plugging in these values for  $t$  and verifying that  $h(-3) = h(-2) = h(1) = 0$ .



Notice that the polynomial in the previous example was degree three, and had three horizontal intercepts and two turning points – places where the graph changes direction. We will now make a general statement without justifying it.

## Intercepts and Turning Points of Polynomials

A polynomial of degree  $n$  will have:

At most  $n$  horizontal intercepts. An odd degree polynomial will always have at least one.

At most  $n-1$  turning points

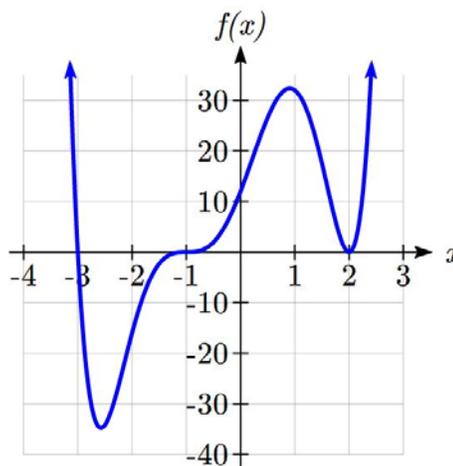
## Try it Now

1. Find the vertical and horizontal intercepts of the function  $f(t) = t^4 - 4t^2$ .

## Graphical Behavior at Intercepts

If we graph the function  $f(x) = (x+3)(x-2)^2(x+1)^3$ , notice that the behavior at each of the horizontal intercepts is different.

At the horizontal intercept  $x = -3$ , coming from the  $(x+3)$  factor of the polynomial, the graph passes directly through the horizontal intercept. The factor is linear (has a power of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a single zero, since the zero corresponds to a single factor of the function.



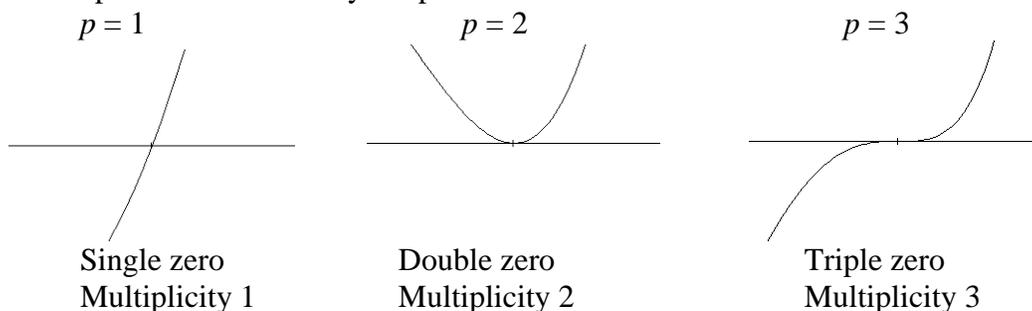
At the horizontal intercept  $x = 2$ , coming from the  $(x - 2)^2$  factor of the polynomial, the graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off of the horizontal axis at the intercept. Since  $(x - 2)^2 = (x - 2)(x - 2)$ , the factor is repeated twice, so we call this a double zero. We could also say the zero has **multiplicity 2**.

At the horizontal intercept  $x = -1$ , coming from the  $(x + 1)^3$  factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same “S” type shape near the intercept that the toolkit  $x^3$  has. We call this a triple zero. We could also say the zero has multiplicity 3.

By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

### Graphical Behavior of Polynomials at Horizontal Intercepts

If a polynomial contains a factor of the form  $(x - h)^p$ , the behavior near the horizontal intercept  $h$  is determined by the power on the factor.



For higher even powers 4,6,8 etc.... the graph will still bounce off of the horizontal axis but the graph will appear flatter with each increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc... the graph will still pass through the horizontal axis but the graph will appear flatter with each increasing odd power as it approaches and leaves the axis.

### Example 6

Sketch a graph of  $f(x) = -2(x + 3)^2(x - 5)$ .

This graph has two horizontal intercepts. At  $x = -3$ , the factor is squared, indicating the graph will bounce at this horizontal intercept. At  $x = 5$ , the factor is not squared, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be  $-2x^3$ , so the long-run behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs get large positive, and the inputs increasing as the inputs get large negative.

To sketch this we consider the following:

As  $x \rightarrow -\infty$  the function  $f(x) \rightarrow \infty$  so we know the graph starts in the 2<sup>nd</sup> quadrant and is decreasing toward the horizontal axis.

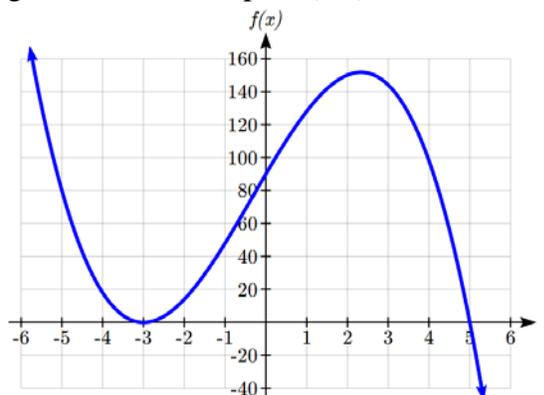
At  $(-3, 0)$  the graph bounces off of the horizontal axis and so the function must start increasing.

At  $(0, 90)$  the graph crosses the vertical axis at the vertical intercept.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis since the graph passes through the next intercept at  $(5, 0)$ .

As  $x \rightarrow \infty$  the function  $f(x) \rightarrow -\infty$  so we know the graph continues to decrease and we can stop drawing the graph in the 4<sup>th</sup> quadrant.

Using technology we can verify that the resulting graph will look like:



## Solving Polynomial Inequalities

One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative. We can solve polynomial inequalities by either utilizing the graph, or by using test values.

### Example 7

$$\text{Solve } (x + 3)(x + 1)^2(x - 4) > 0$$

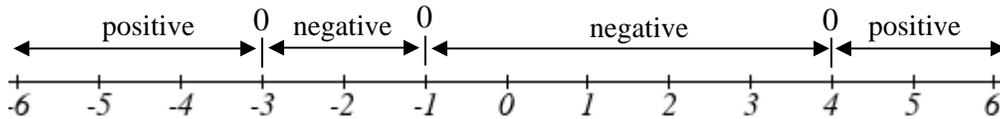
As with all inequalities, we start by solving the equality  $(x + 3)(x + 1)^2(x - 4) = 0$ , which has solutions at  $x = -3, -1$ , and  $4$ . We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function

$f(x) = (x+3)(x+1)^2(x-4)$  at each test value to determine if the function is positive or negative in that interval

Interval	Test $x$ in interval	$f(\text{test value})$	$>0$ or $<0$ ?
$x < -3$	-4	72	$> 0$
$-3 < x < -1$	-2	-6	$< 0$
$-1 < x < 4$	0	-12	$< 0$
$x > 4$	5	288	$> 0$

On a number line this would look like:



From our test values, we can determine this function is positive when  $x < -3$  or  $x > 4$ , or in interval notation,  $(-\infty, -3) \cup (4, \infty)$

We could have also determined on which intervals the function was positive by sketching a graph of the function. We illustrate that technique in the next example

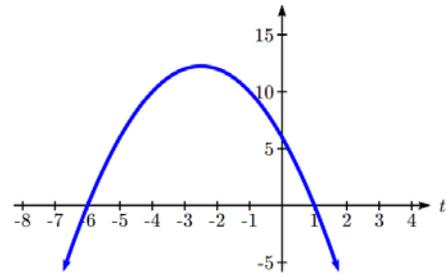
### Example 8

Find the domain of the function  $v(t) = \sqrt{6 - 5t - t^2}$ .

A square root is only defined when the quantity we are taking the square root of, the quantity inside the square root, is zero or greater. Thus, the domain of this function will be when  $6 - 5t - t^2 \geq 0$ .

Again we start by solving the equality  $6 - 5t - t^2 = 0$ . While we could use the quadratic formula, this equation factors nicely to  $(6+t)(1-t) = 0$ , giving horizontal intercepts  $t = 1$  and  $t = -6$ . Sketching a graph of this quadratic will allow us to determine when it is positive.

From the graph we can see this function is positive for inputs between the intercepts. So  $6 - 5t - t^2 \geq 0$  for  $-6 \leq t \leq 1$ , and this will be the domain of the  $v(t)$  function.



## Try it Now

2. Given the function  $g(x) = x^3 - x^2 - 6x$  use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the long run behavior and sketch the graph without technology.

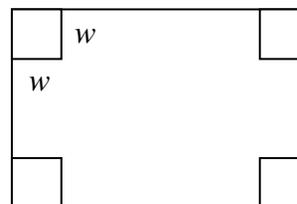
## Estimating Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

## Example 9

An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

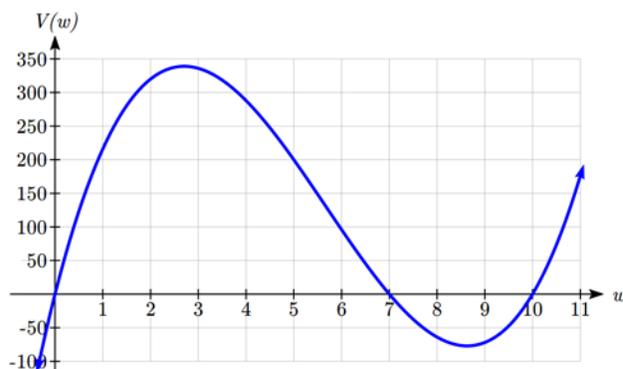
We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable,  $w$ .



Notice that after a square is cut out from each end, it leaves a  $(14-2w)$  cm by  $(20-2w)$  cm rectangle for the base of the box, and the box will be  $w$  cm tall. This gives the volume:

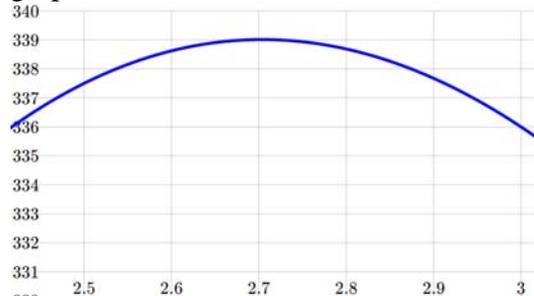
$$V(w) = (14 - 2w)(20 - 2w)w = 280w - 68w^2 + 4w^3$$

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for  $w$ : values from 0 to 7.



From this graph, we can estimate the maximum value is around 340, and occurs when the squares are about 2.75cm square. To improve this estimate, we could use advanced

features of our technology, if available, or simply change our window to zoom in on our graph.



From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the squares are 2.7cm square.

### Try it Now

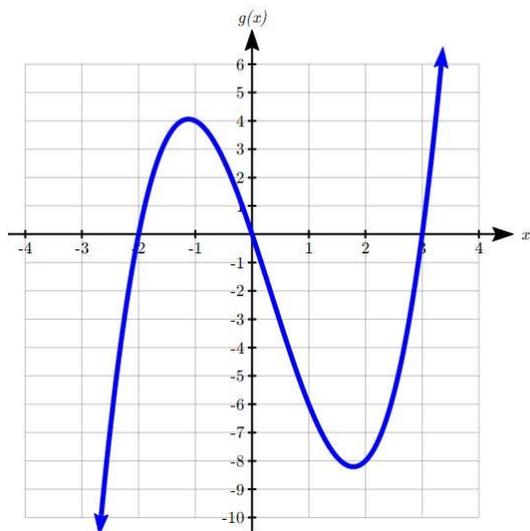
3. Use technology to find the maximum and minimum values on the interval  $[-1, 4]$  of the function  $f(x) = -0.2(x - 2)^3(x + 1)^2(x - 4)$ .

### Important Topics of this Section

- Vocabulary: Polynomials, coefficients, leading coefficient, term
- Degree of a polynomial
- Long Run Behavior
- Short Run Behavior
  - Intercepts (Horizontal & Vertical)
- Methods to find Horizontal intercepts
  - Factoring Methods
  - Factored Forms
  - Technology
- Graphical Behavior at intercepts
  - Single, Double and Triple zeros (or multiplicity 1, 2, and 3 behaviors)
- Solving polynomial inequalities using test values & graphing techniques
- Estimating extrema

## Try it Now Answers

1. Vertical intercept  $(0, 0)$ , Horizontal intercepts  $(0, 0)$ ,  $(-2, 0)$ ,  $(2, 0)$
2. Vertical intercept  $(0, 0)$ , Horizontal intercepts  $(-2, 0)$ ,  $(0, 0)$ ,  $(3, 0)$   
The function is negative on  $(-\infty, -2)$  and  $(0, 3)$   
The function is positive on  $(-2, 0)$  and  $(3, \infty)$   
The leading term is  $x^3$  so as  $x \rightarrow -\infty$ ,  $g(x) \rightarrow -\infty$  and as  $x \rightarrow \infty$ ,  $g(x) \rightarrow \infty$



3. The minimum occurs at approximately the point  $(0, -6.5)$ , and the maximum occurs at approximately the point  $(3.5, 7)$ .

## Section 4.3 Rational Functions

We will begin this section by looking at another economic concept. In addition to total cost and marginal cost, another metric of costs is **average cost**.

### Average Cost

The **average cost** of producing  $q$  items with a total cost of  $C(q)$  is

$$AC(q) = \frac{C(q)}{q}$$

### Example 1

Suppose that manufacturing an item has fixed costs of \$4000 and per-item (marginal) costs of \$20 per item. Find the average cost per item when 250 items are produced.

The total cost function is  $C(q) = 4000 + 20q$ .

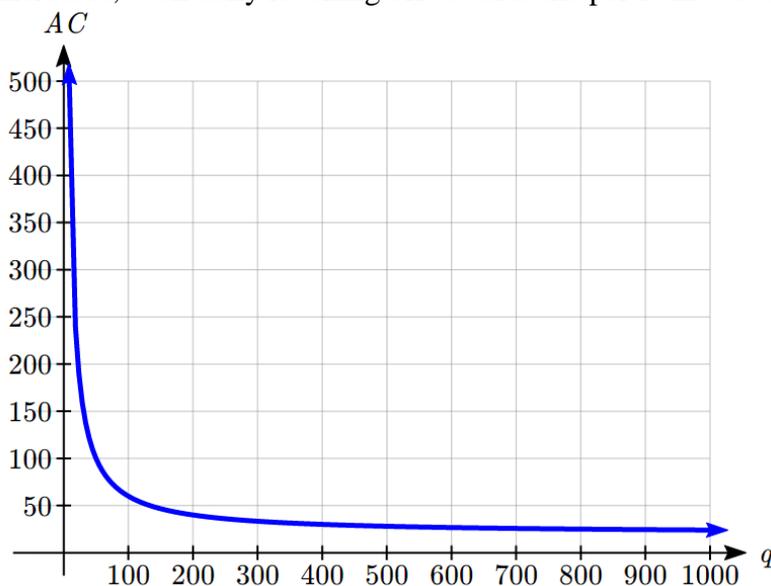
Then average cost is  $AC(q) = \frac{4000 + 20q}{q}$ .

When 250 items are produced, the average cost per item will be

$$AC(250) = \frac{4000 + 20(250)}{250} = \$36$$

The average cost per item is \$36.

Notice that the average cost per item distributes the fixed costs across all the items. If we were to graph the average cost, notice that average costs decrease as production increases, eventually levelling off towards the per-item cost of \$20.



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The average cost functions is one example of a **rational function**.

### Rational Function

A **rational function** is a function that can be written as the ratio of two polynomials,  $P(x)$  and  $Q(x)$ .

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_px^p}{b_0 + b_1x + b_2x^2 + \cdots + b_qx^q}$$

To get a feel for these functions, we will look at a graph.

### Example 2

Graph  $f(x) = \frac{2x+4}{x-3}$

Notice that the graph is undefined at  $x = 3$ , since that would cause us to divide by 0. We can create a table of values to get a sense for the function.

$x$	$f(x)$
-5	.75
-3	.333
-2	0
0	-1.333
2	-8
2.9	-98
2.99	-998
3	undefined
3.1	102
4	12
5	7

Notice that as  $x$  gets close to 3 from the left, the function values are getting larger and larger in the negative direction. When  $x$  is close to 3 from the right, the function values are getting large and positive. We call  $x = 3$  a **vertical asymptote** of the function.

When  $x = -2$ , the function value is 0, giving a **horizontal intercept**. Notice that when  $x = -2$  the numerator of the rational function is zero.

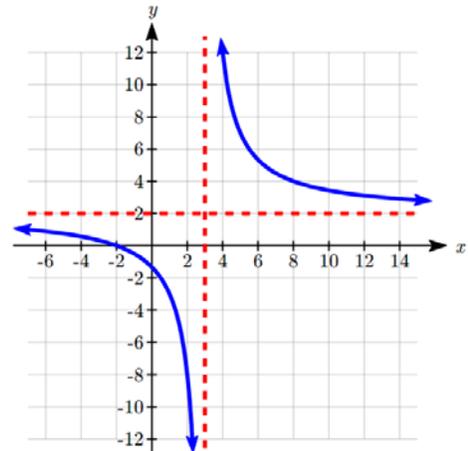
To get a better sense of the long run behavior, we can evaluate the function at additional, larger inputs:

$x$	$f(x)$
-1000	1.99
-100	1.90
-10	1.23
10	3.43
100	2.10
1000	2.01

Notice that as  $x$  gets large on both ends, the function approaches 2 but never actually reaches 2, it just “levels off” as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line  $f(x) = 2$  as the input becomes very large in the negative and positive directions.

Sketching a graph of this function:

The graphs of rational functions are sometimes sketched with the horizontal and vertical asymptotes drawn as dashed lines. It’s important to realize these are *not* part of the graph of the function itself.



### Vertical and Horizontal Asymptotes

A **vertical asymptote** of a graph is a vertical line  $x = a$  where the graph tends towards positive or negative infinity as the inputs approach  $a$ .

A **horizontal asymptote** of a graph is a horizontal line  $y = b$  where the graph approaches the line as the inputs get large.

In the first example looking at average cost, the function had a vertical asymptote at  $q=0$ , and a horizontal asymptote at  $AC=20$ .

### Finding Asymptotes and Intercepts

Given a rational function, as part of investigating the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts, as we have done in the past.

To find vertical asymptotes, we notice that the vertical asymptotes in our examples occur when the denominator of the function is undefined. With one exception, a vertical asymptote will occur whenever the denominator is undefined.

## Example 3

Find the vertical asymptotes of the function  $k(x) = \frac{5 + 2x^2}{2 - x - x^2}$

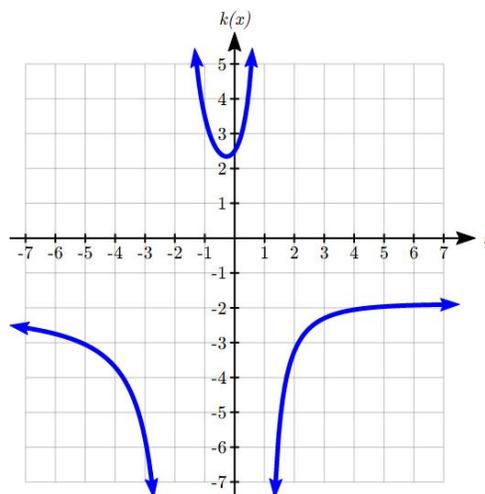
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$2 - x - x^2 = 0$$

$$(2 + x)(1 - x) = 0$$

$$x = -2, 1$$

This indicates two vertical asymptotes, which a look at a graph confirms.



The exception to this rule can occur when both the numerator and denominator of a rational function are zero at the same input.

## Example 4

Find the vertical asymptotes of the function  $k(x) = \frac{x - 2}{x^2 - 4}$ .

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x^2 - 4 = 0$$

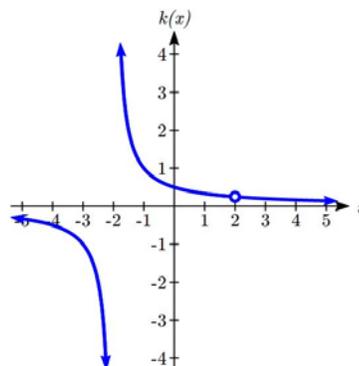
$$x^2 = 4$$

$$x = -2, 2$$

However, the numerator of this function is also equal to zero when  $x = 2$ . Because of this, the function will still be undefined at 2, since  $\frac{0}{0}$  is

undefined, but the graph will not have a vertical asymptote at  $x = 2$ .

The graph of this function will have the vertical asymptote at  $x = -2$ , but at  $x = 2$  the graph will have a hole: a single point where the graph is not defined, indicated by an open circle.



### Vertical Asymptotes and Holes of Rational Functions

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

A **hole** might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. In this case, factor the numerator and denominator and simplify; if the simplified expression still has a zero in the denominator at the original input the original function has a vertical asymptote at the input, otherwise it has a hole.

To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial's long run behavior will mirror that of the leading term. Likewise, a rational function's long run behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

**Case 1:** The degree of the denominator  $>$  degree of the numerator

Example:  $f(x) = \frac{3x + 2}{x^2 + 4x - 5}$

In this case, the long run behavior is  $f(x) \approx \frac{3x}{x^2} = \frac{3}{x}$ . This tells us that as the inputs grow large, this function will behave similarly to the function  $g(x) = \frac{3}{x}$ . As the inputs grow large, the outputs will approach zero, resulting in a horizontal asymptote at  $y = 0$ .

**Case 2:** The degree of the denominator  $<$  degree of the numerator

Example:  $f(x) = \frac{3x^2 + 2}{x - 5}$

In this case, the long run behavior is  $f(x) \approx \frac{3x^2}{x} = 3x$ . This tells us that as the inputs grow large, this function will behave similarly to the function  $g(x) = 3x$ . As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote.

Ultimately, if the numerator is larger than the denominator, the long run behavior of the graph will mimic the behavior of the reduced long run behavior fraction. As

another example if we had the function  $f(x) = \frac{3x^5 - x^2}{x + 3}$  with long run behavior

$f(x) \approx \frac{3x^5}{x} = 3x^4$ , the long run behavior of the graph would look similar to that of an even polynomial.

**Case 3:** The degree of the denominator = degree of the numerator

Example:  $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the long run behavior is  $f(x) \approx \frac{3x^2}{x^2} = 3$ . This tells us that as the inputs grow large, this function will behave like the function  $g(x) = 3$ , which is a horizontal line. This function has a horizontal asymptote at  $y = 3$ .

### Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

Degree of denominator > degree of numerator: Horizontal asymptote at  $y = 0$

Degree of denominator < degree of numerator: No horizontal asymptote

Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

### Example 5

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at  $x = 1, -2,$  and  $5$ , indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as  $x \rightarrow \pm\infty$ ,  $f(x) \rightarrow 0$ . This function will have a horizontal asymptote at  $y = 0$ .

### Try it Now

1. Find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

## Example 6

Nadi Products, an Ebay retailer that resells items purchased from Fiji, has started selling a new-to-market product. Nadi is currently selling 2,000 units a month, which has been growing by 100 units a month. The total market for the product is currently 10,000 units, but is growing by 1000 units a month.

- Determine Nadi's current market share
- Find the horizontal asymptote and interpret it in context of the scenario.

Notice that the sales per month are increasing linearly. Let  $t$  be the number of months, then Nadi's sales per month are  $N = 2000 + 100t$ , and the total market is  $M = 10000 + 1000t$ .

Nadi's market share,  $MS$ , the fraction of the total market that Nadi holds:

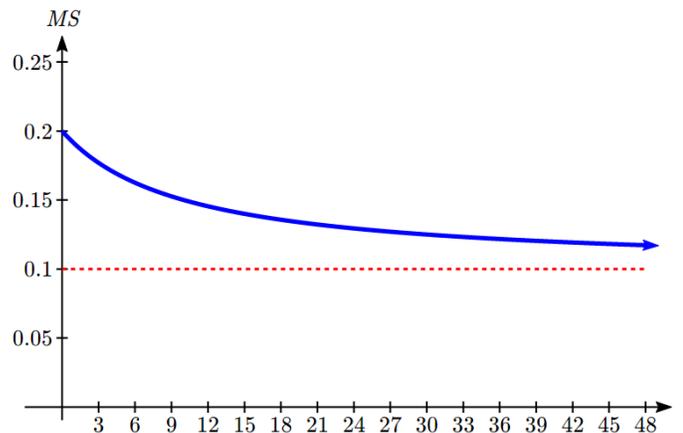
$$MS(t) = \frac{2000 + 100t}{10000 + 1000t}$$

- Nadi's current market share is:

$$MS(0) = \frac{2000 + 100(0)}{10000 + 1000(0)} = \frac{2000}{10000} = 0.20 = 20\%$$

- Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is  $100t$ , with coefficient 100. In the denominator, the leading term is  $1000t$ , with coefficient 1000. The horizontal asymptote will be at the ratio of these values, at  $y = \frac{100}{1000} = 0.10$ .

This tells us that as the input gets large, the output values will approach 0.10. In context, this means that as more time goes by, Nadi's market share will approach 10%.



## Intercepts

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

### Example 7

Find the intercepts of  $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

We can find the vertical intercept by evaluating the function at zero

$$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5}$$

The horizontal intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \quad \text{This is zero when the numerator is zero}$$

$$0 = (x-2)(x+3)$$

$$x = 2, -3$$

### Example 8

Sketch a graph of  $f(x) = \frac{6(x-1)}{(x+2)(x-3)}$ .

We can start our sketch by finding intercepts and asymptotes. Evaluating the function at zero gives the vertical intercept:

$$f(0) = \frac{6(0-1)}{(0+2)(0-3)} = \frac{-6}{-6} = 1$$

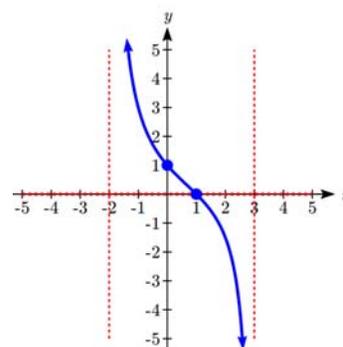
Looking at when the numerator of the function is zero, we can determine the graph will have a horizontal intercept at  $x = 1$ .

Looking at when the denominator of the function is zero, we can determine the graph will have vertical asymptotes at  $x = -2$  and  $x = 3$ .

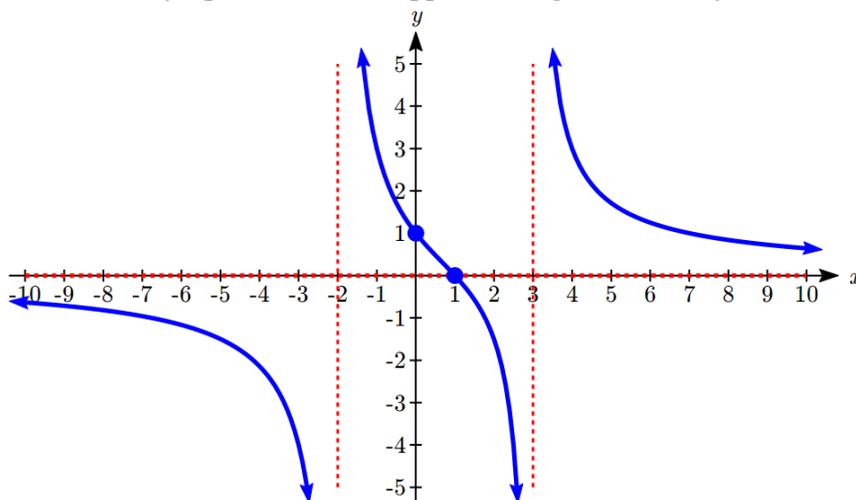
Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at  $y = 0$ .

To sketch the graph, we might start by plotting the intercepts and drawing the asymptotes as dashed lines.

We can now try to sketch a smooth curve passing through both intercepts. As the graph approaches  $x=3$ , the graph must approach negative infinity. As we sketch from  $x=0$  towards  $x=-2$ , we'd expect the graph to approach positive infinity.



To finish the curve, we need to fill in the portions to the right of  $x=3$  and to the left of  $x=-2$ . If we evaluate the function to the right of  $x=3$  we'd see the output is positive, indicating the graph will decrease from positive infinity then level off towards the horizontal asymptote of  $y=0$ . To the left of  $x=-2$  the graph will also level off at the horizontal asymptote, and will approach negative infinity to the left of the asymptote.



### Try it Now

2. Given the function  $f(x) = \frac{2(x+2)}{x-1}$ , find the intercepts and asymptotes and sketch the function.

**Important Topics of this Section**

Horizontal Asymptotes

Vertical Asymptotes

Rational Functions

Finding intercepts, asymptotes, and holes.

Given equation sketch the graph

**Try it Now Answers**

1. Vertical asymptotes at  $x = 2$  and  $x = -3$ ; horizontal asymptote at  $y = 4$

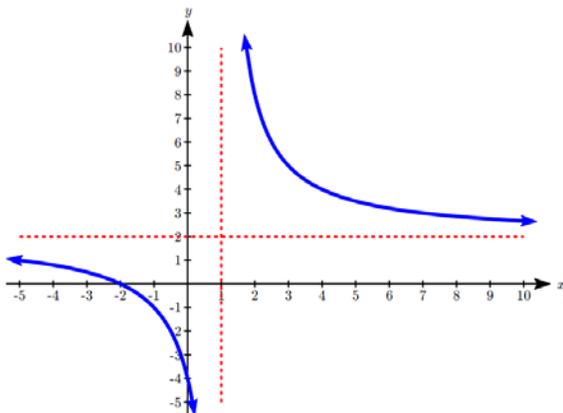
2.

Horizontal asymptote at  $y = 2$ .

Vertical asymptote is at  $x = 1$ .

Vertical intercept at  $(0, 4)$ .

Horizontal intercept  $(2, 0)$ .



## Chapter 5: Exponential and Logarithmic Functions

Section 5.1 Exponential Functions .....	185
Section 5.2 Logarithmic Functions.....	199
Section 5.3 Exponential and Logarithmic Models.....	211

### Section 5.1 Exponential Functions

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year<sup>1</sup>. We might ask if we can find a formula to model the population,  $P$ , as a function of time,  $t$ , in years after 2008, if the population continues to grow at this rate.

In linear growth, we had a constant rate of change – a constant *number* that the output increased for each increase in input. For example, in the equation  $f(x) = 3x + 4$ , the slope tells us the output increases by three each time the input increases by one. This population scenario is different – we have a *percent* rate of change rather than a constant number of people as our rate of change. To see the significance of this difference consider these two companies:

Company A has 100 stores, and expands by opening 50 new stores a year

Company B has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Looking at a few years of growth for these companies:

Year	Stores, company A		Stores, company B
0	100	Starting with 100 each	100
1	$100 + 50 = 150$	They both grow by 50 stores in the first year.	$100 + 50\% \text{ of } 100$ $100 + 0.50(100) = 150$
2	$150 + 50 = 200$	Store A grows by 50, Store B grows by 75	$150 + 50\% \text{ of } 150$ $150 + 0.50(150) = 225$
3	$200 + 50 = 250$	Store A grows by 50, Store B grows by 112.5	$225 + 50\% \text{ of } 225$ $225 + 0.50(225) = 337.5$

Notice that with the percent growth, each year the company is grows by 50% of the current year's total, so as the company grows larger, the number of stores added in a year grows as well.

<sup>1</sup> World Bank, World Development Indicators, as reported on <http://www.google.com/publicdata>, retrieved August 20, 2010

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To try to simplify the calculations, notice that after 1 year the number of stores for company  $B$  was:

$$100 + 0.50(100) \quad \text{or equivalently by factoring}$$

$$100(1 + 0.50) = 150$$

We can think of this as “the new number of stores is the original 100% plus another 50%”.

After 2 years, the number of stores was:

$$150 + 0.50(150) \quad \text{or equivalently by factoring}$$

$$150(1 + 0.50) \quad \text{now recall the 150 came from } 100(1+0.50). \text{ Substituting that,}$$

$$100(1 + 0.50)(1 + 0.50) = 100(1 + 0.50)^2 = 225$$

After 3 years, the number of stores was:

$$225 + 0.50(225) \quad \text{or equivalently by factoring}$$

$$225(1 + 0.50) \quad \text{now recall the 225 came from } 100(1 + 0.50)^2. \text{ Substituting that,}$$

$$100(1 + 0.50)^2(1 + 0.50) = 100(1 + 0.50)^3 = 337.5$$

From this, we can generalize, noticing that to show a 50% increase, each year we multiply by a factor of  $(1+0.50)$ , so after  $n$  years, our equation would be

$$B(n) = 100(1 + 0.50)^n$$

In this equation, the 100 represented the initial quantity, and the 0.50 was the percent growth rate. Generalizing further, we arrive at the general form of exponential functions.

### Exponential Function

An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$$f(x) = a(1 + r)^x \quad \text{or} \quad f(x) = ab^x \quad \text{where } b = 1+r$$

Where

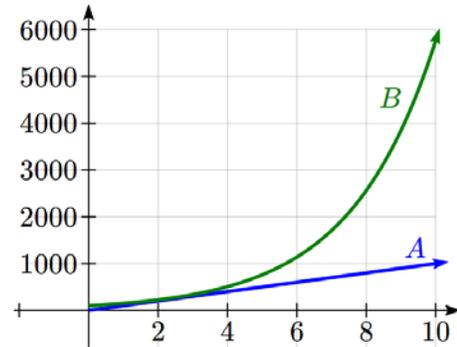
$a$  is the initial or starting value of the function

$r$  is the percent growth or decay rate, written as a decimal

$b$  is the growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit  $b$  to positive values.

To see more clearly the difference between exponential and linear growth, compare the two tables and graphs below, which illustrate the growth of company  $A$  and  $B$  described above over a longer time frame if the growth patterns were to continue.

years	Company A	Company B
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767



### Example 1

Write an exponential function for India's population, and use it to predict the population in 2020.

At the beginning of the chapter we were given India's population of 1.14 billion in the year 2008 and a percent growth rate of 1.34%. Using 2008 as our starting time ( $t = 0$ ), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for  $r$  is 0.0134.

Using the basic formula for exponential growth  $f(x) = a(1 + r)^x$  we can write the formula,  $f(t) = 1.14(1 + 0.0134)^t$

To estimate the population in 2020, we evaluate the function at  $t = 12$ , since 2020 is 12 years after 2008.

$$f(12) = 1.14(1 + 0.0134)^{12} \approx 1.337 \text{ billion people in 2020}$$

### Try it Now

- Given the three statements below, identify which represent exponential functions.
  - The cost of living allowance for state employees increases salaries by 3.1% each year.
  - State employees can expect a \$300 raise each year they work for the state.
  - Tuition costs have increased by 2.8% each year for the last 3 years.

### Example 2

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a \$1000 investment grow to over those 24 months?

First, we must notice that the interest rate is an annual rate, but is compounded monthly, meaning interest is calculated and added to the account monthly.

To find the monthly interest rate, we divide the annual rate of 1.2% by 12 since there are 12 months in a year:  $1.2\%/12 = 0.1\%$ . Each month we will earn 0.1% interest. From this, we can set up an exponential function, with our initial amount of \$1000 and a growth rate of  $r = 0.001$ , and our input  $m$  measured in months.

$$f(m) = 1000\left(1 + \frac{.012}{12}\right)^m$$

$$f(m) = 1000(1 + 0.001)^m$$

After 24 months, the account will have grown to  $f(24) = 1000(1 + 0.001)^{24} = \$1024.28$

### Try it Now

2. Looking at these two equations that represent the balance in two different savings accounts, which account is growing faster, and which account will have a higher balance after 3 years?

$$A(t) = 1000(1.05)^t \qquad B(t) = 900(1.075)^t$$

In all the preceding examples, we saw exponential growth. Exponential functions can also be used to model quantities that are decreasing at a constant percent rate. An example of this is radioactive decay, a process in which radioactive isotopes of certain atoms transform to an atom of a different type, causing a percentage decrease of the original material over time.

### Example 3

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is  $a = 100$  mg, and our growth rate will be negative 13%, since we are decreasing:  $r = -0.13$ . This gives the equation:

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d$$

This can also be explained by recognizing that if 13% decays, then 87% remains.

After one week, 7 days, the quantity remaining would be

$$Q(7) = 100(0.87)^7 = 37.73 \text{ mg of Bismuth-210 remains.}$$

### Try it Now

3. A population of 1000 is decreasing 3% each year. Find the population in 30 years.

**Example 4**

$T(q)$  represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2010. Interpret all of the parts of the equation  $T(2) = 86(1.64)^2 = 231.3056$ .

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts. Since  $b = 1 + r = 1.64$ , we know that every quarter the number of smart phone contracts grows by 64%.  $T(2) = 231.3056$  means that in the 2<sup>nd</sup> quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.

**Finding Equations of Exponential Functions**

In the previous examples, we were able to write equations for exponential functions since we knew the initial quantity and the growth rate. If we do not know the growth rate, but instead know only some input and output pairs of values, we can still construct an exponential function.

**Example 5**

In 2002, a company had 80 retail stores. By 2008, the company had grown to 180 retail stores. If the company is growing exponentially, find a formula for the function.

By defining our input variable to be  $t$ , years after 2002, the information listed can be written as two input-output pairs:  $(0,80)$  and  $(6,180)$ . Notice that by choosing our input variable to be measured as years after the first year value provided, we have effectively “given” ourselves the initial value for the function:  $a = 80$ . This gives us an equation of the form

$$f(t) = 80b^t .$$

Substituting in our second input-output pair allows us to solve for  $b$ :

$$\begin{aligned} 180 &= 80b^6 && \text{Divide by 80} \\ b^6 &= \frac{180}{80} = \frac{9}{4} && \text{Take the 6}^{\text{th}} \text{ root of both sides.} \\ b &= \sqrt[6]{\frac{9}{4}} = 1.1447 \end{aligned}$$

This gives us our equation for the population:

$$f(t) = 80(1.1447)^t$$

Recall that since  $b = 1+r$ , we can interpret this to mean that the population growth rate is  $r = 0.1447$ , and so the population is growing by about 14.47% each year.

In this example, you could also have used  $(9/4)^{(1/6)}$  to evaluate the 6<sup>th</sup> root if your calculator doesn't have an  $n^{\text{th}}$  root button. We chose to use the  $f(x) = ab^x$  form of the exponential function rather than the  $f(x) = a(1+r)^x$  form, but this choice was entirely arbitrary – either form would be fine to use.

When finding equations, the value for  $b$  or  $r$  will usually have to be rounded to be written easily. To preserve accuracy, it is important to not over-round these values. Typically, you want to be sure to preserve at least 3 significant digits in the growth rate. For example, if your value for  $b$  was 1.00317643, you would want to round this no further than to 1.00318.

### Try it Now

4. Find the formula for an exponential passing through (0, 10) and (3, 7).

## Graphs of Exponentials

To get a sense for the behavior of exponentials, let us begin by looking more closely at the function  $f(x) = 2^x$ . Listing a table of values for this function:

$x$	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notice that:

- 1) This function is positive for all values of  $x$ .
- 2) As  $x$  increases, the function grows faster and faster (the rate of change increases).
- 3) As  $x$  decreases, the function values grow smaller, approaching zero.
- 4) This is an example of exponential growth.

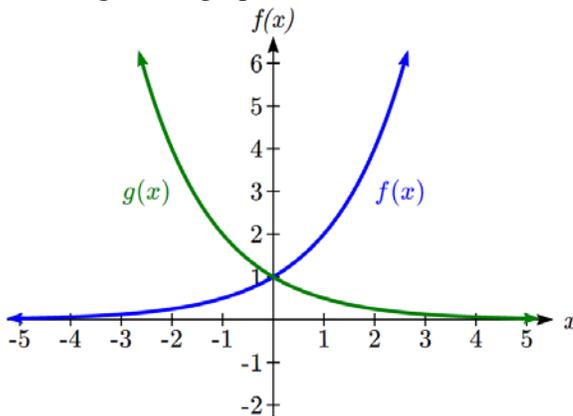
Looking at the function  $g(x) = \left(\frac{1}{2}\right)^x$

$x$	-3	-2	-1	0	1	2	3
$g(x)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Note this function is also positive for all values of  $x$ , but in this case grows as  $x$  decreases, and decreases towards zero as  $x$  increases. This is an example of exponential decay. You may notice from the table that this function appears to be the horizontal reflection of the  $f(x) = 2^x$  table. This is in fact the case:

$$f(-x) = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x = g(x)$$

Looking at the graphs also confirms this relationship:



Consider a function for the form  $f(x) = ab^x$ . Since  $a$ , which we called the initial value in the last section, is the function value at an input of zero,  $a$  will give us the vertical intercept of the graph. From the graphs above, we can see that an exponential graph will have a horizontal asymptote on one side of the graph, and can either increase or decrease, depending upon the growth factor. This horizontal asymptote will also help us determine the long run behavior and is easy to determine from the graph.

The graph will grow when the growth rate is positive, which will make the growth factor  $b$  larger than one. When it's negative, the growth factor will be less than one.

### Graphical Features of Exponential Functions

Graphically, in the function  $f(x) = ab^x$

$a$  is the vertical intercept of the graph

$b$  determines the rate at which the graph grows. When  $a$  is positive,  
the function will increase if  $b > 1$   
the function will decrease if  $0 < b < 1$

The graph will have a horizontal asymptote at  $y = 0$

The graph will be concave up if  $a > 0$ ; concave down if  $a < 0$ .

The domain of the function is all real numbers

The range of the function is  $(0, \infty)$

When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points  $(0, a)$  and  $(1, ab)$ .

The value  $b$  will determine the function's long run behavior:

If  $b > 1$ , as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$  and as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0$ .

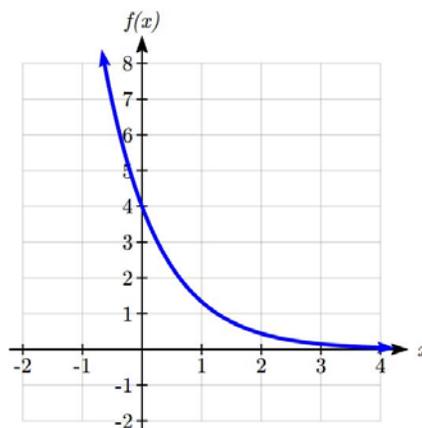
If  $0 < b < 1$ , as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0$  and as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$ .

## Example 6

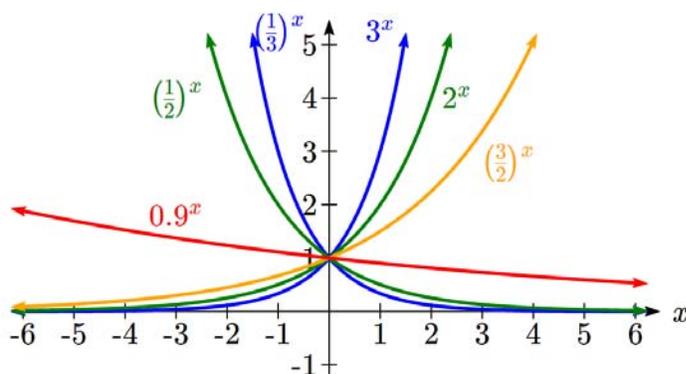
Sketch a graph of  $f(x) = 4\left(\frac{1}{3}\right)^x$

This graph will have a vertical intercept at  $(0,4)$ , and pass through the point  $\left(1, \frac{4}{3}\right)$ . Since  $b < 1$ , the graph will be decreasing towards zero. Since  $a > 0$ , the graph will be concave up.

We can also see from the graph the long run behavior: as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0$  and as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$ .

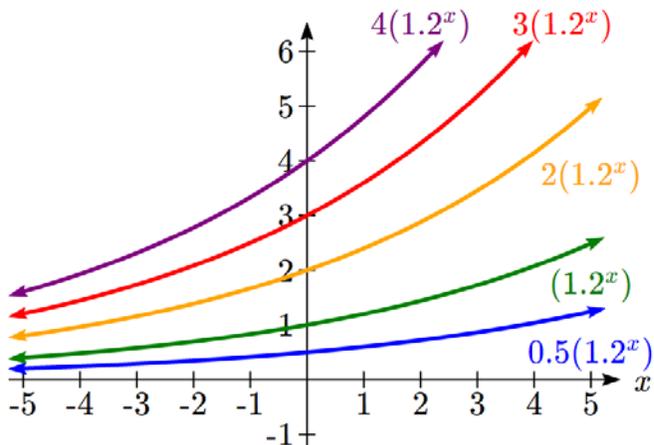


To get a better feeling for the effect of  $a$  and  $b$  on the graph, examine the sets of graphs below. The first set shows various graphs, where  $a$  remains the same and we only change the value for  $b$ .



Notice that the closer the value of  $b$  is to 1, the less steep the graph will be.

In the next set of graphs,  $a$  is altered and our value for  $b$  remains the same.



Notice that changing the value for  $a$  changes the vertical intercept. Since  $a$  is multiplying the  $b^x$  term,  $a$  acts as a vertical stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change and none of these  $a$  values introduced a vertical flip.

### Example 7

Match each equation with its graph.

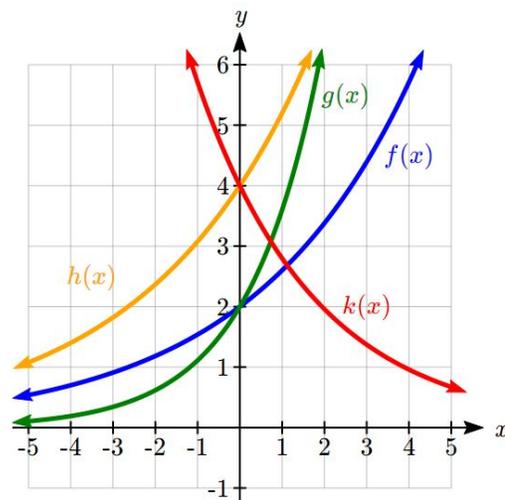
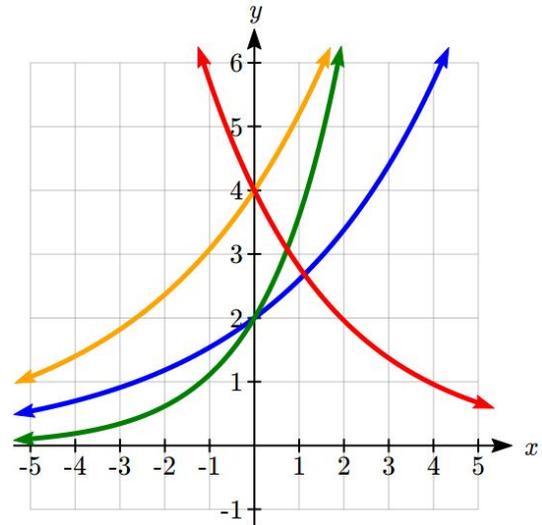
$$f(x) = 2(1.3)^x$$

$$g(x) = 2(1.8)^x$$

$$h(x) = 4(1.3)^x$$

$$k(x) = 4(0.7)^x$$

The graph of  $k(x)$  is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of  $h(x)$  can be identified as the only growing exponential function with a vertical intercept at  $(0,4)$ . The graphs of  $f(x)$  and  $g(x)$  both have a vertical intercept at  $(0,2)$ , but since  $g(x)$  has a larger growth factor, we can identify it as the graph increasing faster.



### Try it Now

5. Graph the following functions on the same axis:  $f(x) = (2)^x$ ;  $g(x) = 2(2)^x$ ;  
 $h(x) = 2(1/2)^x$ .

### Compound Interest

In the bank certificate of deposit (CD) example earlier in the section, we encountered compound interest. Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, utilize compound interest. The term *compounding* comes from the behavior that interest is earned not on the original value, but on the accumulated value of the account. We'll explore compound interest in greater depth in a later chapter, but will introduce basics here.

In the example from earlier, the interest was compounded monthly, so we took the annual interest rate, usually called the **nominal rate** or **annual percentage rate (APR)** and divided by 12, the number of compounds in a year, to find the monthly interest. The exponent was then measured in months.

Generalizing this, we can form a general formula for compound interest. If the APR is written in decimal form as  $r$ , and there are  $k$  compounding periods per year, then the interest per compounding period will be  $r/k$ . Likewise, if we are interested in the value after  $t$  years, then there will be  $kt$  compounding periods in that time.

#### Compound Interest Formula

**Compound Interest** can be calculated using the formula

$$A(t) = a \left( 1 + \frac{r}{k} \right)^{kt}$$

Where

$A(t)$  is the account value

$t$  is measured in years

$a$  is the starting amount of the account, often called the principal

$r$  is the annual percentage rate (APR), also called the nominal rate

$k$  is the number of compounding periods in one year

#### Example 8

If you invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Since we are starting with \$3000,  $a = 3000$

Our interest rate is 3%, so  $r = 0.03$

Since we are compounding quarterly, we are compounding 4 times per year, so  $k = 4$

We want to know the value of the account in 10 years, so we are looking for  $A(10)$ , the value when  $t = 10$ .

$$A(10) = 3000 \left( 1 + \frac{0.03}{4} \right)^{4(10)} = \$4045.05$$

The account will be worth \$4045.05 in 10 years.

**Example 9**

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to \$40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Since the account is earning 6%,  $r = 0.06$

Since interest is compounded twice a year,  $k = 2$

In this problem, we don't know how much we are starting with, so we will be solving for  $a$ , the initial amount needed. We do know we want the end amount to be \$40,000, so we will be looking for the value of  $a$  so that  $A(18) = 40,000$ .

$$40,000 = A(18) = a \left( 1 + \frac{0.06}{2} \right)^{2(18)}$$

$$40,000 = a(2.8983)$$

$$a = \frac{40,000}{2.8983} \approx \$13,801$$

Lily will need to invest \$13,801 to have \$40,000 in 18 years.

**Try it now**

6. Recalculate example 8 from above with monthly compounding.

**A Limit to Compounding**

Because of compounding throughout the year, with compound interest the actual increase in a year is *more* than the annual percentage rate. If \$1,000 were invested at 10%, the table below shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	\$1100
Semiannually	\$1102.50
Quarterly	\$1103.81
Monthly	\$1104.71
Daily	\$1105.16

The amount we earn increases as we increase the compounding frequency. Notice, though, that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that although increasing the frequency of compounding will increase our result, there is an upper limit to this process.

To see this, let us examine the value of \$1 invested at 100% interest for 1 year.

Frequency	Value
Annual	\$2
Semiannually	\$2.25
Quarterly	\$2.441406
Monthly	\$2.613035
Daily	\$2.714567
Hourly	\$2.718127
Once per minute	\$2.718279
Once per second	\$2.718282

These values do indeed appear to be approaching an upper limit. This value ends up being so important that it gets represented by its own letter, much like how  $\pi$  represents a number.

#### Euler's Number: $e$

$e$  is the letter used to represent the value that  $\left(1 + \frac{1}{k}\right)^k$  approaches as  $k$  gets big.

$$e \approx 2.718282$$

Because  $e$  is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of  $e$ , usually labeled  $e^x$ . Some computer software instead defines a function  $\exp(x)$ , where  $\exp(x) = e^x$ .

Because  $e$  arises when the time between compounds becomes very small,  $e$  allows us to define **continuous growth** and allows us to define a new toolkit function,  $f(x) = e^x$ .

#### Continuous Growth Formula

**Continuous Growth** can be calculated using the formula

$$f(x) = ae^{rx}$$

where

$a$  is the starting amount

$r$  is the continuous growth rate

This type of equation is commonly used when describing quantities that change more or less continuously, like chemical reactions, growth of large populations, and radioactive decay. It is also used extensively in calculus.

## Example 10

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative:  $r = -0.173$

$$f(3) = 100e^{-0.173(3)} \approx 59.512 \text{ mg of Radon-222 will remain.}$$

## Try it Now

7. Interpret the following:  $S(t) = 20e^{0.12t}$  if  $S(t)$  represents the growth of a substance in grams, and time is measured in days.

Continuous growth is also often applied to compound interest, allowing us to talk about continuous compounding.

## Example 11

If \$1000 is invested in an account earning 10% compounded continuously, find the value after 1 year.

Here, the continuous growth rate is 10%, so  $r = 0.10$ . We start with \$1000, so  $a = 1000$ . To find the value after 1 year,

$$f(1) = 1000e^{0.10(1)} \approx \$1105.17$$

Notice this is a \$105.17 increase for the year. As a percent increase, this is

$$\frac{105.17}{1000} = 0.10517 = 10.517\% \text{ increase over the original } \$1000.$$

Notice that this value is slightly larger than the amount generated by daily compounding in the table computed earlier.

The continuous growth rate is like the nominal growth rate (or APR) – it reflects the growth rate before compounding takes effect. This is different than the annual growth rate used in the formula  $f(x) = a(1+r)^x$ , which is like the annual percentage yield – it reflects the *actual* amount the output grows in a year.

While the continuous growth rate in the example above was 10%, the actual annual yield was 10.517%. This means we could write two different looking but equivalent formulas for this account's growth:

$$f(t) = 1000e^{0.10t} \quad \text{using the 10\% continuous growth rate}$$

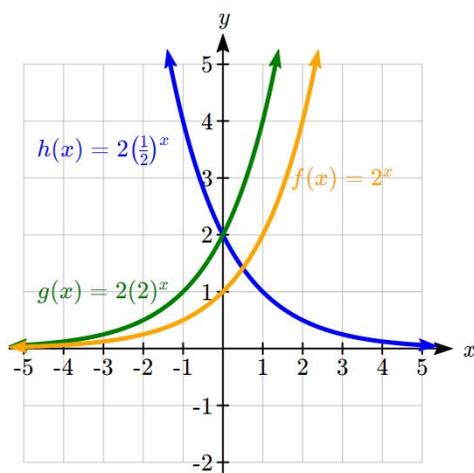
$$f(t) = 1000(1.10517)^t \quad \text{using the 10.517\% actual annual yield rate.}$$

### Important Topics of this Section

Percent growth  
 Exponential functions  
     Finding formulas  
     Interpreting equations  
     Graphs – Intercept and Growth factor  
 Exponential Growth & Decay  
 Long run behavior  
 Compound interest  
 Continuous Growth

### Try it Now Answers

1. A & C are exponential functions, they grow by a % not a constant number.
2. B(t) is growing faster, but after 3 years A(t) still has a higher account balance
3.  $1000(0.97)^{30} = 401.0071$ , or about 401 people.
4.  $f(x) = 10(0.8879)^x$ .



- 5.
6. \$4048.06
7. An initial substance weighing 20g is growing at a continuous rate of 12% per day.

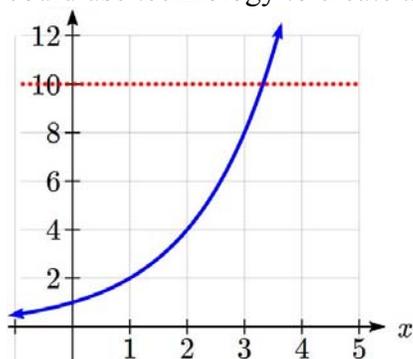
## Section 5.2 Logarithmic Functions

A population of 50 flies is expected to double every week, leading to a function of the form  $f(x) = 50(2)^x$ , where  $x$  represents the number of weeks that have passed. When will this population reach 500? Trying to solve this problem leads to:

$$500 = 50(2)^x \quad \text{Dividing both sides by 50 to isolate the exponential}$$

$$10 = 2^x$$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation  $2^x = 10$  above. We know that  $2^3 = 8$  and  $2^4 = 16$ , so it is clear that  $x$  must be some value between 3 and 4 since  $g(x) = 2^x$  is increasing. We could use technology to create a table of values or graph to better estimate the solution.



From the graph, we could better estimate the solution to be around 3.3. This result is still fairly unsatisfactory, since it would be nice to have a function that gives the answer. None of the functions we have already discussed would work, so we must introduce a new function, named **log**, as the function that “undoes” an exponential function, like how a square root “undoes” a square. Since exponential functions have different bases, we will define corresponding logarithms of different bases as well.

### Logarithm

**The logarithm** (base  $b$ ) function, written  $\log_b(x)$ , “undoes” exponential function  $b^x$ .

The statement  $b^a = c$  is equivalent to the statement  $\log_b(c) = a$ .

Since the logarithm and exponential “undo” each other (in technical terms, they are inverses), it follows that:

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### Properties of Logs: Inverse Properties

$$\log_b(b^x) = x$$

$$b^{\log_b x} = x$$

Since log is a function, it is most correctly written as  $\log_b(c)$ , using parentheses to denote function evaluation, just as we would with  $f(c)$ . However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as  $\log_b c$ .

#### Example 1

Write these exponential equations as logarithmic equations:

$$2^3 = 8$$

$$5^2 = 25$$

$$10^{-4} = \frac{1}{10000}$$

$$2^3 = 8 \quad \text{is equivalent to } \log_2(8) = 3$$

$$5^2 = 25 \quad \text{is equivalent to } \log_5(25) = 2$$

$$10^{-4} = \frac{1}{10000} \quad \text{is equivalent to } \log_{10}\left(\frac{1}{10000}\right) = -4$$

#### Example 2

Write these logarithmic equations as exponential equations:

$$\log_6(\sqrt{6}) = \frac{1}{2}$$

$$\log_3(9) = 2$$

$$\log_6(\sqrt{6}) = \frac{1}{2} \quad \text{is equivalent to } 6^{1/2} = \sqrt{6}$$

$$\log_3(9) = 2 \quad \text{is equivalent to } 3^2 = 9$$

#### Try it Now

- Write the exponential equation  $4^2 = 16$  as a logarithmic equation.

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

## Example 3

Solve  $\log_4(x) = 2$  for  $x$ .

By rewriting this expression as an exponential,  $4^2 = x$ , so  $x = 16$

## Example 4

Solve  $2^x = 10$  for  $x$ .

By rewriting this expression as a logarithm, we get  $x = \log_2(10)$

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful – often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

## Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written  $\log(x)$ .

The **natural log** is the logarithm with base  $e$ , and is typically written  $\ln(x)$ .

## Example 5

Evaluate  $\log(1000)$  using the definition of the common log.

To evaluate  $\log(1000)$ , we can say  
 $x = \log(1000)$ , then rewrite into exponential  
 form using the common log base of 10.  
 $10^x = 1000$

From this, we might recognize that 1000 is the  
 cube of 10, so  $x = 3$ .

We also can use the inverse property of logs to  
 write  $\log_{10}(10^3) = 3$

## Values of the common log

number	number as exponential	$\log(\text{number})$
1000	$10^3$	3
100	$10^2$	2
10	$10^1$	1
1	$10^0$	0
0.1	$10^{-1}$	-1
0.01	$10^{-2}$	-2
0.001	$10^{-3}$	-3

## Try it Now

2. Evaluate  $\log(1,000,000)$ .

**Example 6**

Evaluate  $\ln(\sqrt{e})$ .

We can rewrite  $\ln(\sqrt{e})$  as  $\ln(e^{1/2})$ . Since  $\ln$  is a log base  $e$ , we can use the inverse property for logs:  $\ln(e^{1/2}) = \log_e(e^{1/2}) = \frac{1}{2}$ .

**Example 7**

Evaluate  $\log(500)$  using your calculator or computer.

Using a computer, we can evaluate  $\log(500) \approx 2.69897$

**Graphs of Logarithms**

Recall that the exponential function  $f(x) = 2^x$  produces this table of values

$x$	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Since the logarithmic function “undoes” the exponential,  $g(x) = \log_2(x)$  produces the table of values

$x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

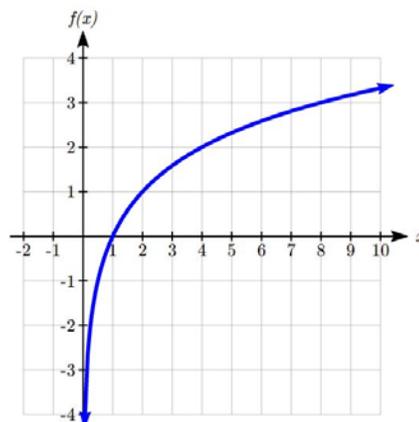
In this second table, notice that

- 1) As the input increases, the output increases.
- 2) As input increases, the output increases more slowly.
- 3) Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is  $(0, \infty)$ .
- 4) Since the exponential function can accept all real numbers as inputs, the logarithm can output any real number, so the range is all real numbers or  $(-\infty, \infty)$ .

Sketching the graph, notice that as the input approaches zero from the right, the output of the function grows very large in the negative direction, indicating a vertical asymptote at  $x = 0$ .

In symbolic notation we write

as  $x \rightarrow 0^+$ ,  $f(x) \rightarrow -\infty$ , and as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$

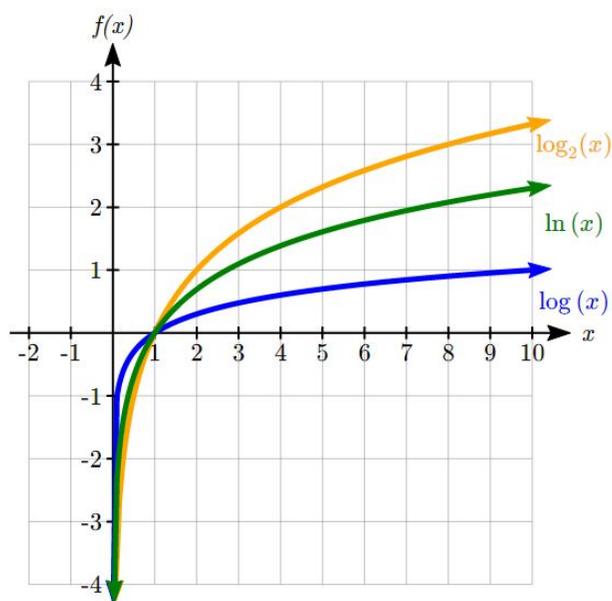


### Graphical Features of the Logarithm

- Graphically, in the function  $g(x) = \log_b(x)$
- The graph has a horizontal intercept at  $(1, 0)$
- The graph has a vertical asymptote at  $x = 0$
- The graph is increasing and concave down
- The domain of the function is  $x > 0$ , or  $(0, \infty)$
- The range of the function is all real numbers, or  $(-\infty, \infty)$

When sketching a general logarithm with base  $b$ , it can be helpful to remember that the graph will pass through the points  $(1, 0)$  and  $(b, 1)$ .

To get a feeling for how the base affects the shape of the graph, examine the graphs below.



Notice that the larger the base, the slower the graph grows. For example, the common log graph, while it grows without bound, it does so very slowly. For example, to reach an output of 8, the input must be 100,000,000.

Another important observation made was the domain of the logarithm. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a log.

#### Example 8

Find the domain of the function  $f(x) = \log(5 - 2x)$

The logarithm is only defined with the input is positive, so this function will only be defined when  $5 - 2x > 0$ . Solving this inequality,

$$-2x > -5$$

$$x < \frac{5}{2}$$

The domain of this function is  $x < \frac{5}{2}$ , or in interval notation,  $\left(-\infty, \frac{5}{2}\right)$

## Logarithm Properties

To utilize the common or natural logarithm functions to evaluate expressions like  $\log_2(10)$ , we need to establish some additional properties.

### Properties of Logs: Exponent Property

$$\log_b(A^r) = r \log_b(A)$$

To show why this is true, we offer a proof.

Since the logarithmic and exponential functions are inverses,  $b^{\log_b A} = A$ .

$$\text{So } A^r = \left(b^{\log_b A}\right)^r$$

Utilizing the exponential rule that states  $(x^p)^q = x^{pq}$ ,

$$A^r = \left(b^{\log_b A}\right)^r = b^{r \log_b A}$$

$$\text{So then } \log_b(A^r) = \log_b\left(b^{r \log_b A}\right)$$

Again utilizing the inverse property on the right side yields the result

$$\log_b(A^r) = r \log_b A$$

### Example 9

Rewrite  $\log_3(25)$  using the exponent property for logs.

Since  $25 = 5^2$ ,

$$\log_3(25) = \log_3(5^2) = 2 \log_3 5$$

### Example 10

Rewrite  $4 \ln(x)$  using the exponent property for logs.

Using the property in reverse,  $4 \ln(x) = \ln(x^4)$

## Try it Now

3. Rewrite using the exponent property for logs:  $\ln\left(\frac{1}{x^2}\right)$ .

The exponent property allows us to find a method for changing the base of a logarithmic expression.

## Properties of Logs: Change of Base

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)}$$

Proof:

Let  $\log_b(A) = x$ . Rewriting as an exponential gives  $b^x = A$ . Taking the log base  $c$  of both sides of this equation gives

$$\log_c b^x = \log_c A$$

Now utilizing the exponent property for logs on the left side,

$$x \log_c b = \log_c A$$

Dividing, we obtain

$$x = \frac{\log_c A}{\log_c b} \quad \text{or replacing our expression for } x, \log_b A = \frac{\log_c A}{\log_c b}$$

With this change of base formula, we can finally find a good decimal approximation to our question from the beginning of the section.

## Example 11

Evaluate  $\log_2(10)$  using the change of base formula.

According to the change of base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base  $e$ :

$$\log_2 10 = \frac{\log_e 10}{\log_e 2} = \frac{\ln 10}{\ln 2}$$

Using our calculators to evaluate this,

$$\frac{\ln 10}{\ln 2} \approx \frac{2.30259}{0.69315} \approx 3.3219$$

This finally allows us to answer our original question – the population of flies we discussed at the beginning of the section will take 3.32 weeks to grow to 500.

## Example 12

Evaluate  $\log_5(100)$  using the change of base formula.

We can rewrite this expression using any other base. If our calculators are able to evaluate the common logarithm, we could rewrite using the common log, base 10.

$$\log_5(100) = \frac{\log_{10} 100}{\log_{10} 5} \approx \frac{2}{0.69897} = 2.861$$

While we were able to solve the basic exponential equation  $2^x = 10$  by rewriting in logarithmic form and then using the change of base formula to evaluate the logarithm, the proof of the change of base formula illuminates an alternative approach to solving exponential equations.

## Solving exponential equations:

1. Isolate the exponential expressions when possible
2. Take the logarithm of both sides
3. Utilize the exponent property for logarithms to pull the variable out of the exponent
4. Use algebra to solve for the variable.

## Example 13

Solve  $2^x = 10$  for  $x$ .

Using this alternative approach, rather than rewrite this exponential into logarithmic form, we will take the logarithm of both sides of the equation. Since we often wish to evaluate the result to a decimal answer, we will usually utilize either the common log or natural log. For this example, we'll use the natural log:

$$\ln(2^x) = \ln(10) \quad \text{Utilizing the exponent property for logs,}$$

$$x \ln(2) = \ln(10) \quad \text{Now dividing by } \ln(2),$$

$$x = \frac{\ln(10)}{\ln(2)} \approx 2.861$$

Notice that this result matches the result we found using the change of base formula.

## Example 14

In the first section, we predicted the population (in billions) of India  $t$  years after 2008 by using the function  $f(t) = 1.14(1 + 0.0134)^t$ . If the population continues following this trend, when will the population reach 2 billion?

We need to solve for the  $t$  so that  $f(t) = 2$

$$2 = 1.14(1.0134)^t \quad \text{Divide by 1.14 to isolate the exponential expression}$$

$$\frac{2}{1.14} = 1.0134^t \quad \text{Take the logarithm of both sides of the equation}$$

$$\ln\left(\frac{2}{1.14}\right) = \ln(1.0134^t) \quad \text{Apply the exponent property on the right side}$$

$$\ln\left(\frac{2}{1.14}\right) = t \ln(1.0134) \quad \text{Divide both sides by } \ln(1.0134)$$

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln(1.0134)} \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

### Try it Now

4. Solve  $5(0.93)^x = 10$ .

### Additional Properties of Logarithms

Some situations cannot be addressed using the properties already discussed. For these, we need some additional properties.

#### Properties of Logs

##### Sum of Logs Property:

$$\log_b(A) + \log_b(C) = \log_b(AC)$$

##### Difference of Logs Property:

$$\log_b(A) - \log_b(C) = \log_b\left(\frac{A}{C}\right)$$

It's just as important to know what properties logarithms do *not* satisfy as to memorize the valid properties listed above. In particular, the logarithm is not a linear function, which means that it does not distribute:  $\log(A + B) \neq \log(A) + \log(B)$ .

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

## Example 15

Write  $\log_3(5) + \log_3(8) - \log_3(2)$  as a single logarithm.

Using the sum of logs property on the first two terms,

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to  $\log_3(40) - \log_3(2)$

Then using the difference of logs property,

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

## Example 16

Evaluate  $2\log(5) + \log(4)$  without a calculator by first rewriting as a single logarithm.

On the first term, we can use the exponent property of logs to write

$$2\log(5) = \log(5^2) = \log(25)$$

With the expression reduced to a sum of two logs,  $\log(25) + \log(4)$ , we can utilize the sum of logs property

$$\log(25) + \log(4) = \log(4 \cdot 25) = \log(100)$$

Since  $100 = 10^2$ , we can evaluate this log without a calculator:

$$\log(100) = \log(10^2) = 2$$

## Example 17

Rewrite  $\ln\left(\frac{x^4 y}{7}\right)$  as a sum or difference of logs

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$\ln\left(\frac{x^4 y}{7}\right) = \ln(x^4 y) - \ln(7)$$

Then seeing the product in the first term, we use the sum property

$$\ln(x^4 y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we could use the exponent property on the first term

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

## Try it Now

5. Without a calculator evaluate by first rewriting as a single logarithm:

$$\log_2(8) + \log_2(4)$$

**Log properties in solving equations**

The logarithm properties often arise when solving problems involving logarithms.

## Example 18

Solve  $\log(50x + 25) - \log(x) = 2$ .

In order to rewrite in exponential form, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{50x + 25}{x}\right) = 2$$

Rewriting in exponential form reduces this to an algebraic equation:

$$\frac{50x + 25}{x} = 10^2 = 100$$

Solving,

$$50x + 25 = 100x$$

$$25 = 50x$$

$$x = \frac{25}{50} = \frac{1}{2}$$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

## Try it Now

6. Solve  $\log(x^2 - 4) = 1 + \log(x + 2)$ .

**Important Topics of this Section**

The Logarithmic function as the inverse of the exponential function

Writing logarithmic & exponential expressions

Properties of logs

    Inverse properties

    Exponent property

    Change of base

    Sum and difference of logs properties

Common log

Natural log

Graph of the logarithmic function (domain and range)

Solving exponential equations

Solving equations using log rules

**Try it Now Answers**

1.  $\log_4(16) = 2$

2. 6

3.  $-2\ln(x)$

4.  $\frac{\ln(2)}{\ln(0.93)} \approx -9.5513$

5. 5

6. 12

## Section 5.3 Exponential and Logarithmic Models

While we have explored some basic applications of exponential and logarithmic functions, in this section we explore some important applications in more depth.

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

### Example 1a

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year<sup>2</sup>. If these trends continue, when will the population of Kenya match that of Sudan?

We start by writing an equation for each population in terms of  $t$ , the number of years after 2008.

$$\text{Kenya}(t) = 38.8(1 + 0.0264)^t$$

$$\text{Sudan}(t) = 41.3(1 + 0.0224)^t$$

To find when the populations will be equal, we can set the equations equal

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

For our first approach, we take the log of both sides of the equation

$$\log(38.8(1.0264)^t) = \log(41.3(1.0224)^t)$$

Utilizing the sum property of logs, we can rewrite each side,

$$\log(38.8) + \log(1.0264^t) = \log(41.3) + \log(1.0224^t)$$

Then utilizing the exponent property, we can pull the variables out of the exponent

$$\log(38.8) + t \log(1.0264) = \log(41.3) + t \log(1.0224)$$

Moving all the terms involving  $t$  to one side of the equation and the rest of the terms to the other side,

$$t \log(1.0264) - t \log(1.0224) = \log(41.3) - \log(38.8)$$

Factoring out the  $t$  on the left,

$$t(\log(1.0264) - \log(1.0224)) = \log(41.3) - \log(38.8)$$

<sup>2</sup> World Bank, World Development Indicators, as reported on <http://www.google.com/publicdata>, retrieved August 24, 2010

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Dividing to solve for  $t$

$$t = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)} \approx 15.991 \text{ years until the populations will be equal.}$$

### Example 1b

Solve the problem above by rewriting before taking the log.

Starting at the equation

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

Divide to move the exponential terms to one side of the equation and the constants to the other side

$$\frac{1.0264^t}{1.0224^t} = \frac{41.3}{38.8}$$

Using exponent rules to group on the left,

$$\left(\frac{1.0264}{1.0224}\right)^t = \frac{41.3}{38.8}$$

Taking the log of both sides

$$\log\left(\left(\frac{1.0264}{1.0224}\right)^t\right) = \log\left(\frac{41.3}{38.8}\right)$$

Utilizing the exponent property on the left,

$$t \log\left(\frac{1.0264}{1.0224}\right) = \log\left(\frac{41.3}{38.8}\right)$$

Dividing gives

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} \approx 15.991 \text{ years}$$

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)}$$

While both methods work equally well, it often requires fewer steps to utilize algebra before taking logs, rather than relying solely on log properties.

### Try it Now

1. Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

### Radioactive Decay

In an earlier section, we discussed radioactive decay – the idea that radioactive isotopes change over time. One of the common terms associated with radioactive decay is half-life.

### Half Life

The **half-life** of a radioactive isotope is the time it takes for half the substance to decay.

Given the basic exponential growth/decay equation  $h(t) = ab^t$ , half-life can be found by solving for when half the original amount remains; by solving  $\frac{1}{2}a = a(b)^t$ , or more simply  $\frac{1}{2} = b^t$ . Notice how the initial amount is irrelevant when solving for half-life.

### Example 2

Bismuth-210 is an isotope that decays by about 13% each day. What is the half-life of Bismuth-210?

We were not given a starting quantity, so we could either make up a value or use an unknown constant to represent the starting amount. To show that starting quantity does not affect the result, let us denote the initial quantity by the constant  $a$ . Then the decay of Bismuth-210 can be described by the equation  $Q(d) = a(0.87)^d$ .

To find the half-life, we want to determine when the remaining quantity is half the original:  $\frac{1}{2}a$ . Solving,

$$\begin{array}{ll} \frac{1}{2}a = a(0.87)^d & \text{Dividing by } a, \\ \frac{1}{2} = 0.87^d & \text{Take the log of both sides} \end{array}$$

$$\log\left(\frac{1}{2}\right) = \log(0.87^d) \quad \text{Use the exponent property of logs}$$

$$\log\left(\frac{1}{2}\right) = d \log(0.87) \quad \text{Divide to solve for } d$$

$$d = \frac{\log\left(\frac{1}{2}\right)}{\log(0.87)} \approx 4.977 \text{ days}$$

This tells us that the half-life of Bismuth-210 is approximately 5 days.

### Example 3

Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, how much will remain after 30 years? 60 years? 90 years?

Since the half-life is 30 years, after 30 years, half the original amount, 100mg, will remain.

After 60 years, another 30 years have passed, so during that second 30 years, another half of the substance will decay, leaving 50mg.

After 90 years, another 30 years have passed, so another half of the substance will decay, leaving 25mg.

### Example 4

Carbon-14 is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. Carbon-14 has a half-life of 5730 years. If a bone fragment is found that contains 20% of its original carbon-14, how old is the bone?

To find how old the bone is, we first will need to find an equation for the decay of the carbon-14. We could either use a continuous or annual decay formula, but opt to use the continuous decay formula since it is more common in scientific texts. The half life tells us that after 5730 years, half the original substance remains. Solving for the rate,

$$\frac{1}{2}a = ae^{r5730} \quad \text{Dividing by } a$$

$$\frac{1}{2} = e^{r5730} \quad \text{Taking the natural log of both sides}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{r5730}) \quad \text{Use the inverse property of logs on the right side}$$

$$\ln\left(\frac{1}{2}\right) = 5730r \quad \text{Divide by 5730}$$

$$r = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.000121$$

Now we know the decay will follow the equation  $Q(t) = ae^{-0.000121t}$ . To find how old the bone fragment is that contains 20% of the original amount, we solve for  $t$  so that  $Q(t) = 0.20a$ .

$$\begin{aligned} 0.20a &= ae^{-0.000121t} \\ 0.20 &= e^{-0.000121t} \\ \ln(0.20) &= \ln(e^{-0.000121t}) \\ \ln(0.20) &= -0.000121t \\ t &= \frac{\ln(0.20)}{-0.000121} \approx 13301 \text{ years} \end{aligned}$$

The bone fragment is about 13,300 years old.

### Try it Now

2. In Example 3, we learned that Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, how long will it take until only 1 milligram remains?

For decaying quantities, we asked how long it takes for half the substance to decay. For growing quantities we might ask how long it takes for the quantity to double.

### Doubling Time

The **doubling time** of a growing quantity is the time it takes for the quantity to double.

Given the basic exponential growth equation  $h(t) = ab^t$ , doubling time can be found by solving for when the original quantity has doubled; by solving  $2a = a(b)^x$ , or more simply  $2 = b^x$ . Again notice how the initial amount is irrelevant when solving for doubling time.

### Example 5

If you invest money at 8% compounded quarterly, how long will it take your money to double?

Using the compound interest equation, we can write  $A(t) = P\left(1 + \frac{0.08}{4}\right)^{4t} = P(1.02)^{4t}$ .

To find the doubling time, we look for the time until we have twice the original amount, so when  $A(t) = 2P$ . Notice we don't need to know how much was invested.

$$2P = P(1.02)^{4t}$$

$$2 = (1.02)^{4t}$$

$$\log(2) = \log(1.02^{4t})$$

$$\log(2) = 4t \log(1.02)$$

$$t = \frac{\log(2)}{4\log(1.02)} \approx 8.751 \text{ years.}$$

It will take about 8.75 years for the investment double in value.

### Example 6

Use of a new social networking website has been growing exponentially, with the number of new members doubling every 5 months. If the site currently has 120,000 users and this trend continues, how many users will the site have in 1 year?

We can use the doubling time to find a function that models the number of site users, and then use that equation to answer the question. While we could use an arbitrary  $a$  as we have before for the initial amount, in this case, we know the initial amount was 120,000.

If we use a continuous growth equation, it would look like  $N(t) = 120e^{rt}$ , measured in thousands of users after  $t$  months. Based on the doubling time, there would be 240 thousand users after 5 months. This allows us to solve for the continuous growth rate:

$$240 = 120e^{r5}$$

$$2 = e^{r5}$$

$$\ln 2 = 5r$$

$$r = \frac{\ln 2}{5} \approx 0.1386$$

Now that we have an equation,  $N(t) = 120e^{0.1386t}$ , we can predict the number of users after 12 months:

$$N(12) = 120e^{0.1386(12)} = 633.140 \text{ thousand users.}$$

So after 1 year, we would expect the site to have around 633,140 users.

### Try it Now

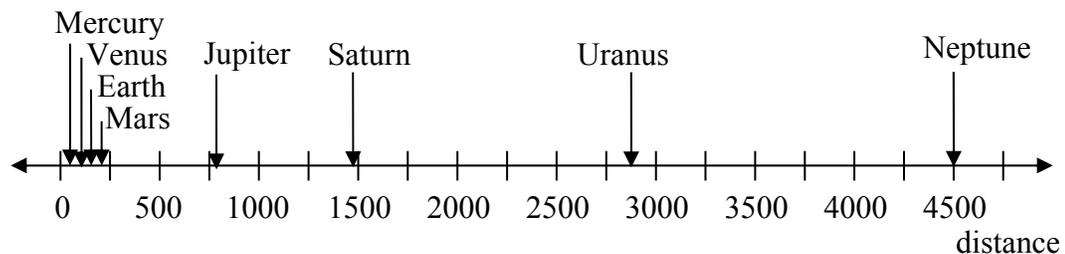
3. If tuition at a college is increasing by 6.6% each year, how many years will it take for tuition to double?

### Logarithmic Scales

For quantities that vary greatly in magnitude, a standard linear scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

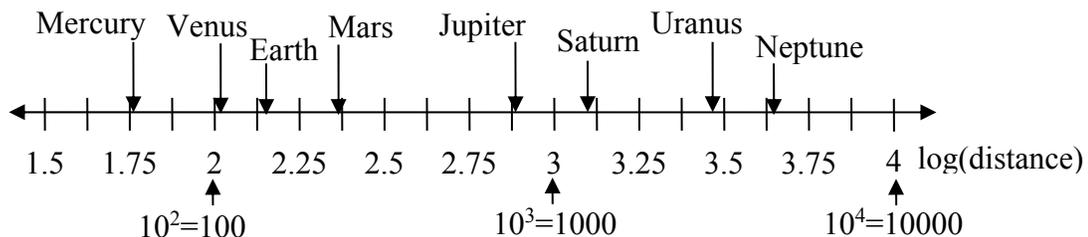
Planet	Distance (millions of km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale – one with equally spaced values – these values get bunched up.

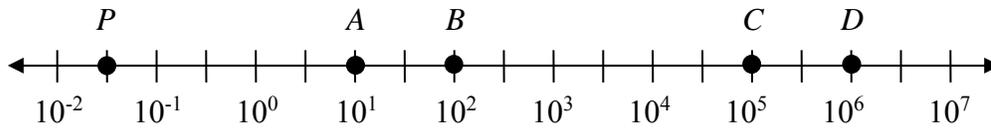


However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent.

Planet	Distance (millions of km)	$\log(\text{distance})$
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65



Sometimes, as shown above, the scale on a logarithmic number line will show the log values, but more commonly the original values are listed as powers of 10, as shown below.



#### Example 7

Estimate the value of point  $P$  on the log scale above

The point  $P$  appears to be half way between  $-2$  and  $-1$  in log value, so if  $V$  is the value of this point,

$$\log(V) \approx -1.5$$

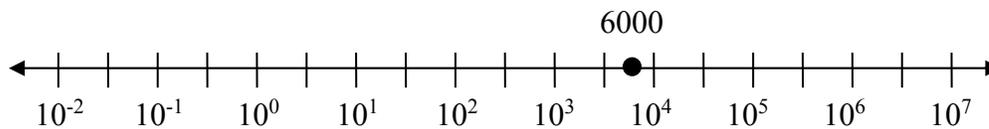
Rewriting in exponential form,

$$V \approx 10^{-1.5} = 0.0316$$

#### Example 8

Place the number 6000 on a logarithmic scale.

Since  $\log(6000) \approx 3.8$ , this point would belong on the log scale about here:



#### Try it Now

4. Plot the data in the table below on a logarithmic scale<sup>3</sup>.

Source of Sound/Noise	Approximate Sound Pressure in $\mu\text{Pa}$ (micro Pascals)
Launching of the Space Shuttle	2,000,000,000
Full Symphony Orchestra	2,000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio	200
Softest Sound a human can hear	20

<sup>3</sup> From [http://www.epd.gov.hk/epd/noise\\_education/web/ENG\\_EPd\\_HTML/m1/intro\\_5.html](http://www.epd.gov.hk/epd/noise_education/web/ENG_EPd_HTML/m1/intro_5.html), retrieved Oct 2, 2010

It is also common to see two dimensional graphs with one or both axes using a logarithmic scale. One common use of a logarithmic scale on the vertical axis is to graph quantities that are changing exponentially, since it helps reveal relative differences. This is commonly used in stock charts, since values historically have grown exponentially over time. Both stock charts below show the Dow Jones Industrial Average, from 1928 to 2010.

Linear scale:



Logarithmic scale:



Both charts have a linear horizontal scale, but the first graph has a linear vertical scale, while the second has a logarithmic vertical scale. The first scale is the one we are more familiar with, and shows what appears to be a strong exponential trend, at least up until the year 2000.

### Example 9

There were stock market drops in 1929 and 2008. Which was larger?

In the first graph, the stock market drop around 2008 looks very large, and in terms of dollar values, it was indeed a large drop. However the second graph shows relative changes, and the drop in 2009 seems less major on this graph, and in fact the drop starting in 1929 was, percentage-wise, much more significant.

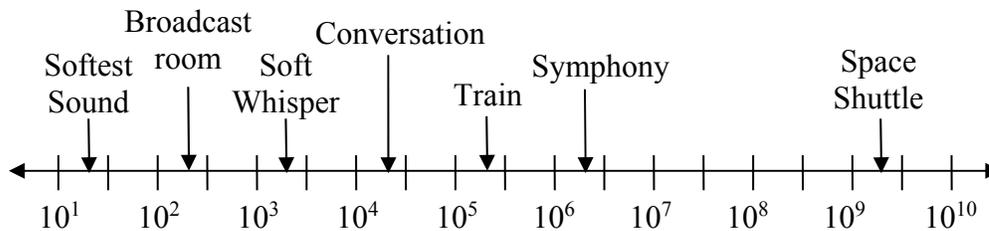
Specifically, in 2008, the Dow value dropped from about 14,000 to 8,000, a drop of 6,000. This is obviously a large value drop, and amounts to about a 43% drop. In 1929, the Dow value dropped from a high of around 380 to a low of 42 by July of 1932. While value-wise this drop of 338 is much smaller than the 2008 drop, it corresponds to a 89% drop, a much larger relative drop than in 2008. The logarithmic scale shows these relative changes.

### Important Topics of this Section

Radioactive decay  
Half life  
Doubling time  
Logarithmic Scales

### Try it Now Answers

- 4.1874 weeks
- 229.3157, or approximately 229 years
- It will take 10.845 years, or approximately 11 years, for tuition to double.
- 



## Chapter 6: Finance

Section 6.1: Simple and Compound Interest.....	221
Section 6.2: Annuities.....	230
Section 6.3: Payout Annuities.....	235
Section 6.4: Loans.....	239
Section 6.5: Multistage Finance Problems.....	246

### Section 6.1: Simple and Compound Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed \$100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100:  $\$100(0.05) = \$5$ . The total amount you would repay would be \$105, the original principal plus the interest.

#### Simple One-time Interest

$$I = Pr$$

$$A = P + I = P + Pr = P(1 + r)$$

$I$  is the interest

$A$  is the end amount: principal plus interest

$P$  is the principal (starting amount)

$r$  is the interest rate in decimal form. Example: 5% = 0.05

#### Example 1

A friend asks to borrow \$300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

$P = \$300$                       the principal

$r = 0.03$                         3% rate

$I = \$300(0.03) = \$9$ . You will earn \$9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

## Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Each year, you would earn 5% interest:  $\$1000(0.05) = \$50$  in interest. So over the course of five years, you would earn a total of \$250 in interest. When the bond matures, you would receive back the \$1,000 you originally paid, leaving you with a total of \$1,250.

We can generalize this idea of simple interest over time.

## Simple Interest over Time

$$I = Prt$$

$$A = P + I = P + Prt = P(1 + rt)$$

$I$  is the interest

$A$  is the end amount: principal plus interest

$P$  is the principal (starting amount)

$r$  is the interest rate in decimal form

$t$  is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

## APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.

A 4% annual rate paid quarterly would be divided into four 1% payments.

## Example 3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a \$1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P = \$1000$                       the principal

$r = 0.02$                         2% rate per half-year

$t = 8$                               4 years = 8 half-years

$I = \$1000(0.02)(8) = \$160$ . You will earn \$160 interest total over the four years.

## Try it Now

1. A loan company charges \$30 interest for a one month loan of \$500. Find the annual interest rate they are charging.

## Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**. We looked at this situation earlier, in the chapter on exponential growth.

## Compound Interest

$$A = P \left( 1 + \frac{r}{k} \right)^{kt}$$

$A$  is the balance in the account after  $t$  years.

$P$  is the starting balance of the account (also called initial deposit, or principal)

$r$  is the annual interest rate in decimal form

$k$  is the number of compounding periods in one year.

- If the compounding is done annually (once a year),  $k = 1$ .
- If the compounding is done quarterly,  $k = 4$ .
- If the compounding is done monthly,  $k = 12$ .
- If the compounding is done daily,  $k = 365$ .

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

## Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

In this example,

$P = \$3000$                       the initial deposit

$r = 0.06$                         6% annual rate

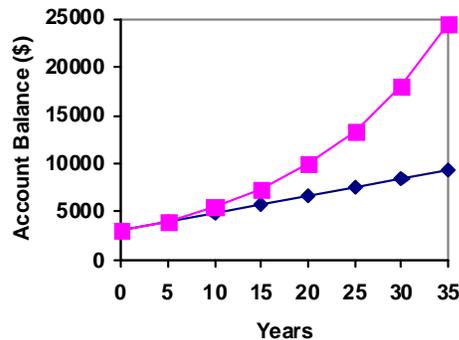
$k = 12$                           12 months in 1 year

$t = 20$                           since we're looking for how much we'll have after 20 years

So  $A = 3000 \left( 1 + \frac{0.06}{12} \right)^{20 \cdot 12} = \$9930.61$  (round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest (\$15 per month)	6% compounded monthly = 0.5% each month.
5	\$3900	\$4046.55
10	\$4800	\$5458.19
15	\$5700	\$7362.28
20	\$6600	\$9930.61
25	\$7500	\$13394.91
30	\$8400	\$18067.73
35	\$9300	\$24370.65



As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

### Evaluating exponents on the calculator

When we need to calculate something like  $5^3$  it is easy enough to just multiply  $5 \cdot 5 \cdot 5 = 125$ . But when we need to calculate something like  $1.005^{240}$ , it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$$\boxed{\wedge}, \boxed{y^x}, \text{ or } \boxed{x^y}.$$

To evaluate  $1.005^{240}$  we'd type  $1.005 \boxed{\wedge} 240$ , or  $1.005 \boxed{y^x} 240$ . Try it out - you should get something around 3.3102044758.

### Example 5

You know that you will need \$40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

In this example,  
We're looking for  $P$ .

$r = 0.04$                       4%  
 $k = 4$                             4 quarters in 1 year  
 $t = 18$                             Since we know the balance in 18 years  
 $A = \$40,000$                     The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for  $P$ .

$$40000 = P \left( 1 + \frac{0.04}{4} \right)^{18 \times 4}$$

$$40000 = P(2.0471)$$

$$P = \frac{40000}{2.0471} = \$19539.84$$

So you would need to deposit \$19,539.84 now to have \$40,000 in 18 years.

### Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a “close enough” answer, but keeping more digits is always better. If your calculator allows it, do all your calculations without rounding in the calculator and only round the final answer.

### Example 6

To see why not over-rounding is so important, suppose you were investing \$1000 at 5% interest compounded monthly for 30 years.

$P = \$1000$                       the initial deposit  
 $r = 0.05$                         5%  
 $k = 12$                             12 months in 1 year  
 $t = 30$                             since we're looking for the amount after 30 years

If we first compute  $r/k$ , we find  $0.05/12 = 0.00416666666667$

Here is the effect of rounding this to different values:

$r/k$ rounded to:	Gives $A$ to be:	Error
0.004	\$4208.59	\$259.15
0.0042	\$4521.45	\$53.71
0.00417	\$4473.09	\$5.35
0.004167	\$4468.28	\$0.54
0.0041667	\$4467.80	\$0.06
no rounding	\$4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough - \$5 off of \$4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

## Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$A = 1000 \left( 1 + \frac{0.05}{12} \right)^{12 \cdot 30}$$

We can quickly calculate  $12 \times 30 = 360$ , giving  $A = 1000 \left( 1 + \frac{0.05}{12} \right)^{360}$ .

Now we can use the calculator.

Type this	Calculator shows
0.05 $\div$ 12 $=$	0.00416666666667
$+$ 1 $=$	1.00416666666667
$y^x$ 360 $=$	4.46774431400613
$\times$ 1000 $=$	4467.74431400613

The previous steps were assuming you have a “one operation at a time” calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

$$1000 \times ( 1 + 0.05 \div 12 ) y^x 360 =$$

Or

$$1000 \times ( 1 + 0.05 \div 12 ) ^ 360 =$$

## Try it Now

2. If \$70,000 are invested at 7% compounded monthly for 25 years, find the end balance.

Because of compounding throughout the year, with compound interest the actual increase in a year is *more* than the annual percentage rate. If \$1,000 were invested at 10%, the table below shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	\$1100
Semiannually	\$1102.50
Quarterly	\$1103.81
Monthly	\$1104.71
Daily	\$1105.16

If we were to compute the actual percentage increase for the daily compounding, there was an increase of \$105.16 from an original amount of \$1,000, for a percentage increase of  $\frac{105.16}{1000} = 0.10516 = 10.516\%$  increase. This quantity is called the **annual percentage yield (APY)**.

Notice that given any starting amount, the amount after 1 year would be

$A = P\left(1 + \frac{r}{k}\right)^k$ . To find the total change, we would subtract the original amount, then to find the percentage change we would divide that by the original amount:

$$\frac{P\left(1 + \frac{r}{k}\right)^k - P}{P} = \left(1 + \frac{r}{k}\right)^k - 1$$

### Annual Percentage Yield

The **annual percentage yield** is the actual percent a quantity increases in one year. It can be calculated as

$$APY = \left(1 + \frac{r}{k}\right)^k - 1$$

Notice this is equivalent to finding the value of \$1 after 1 year, and subtracting the original dollar.

### Example 7

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is offering a better rate?

We can compare these rates using the annual percentage yield – the actual percent increase in a year.

$$\text{Bank A: } APY = \left(1 + \frac{0.012}{4}\right)^4 - 1 = 0.012054 = 1.2054\%$$

$$\text{Bank B: } APY = \left(1 + \frac{0.011}{12}\right)^{12} - 1 = 0.011056 = 1.1056\%$$

Bank B's monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

## Example 8

If you invest \$2000 at 6% compounded monthly, how long will it take the account to double in value?

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

$P = \$2000$                       the initial deposit  
 $r = 0.06$                         6% annual rate  
 $k = 12$                           12 months in 1 year

So our general equation is  $A = 2000 \left(1 + \frac{0.06}{12}\right)^{12t}$ . We also know that we want our ending amount to be double of \$2000, which is \$4000, so we're looking for  $t$  so that  $A = 4000$ . To solve this, we set our equation for  $A$  equal to 4000.

$$4000 = 2000 \left(1 + \frac{0.06}{12}\right)^{12t} \quad \text{Divide both sides by 2000}$$

$$2 = (1.005)^{12t} \quad \text{To solve for the exponent, take the log of both sides}$$

$$\log(2) = \log\left((1.005)^{12t}\right) \quad \text{Use the exponent property of logs on the right side}$$

$$\log(2) = 12t \log(1.005) \quad \text{Now we can divide both sides by } 12\log(1.005)$$

$$\frac{\log(2)}{12\log(1.005)} = t \quad \text{Approximating this to a decimal}$$

$$t = 11.581$$

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded  $\log(2)$  to 0.301 and  $\log(1.005)$  to 0.00217, then your final answer would have been about 11.577 years.

## Important Topics of this Section

APR  
 Simple interest  
 Compound interest  
 Compounding frequency  
 APY  
 Evaluating on a calculator

## Try it Now Answers

1.

 $I = \$30$  of interest $P = \$500$  principal $r =$  unknown $t = 1$  month

Using  $I = Prt$ , we get  $30 = 500 \cdot r \cdot 1$ . Solving, we get  $r = 0.06$ , or 6%. Since the time was monthly, this is the monthly interest. The annual rate would be 12 times this: 72% interest.

$$2. A = 70,000 \left( 1 + \frac{0.07}{12} \right)^{12(25)} = 400,779.27$$

## Section 6.2: Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

Suppose we will deposit \$100 each month into an account paying 6% interest. How much will we have after a year? We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$$r = 0.06 \text{ (6\%)}$$

$$k = 12 \text{ (12 compounds/deposits per year)}$$

$$d = \$100 \text{ (our deposit per month)}$$

$$t = 1 \text{ year}$$

With **ordinary** annuities we assume the payment is made at the end of the period. The \$100 we deposit at the end of the first month will earn interest for 11 months and at the end of the

$$\text{year will be worth } A = 100 \left( 1 + \frac{0.06}{12} \right)^{11} = 100(1.005)^{11}.$$

The \$100 deposited at the end of the second month will have 10 months to grow, and will be worth  $A = 100(1.005)^{10}$  at the end of the year. This pattern continues down to the last deposit, which has no time to compound, and will be worth  $A = 100$ .

In total, we will have accumulated:

$$A = 100(1.005)^{11} + 100(1.005)^{10} + \cdots + 100(1.005)^2 + 100(1.005)^1 + 100$$

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005A = 1.005(100(1.005)^{11} + 100(1.005)^{10} + \cdots + 100(1.005)^2 + 100(1.005) + 100)$$

Distributing on the right side of the equation gives

$$1.005A = 100(1.005)^{12} + 100(1.005)^{11} + \cdots + 100(1.005)^3 + 100(1.005)^2 + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$1.005A = 100(1.005)^{12} + 100(1.005)^{11} + \cdots + 100(1.005)$$

$$A = 100(1.005)^{11} + \cdots + 100(1.005) + 100$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005A - A = 100(1.005)^{12} - 100$$

Now we solve this equation for A.

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$$0.005A = 100\left((1.005)^{12} - 1\right)$$

$$A = \frac{100\left((1.005)^{12} - 1\right)}{0.005}$$

Recall 0.005 was  $r/k$ , 100 was the deposit  $d$ , and 12 was the number of months,  $kt$ . Generalizing this result, we get the saving annuity formula.

### Annuity Formula

$$A = \frac{d\left(\left(1 + \frac{r}{k}\right)^{kt} - 1\right)}{\left(\frac{r}{k}\right)}$$

$A$  is the balance in the account after  $t$  years.

$d$  is the regular deposit (the amount you deposit each year, each month, etc.)

$r$  is the annual interest rate in decimal form.

$k$  is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding,  $k = 12$ .

If you make your deposits every year, use yearly compounding,  $k = 1$ .

If you make your deposits every quarter, use quarterly compounding,  $k = 4$ .

Etc.

### When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

## Example 1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

In this example,

$d = \$100$  the monthly deposit

$r = 0.06$  6% annual rate

$k = 12$  since we're doing monthly deposits, we'll compound monthly

$t = 20$  we want the amount after 20 years

Putting this into the equation:

$$A = \frac{100 \left( \left( 1 + \frac{0.06}{12} \right)^{20(12)} - 1 \right)}{\left( \frac{0.06}{12} \right)}$$

$$A = \frac{100 \left( (1.005)^{240} - 1 \right)}{(0.005)}$$

$$A = \frac{100(3.310 - 1)}{(0.005)}$$

$$A = \frac{100(2.310)}{(0.005)} = \$46,200$$

The account will grow to \$46,200 after 20 years.

Notice that you deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is  $\$46,200 - \$24,000 = \$22,200$ .

## Example 2

You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

In this example,

We're looking for  $d$ .

$r = 0.08$  8% annual rate

$k = 12$  since we're depositing monthly

$t = 30$  30 years

$A = \$200,000$  The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for  $d$ .

$$200,000 = \frac{d \left( \left( 1 + \frac{0.08}{12} \right)^{30(12)} - 1 \right)}{\left( \frac{0.08}{12} \right)}$$

$$200,000 = \frac{d \left( (1.00667)^{360} - 1 \right)}{(0.00667)}$$

$$200,000 = d(1491.57)$$

$$d = \frac{200,000}{1491.57} = \$134.09$$

So you would need to deposit \$134.09 each month to have \$200,000 in 30 years if your account earns 8% interest

### Try it Now

1. A more conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

### Example 3

If you invest \$100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to \$10,000?

This is a savings annuity problem since we are making regular deposits into the account.

$d = \$100$  the monthly deposit

$r = 0.03$  3% annual rate

$k = 12$  since we're doing monthly deposits, we'll compound monthly

We don't know  $t$ , but we want  $A$  to be \$10,000.

Putting this into the equation:

$$10,000 = \frac{100 \left( \left( 1 + \frac{0.03}{12} \right)^{12t} - 1 \right)}{\left( \frac{0.03}{12} \right)}$$

Simplifying the fractions a bit

$$10,000 = \frac{100 \left( (1.0025)^{12t} - 1 \right)}{0.0025}$$

We want to isolate the exponential term,  $1.0025^{12t}$ , so multiply both sides by 0.0025

$$25 = 100\left((1.0025)^{12t} - 1\right) \quad \text{Divide both sides by 100}$$

$$0.25 = (1.0025)^{12t} - 1 \quad \text{Add 1 to both sides}$$

$$1.25 = (1.0025)^{12t} \quad \text{Now take the log of both sides}$$

$$\log(1.25) = \log\left((1.0025)^{12t}\right) \quad \text{Use the exponent property of logs}$$

$$\log(1.25) = 12t \log(1.0025) \quad \text{Divide by } 12\log(1.0025)$$

$$\frac{\log(1.25)}{12\log(1.0025)} = t \quad \text{Approximating to a decimal}$$

$$t = 7.447 \text{ years}$$

It will take about 7.447 years to grow the account to \$10,000.

### Important Topics of this Section

Finding the future value of an annuity

Finding deposits needed to fund an annuity

### Try it Now Answers

1.

$d = \$5$                       the daily deposit

$r = 0.03$                      3% annual rate

$k = 365$                      since we're doing daily deposits, we'll compound daily

$t = 10$                         we want the amount after 10 years

$$A = \frac{5\left(\left(1 + \frac{0.03}{365}\right)^{365 \times 10} - 1\right)}{\frac{0.03}{365}} = \$21,282.07$$

We would have deposited a total of  $\$5 \cdot 365 \cdot 10 = \$18,250$ , so \$3,032.07 is from interest

### Section 6.3: Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved \$500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

#### Payout Annuity Formula

$$P = \frac{w \left( 1 - \left( 1 + \frac{r}{k} \right)^{-kt} \right)}{\left( \frac{r}{k} \right)}$$

$P$  is the balance in the account at the beginning (starting amount, or principal).

$w$  is the regular withdrawal (the amount you take out each year, each month, etc.)

$r$  is the annual interest rate (in decimal form. Example: 5% = 0.05)

$k$  is the number of compounding periods in one year.

$t$  is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

#### When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

## Example 1

After retiring, you want to be able to take \$1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

In this example,

$w = \$1000$                       the monthly withdrawal  
 $r = 0.06$                         6% annual rate  
 $k = 12$                             since we're doing monthly withdrawals, we'll compound monthly  
 $t = 20$                             since we're taking withdrawals for 20 years

We're looking for  $P$ ; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P = \frac{1000 \left( 1 - \left( 1 + \frac{0.06}{12} \right)^{-20(12)} \right)}{\left( \frac{0.06}{12} \right)}$$

$$P = \frac{1000 \times (1 - (1.005)^{-240})}{(0.005)}$$

$$P = \frac{1000 \times (1 - 0.302)}{(0.005)} = \$139,600$$

You will need to have \$139,600 in your account when you retire.

Notice that you withdrew a total of \$240,000 (\$1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is  $\$240,000 - \$139,600 = \$100,400$  in interest.

## Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this  $\boxed{(-)}$ , some with  $\boxed{+/-}$ . The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate  $1.005^{-240}$  you'd type something like:  $1.005 \boxed{\wedge} \boxed{(-)} 240$

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit:  $1.005 \boxed{y^x} 240 \boxed{(-)} =$

Give it a try - you should get  $1.005^{-240} = 0.302096$

**Example 2**

You know you will have \$500,000 in your account when you retire. You want to be able to take monthly withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

In this example,

We're looking for  $w$ .

$r = 0.08$                       8% annual rate

$k = 12$                          since we're withdrawing monthly

$t = 30$                          30 years

$P = \$500,000$                 we are beginning with \$500,000

In this case, we're going to have to set up the equation, and solve for  $w$ .

$$500,000 = \frac{w \left( 1 - \left( 1 + \frac{0.08}{12} \right)^{-30(12)} \right)}{\left( \frac{0.08}{12} \right)}$$

$$500,000 = \frac{w \left( 1 - (1.00667)^{-360} \right)}{(0.00667)}$$

$$500,000 = w(136.232)$$

$$w = \frac{500,000}{136.232} = \$3670.21$$

You would be able to withdraw \$3,670.21 each month for 30 years.

**Try it Now**

1. A donor gives \$100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

**Important Topics of this Section**

Find the present value of an annuity (principal required)

Find the payments that can be made from a payout annuity

## Try it Now Answers

1.

 $w = \text{unknown}$  $r = 0.04$                       4% annual rate $k = 1$                               since we're doing annual scholarships $t = 20$                               20 years $P = 100,000$                       we're starting with \$100,000

$$100,000 = \frac{w \left( 1 - \left( 1 + \frac{0.04}{1} \right)^{-20(1)} \right)}{\frac{0.04}{1}}$$

Solving for  $w$  gives \$7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that,  $\$100,000(0.04) = \$4,000$  a year would have been available.

## Section 6.4: Loans

In the last section, you learned about payout annuities.

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

### Loans Formula

$$P = \frac{PMT \left( 1 - \left( 1 + \frac{r}{k} \right)^{-kt} \right)}{\left( \frac{r}{k} \right)}$$

$P$  is the balance in the account at the beginning (the principal, or amount of the loan).

$PMT$  is your loan payment (your monthly payment, annual payment, etc)

$r$  is the annual interest rate in decimal form.

$k$  is the number of compounding periods in one year.

$t$  is the length of the loan, in years

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

### When do you use this

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Loans: Many payments

## Example 1

You can afford \$200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with \$200 per month?

In this example,

$PMT = \$200$             the monthly loan payment

$r = 0.03$                 3% annual rate

$k = 12$                   since we're doing monthly payments, we'll compound monthly

$t = 5$                     since we're making monthly payments for 5 years

We're looking for  $P$ , the starting amount of the loan.

$$P = \frac{200 \left( 1 - \left( 1 + \frac{0.03}{12} \right)^{-5(12)} \right)}{\left( \frac{0.03}{12} \right)}$$

$$P = \frac{200 \left( 1 - (1.0025)^{-60} \right)}{(0.0025)}$$

$$P = \frac{200(1 - 0.861)}{(0.0025)} = \$11,120$$

You can afford a \$11,120 loan.

You will pay a total of \$12,000 (\$200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying \$12,000 - \$11,120 = \$880 interest total.

## Example 2

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

In this example,

We're looking for  $PMT$ .

$r = 0.06$                 6% annual rate

$k = 12$                   since we're paying monthly

$t = 30$                   30 years

$P = \$140,000$         the starting loan amount

In this case, we're going to have to set up the equation, and solve for  $PMT$ .

$$140,000 = \frac{PMT \left( 1 - \left( 1 + \frac{0.06}{12} \right)^{-30(12)} \right)}{\left( \frac{0.06}{12} \right)}$$

$$140,000 = \frac{PMT \left( 1 - (1.005)^{-360} \right)}{(0.005)}$$

$$140,000 = PMT(166.792)$$

$$PMT = \frac{140,000}{166.792} = \$839.37$$

You will make payments of \$839.37 per month for 30 years.

You're paying a total of \$302,173.20 to the loan company: \$839.37 per month for 360 months. You are paying a total of \$302,173.20 - \$140,000 = \$162,173.20 in interest over the life of the loan.

To see what is happening to the loan balance over time, we could build an **amortization schedule**, a spreadsheet that shows the details of the payoff. To create one for the previous example, each month we'd calculated the interest charge as  $3\%/12 = 0.25\%$  interest on the remaining loan balance. Payment beyond the interest charge goes towards principal and reduces the remaining loan balance. The first several months are shown here:

Month	Payment	Interest Charge	Portion to Principal	Remaining Loan Balance
0	--	--	--	\$140,000.00
1	\$839.37	\$350.00	\$489.37	139,510.63
2	839.37	348.78	490.59	139,020.04
3	839.37	347.55	491.82	138,528.22
4	839.37	346.32	493.05	138,035.17

Notice that the interest charge decreases each month as the loan balance decreases.

#### Try it Now

1. Janine bought \$3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

## Example 3

Consider the \$140,000 mortgage at 6% from the previous example. If the homeowner increased their payments to \$1000 per month, how long will it take them to pay off the loan?

$r = 0.06$                       6% annual rate  
 $k = 12$                          since we're paying monthly  
 $P = \$140,000$                 the starting loan amount  
 $PMT = \$1000$                 the given payment amount  
 We are solving for  $t$ , the amount of time it will take to pay off.

$$140,000 = \frac{1000 \left( 1 - \left( 1 + \frac{0.06}{12} \right)^{-12t} \right)}{\left( \frac{0.06}{12} \right)}$$

$$140,000 = \frac{1000 \left( 1 - (1.005)^{-12t} \right)}{(0.005)}$$

Multiply both sides by 0.005

$$700 = 1000 \left( 1 - (1.005)^{-12t} \right)$$

Divide both sides by 1000

$$0.7 = 1 - (1.005)^{-12t}$$

Subtract 1 from both sides

$$-0.3 = -(1.005)^{-12t}$$

Multiply by -1

$$0.3 = (1.005)^{-12t}$$

Take the log of both sides

$$\log(0.3) = \log \left( (1.005)^{-12t} \right)$$

Use the exponent property of logs

$$\log(0.3) = -12t \log(1.005)$$

Divide by  $-12\log(1.005)$

$$t = \frac{\log(0.3)}{-12\log(1.005)} \approx 20.12$$

The loan will be paid off in about 20 years.

By increasing the payment by about 20%, they were able to decrease the length of the loan by about 33%, and reduce interest paid by more than \$60,000.

## Try it Now

2. Joel is considering putting a \$1,000 laptop purchase on his credit card, which has an interest rate of 12% compounded monthly. How long will it take him to pay off the purchase if he makes payments of \$30 a month?

## Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were \$1,000 a month, after a year you will *not* have paid off \$12,000 of the loan balance.

To determine the remaining loan balance, we can think “how much loan will these loan payments be able to pay off in the remaining time on the loan?”

### Example 4

If a mortgage at a 6% interest rate has payments of \$1,000 a month, how much will the loan balance be 10 years from the end the loan?

To determine this, we are looking for the amount of the loan that can be paid off by \$1,000 a month payments in 10 years. In other words, we're looking for  $P$  when

$PMT = \$1,000$  the monthly loan payment

$r = 0.06$  6% annual rate

$k = 12$  since we're doing monthly payments, we'll compound monthly

$t = 10$  since we're making monthly payments for 10 more years

$$P = \frac{1000 \left( 1 - \left( 1 + \frac{0.06}{12} \right)^{-10(12)} \right)}{\left( \frac{0.06}{12} \right)}$$

$$P = \frac{1000 \left( 1 - (1.005)^{-120} \right)}{(0.005)}$$

$$P = \frac{1000(1 - 0.5496)}{(0.005)} = \$90,073.45$$

The loan balance with 10 years remaining on the loan will be \$90,073.45

Often times answering remaining balance questions requires two steps:

- 1) Calculating the monthly payments on the loan
- 2) Calculating the remaining loan balance based on the *remaining time* on the loan

## Example 5

A couple purchases a home with a \$180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years?

First we will calculate their monthly payments.

We're looking for  $PMT$ .

$r = 0.04$                       4% annual rate  
 $k = 12$                         since they're paying monthly  
 $t = 30$                          30 years  
 $P = \$180,000$                 the starting loan amount

We set up the equation and solve for  $PMT$ .

$$180,000 = \frac{PMT \left( 1 - \left( 1 + \frac{0.04}{12} \right)^{-30(12)} \right)}{\left( \frac{0.04}{12} \right)}$$

$$180,000 = \frac{PMT \left( 1 - (1.00333)^{-360} \right)}{(0.00333)}$$

$$180,000 = PMT(209.562)$$

$$PMT = \frac{180,000}{209.562} = \$858.93$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

$PMT = \$858.93$                 the monthly loan payment we calculated above  
 $r = 0.04$                         4% annual rate  
 $k = 12$                          since they're doing monthly payments  
 $t = 25$                          since they'd be making monthly payments for 25 more years

$$P = \frac{858.93 \left( 1 - \left( 1 + \frac{0.04}{12} \right)^{-25(12)} \right)}{\left( \frac{0.04}{12} \right)}$$

$$P = \frac{858.93 \left( 1 - (1.00333)^{-300} \right)}{(0.00333)}$$

$$P = \frac{858.93(1 - 0.369)}{(0.00333)} \approx \$162,758$$

The loan balance after 5 years, with 25 years remaining on the loan, will be \$162,758

Over that 5 years, the couple has paid off \$180,000 - \$162,758 = \$17,242 of the loan balance. They have paid a total of \$858.93 a month for 5 years (60 months), for a total of \$51,535.80, so \$51,535.80 - \$17,242 = \$34,292.80 of what they have paid so far has been interest.

### Important Topics of this Section

- Find the amount of a loan given the payments
- Find the payments required given the loan amount
- Find the remaining balance of a loan

### Try it Now Answers

1.

$PMT$  = unknown

$r = 0.16$                       16% annual rate

$k = 12$                             since we're making monthly payments

$t = 2$                                 2 years to repay

$P = 3,000$                         we're starting with a \$3,000 loan

$$3,000 = \frac{PMT \left( 1 - \left( 1 + \frac{0.16}{12} \right)^{-2(12)} \right)}{\frac{0.16}{12}}$$

Solving for  $PMT$  gives \$146.89 as monthly payments.

In total, she will pay \$3,525.36 to the store, meaning she will pay \$525.36 in interest over the two years.

2.

$PMT = \$30$                       The monthly payments

$r = 0.12$                             12% annual rate

$k = 12$                                 since we're making monthly payments

$P = 1,000$                         we're starting with a \$1,000 loan

We are solving for  $t$ , the time to pay off the loan

$$1,000 = \frac{30 \left( 1 - \left( 1 + \frac{0.12}{12} \right)^{-12t} \right)}{\frac{0.12}{12}}$$

Solving for  $t$  gives 3.396. It will take about 3.4 years to pay off the purchase.

## Section 6.5: Multistage Finance Problems

### Which equation to use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly) then you're looking at a basic Annuity problem. Basic annuities are when you are saving money. Usually in an annuity problem, your account starts empty, and has money in the future.

If you're pulling money out of the account on a regular basis, then you're looking at a Payout Annuity problem. Payout annuities are used for things like retirement income, where you start with money in your account, pull money out on a regular basis, and your account ends up empty in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and to determine what approach will best allow you to solve the problem.

### Try it Now

1. For each of the following scenarios, determine if it is a compound interest problem, a savings annuity problem, a payout annuity problem, or a loans problem. Then solve each problem.

- Marcy received an inheritance of \$20,000, and invested it at 6% interest. She is going to use it for college, withdrawing money for tuition and expenses each quarter. How much can she take out each quarter if she has 3 years of school left?
- Paul wants to buy a new car. Rather than take out a loan, he decides to save \$200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
- Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have \$30,000 in the account in 6 years. How much should Keisha deposit into the account?
- Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of \$100, how much new equipment can she buy?
- How much would you need to save every month in an account earning 4% interest to have \$5,000 saved up in two years?

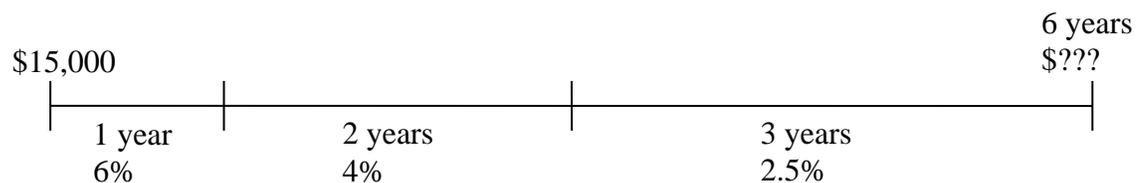
### Multistage Problems

Many finance problems in real life can't be solved using a single application of any of the finance formulas we have learned in the prior sections. Instead, we have to combine multiple calculations. The calculations can continue to be done using formulas, but often times people will use calculators with specialty functions for doing the calculations. In this section we determine the values needed to use the formulas, but not show the details of the solving; you could instead solve each portion using a specialty calculator if you have access to one.

#### Example 1

Suppose you deposited \$15,000 into a bank account offering a special 6% APR. After one year, they lower the promotional APR to 4%, then two years after that lower it again to 2.5%. If the account compounds monthly, what will your account balance be 6 years after the initial deposit?

We can imagine this situation as three distinct time phases, each with a different interest rate.



To handle the change of interest rates, it is easiest to imagine that each time the interest rate changes we withdraw all our money then redeposit it into a new account at a different rate.

For the first year:

$$P = \$15,000, \quad r = 0.06, \quad t = 1 \text{ year}, \quad k = 12$$

Using the compound interest equation, we find the balance after 1 year would be:

$$A = \$15,925.17$$

We can then use this as the principal for the next two years at a different rate, as if we had withdrawn that money and reinvested it at 4%:

$$P = \$15,925.17, r = 0.04, t = 2 \text{ years}, k = 12$$

Using the compound interest equation, we find the balance after these next 2 years would be:

$$A = \$17,249.24$$

Using that as the principal for the last three years:

$$P = \$17,249.24, r = 0.025, t = 3 \text{ years}, k = 12$$

Using the compound interest equation, we find the balance at the end would be:

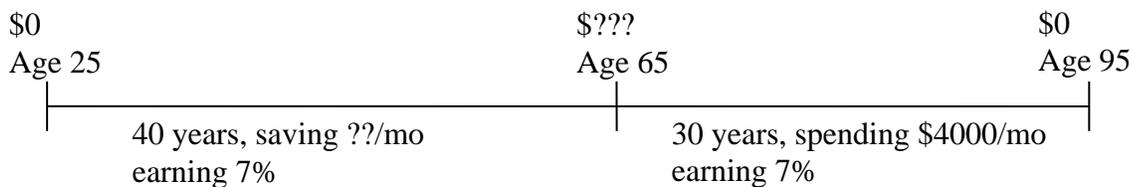
$$A = \$18,591.23$$

After the 6 years, the final balance would be \$18,591.23

### Example 2

Jenny plans to retire at age 65, and estimates she will need \$48,000 a year in retirement for 30 years. To save for retirement, she plans to put money every month into her 401k plan at work. If Jenny is currently 25, how much will she need to save each month to meet her retirement goal, assuming her retirement accounts earn an average of 7% per year.

We can think about this as two separate phases: the accumulation phase and the spending phase. We assume Jenny starts with no money at the beginning, and will completely spend down her account by the time she's 95.



To decide which phase to start with, we look to see which we have the most information for. In the first phase, we don't know how much she will save each month, and we don't know the ending balance of the account. For the second phase, we know everything except the starting balance of the account, so we will start the problem in the second phase.

Basically, we are going to work the problem backwards, using her spending requirement to determine how much she needs at retirement, then use that value to determine how much she needs to save each month.

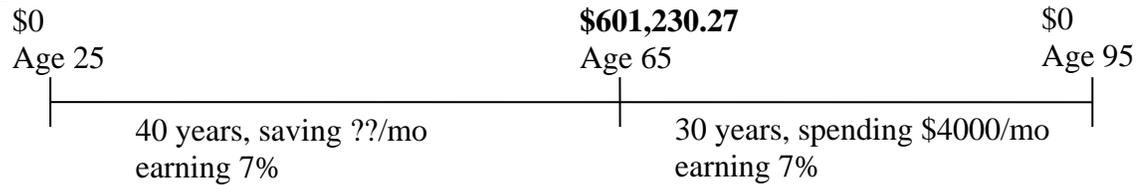
The second phase is a payout annuity problem, where:

$$w = \$4,000, r = 0.07, t = 30, k = 12$$

Using the payout annuity formula to find the principal gives:

$$P = \$601,230.27$$

This is how much Jenny will need at retirement to meet her spending needs, and fills in a part of our timeline:



Now we can work on the first phase, which is a savings annuity problem where the desired end balance is the \$601,230.27 we just found.

$$A = \$601,230.27, \quad r = 0.07, \quad t = 40, \quad k = 12$$

Using the savings annuity formula to solve for the deposit amount gives:

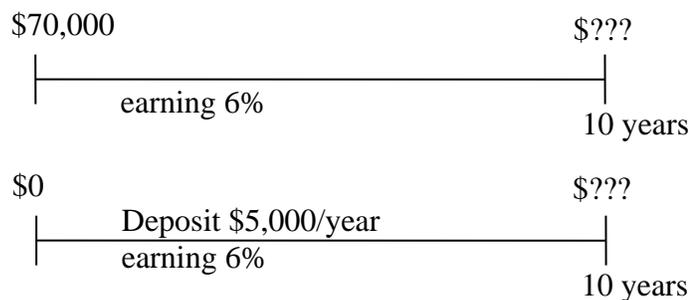
$$d = 229.06$$

Jenny will need to save \$229.06 per month to meet her retirement goals.

### Example 3

Ula just got a new job. She had \$70,000 in a 401k at her old job that she rolled over into an IRA. She plans to continue contributing \$5,000 a year into her IRA. If her account earns 6% compounded annually, how much will she have in her IRA in 10 years?

The challenge with this problem is that our savings annuity formula assumed that our account balance started at zero. While we could try to build a new formula to handle this situation, we can more easily envision it as two different money streams, as if the original \$70,000 is invested in a different account than future contributions.



In the first stream of money, we have \$70,000 earning 6% interest for 10 years, which is a basic compound interest problem:

$$P = \$70,000, \quad r = 0.06, \quad t = 10, \quad k = 1$$

Using the compound interest formula to solve for the ending balance:

$$A = \$125,359.34$$

In the second stream of money, we are depositing \$5,000 a year, a savings annuity problem:

$$d = \$5,000, \quad r = 0.06, \quad t = 10, \quad k = 1$$

Using the savings annuity formula to find the ending balance:

$$A = \$65,903.97$$

Now that we know the ending balance of each stream of money, we can add those together to find the total balance of the account at the end of the ten years:

$$\$125,359.34 + \$65,903.97 = \$191,263.31$$

#### Example 4

Five years ago a couple purchased a home for \$260,000, making a 20% down payment and financing the rest with a 30-year adjustable rate mortgage fixed at 3% for the first 5 years. Now that the fixed rate period is up, the couple is facing a higher adjustable rate. They now plan to refinance into a fixed rate 30-year mortgage at 4%. What will their new monthly payments be? Assume there are no costs associated with the refinance.

To determine the monthly payments for the refinance, we first need to know how much the refinanced loan amount will be. The amount refinanced will be the same as the remaining loan balance on the original loan after 5 years. We will start by finding the details of the original loan.

The 20% down payment will be  $\$260,000(0.20) = \$52,000$ , leaving \$208,000 to be financed.

We start by finding the payments on the original loan:

$$P = \$208,000, \quad r = 0.03, \quad t = 30, \quad k = 12$$

Using the loan formula to solve for the payment,

$$PMT = \$876.94$$

Now we can find the remaining balance on the original loan after 5 years. As discussed in the last section, we can find this by determining the amount of loan that can be paid off with payments of \$876.94 in the remaining 25 years:

$$PMT = \$876.94, \quad r = 0.03, \quad t = 25, \quad k = 12$$

Using the loan formula to solve for the remaining balance,

$$P = \$184,926.00$$

So after 5 years, the remaining balance on the loan is \$184,926.00, and that is the amount we will refinance in a new 30 year mortgage at the fixed 4% rate:

$$P = \$184,926.00, \quad r = 0.04, \quad t = 30, \quad k = 12$$

Solving for the new payment amount:

$$PMT = \$882.87$$

Because of the higher interest rate the monthly payment increased, even though the loan amount was less at refinance. Most likely this increase is much smaller than the one the couple would have faced from the original loan's rate adjusting.

Notice also that because of the refinance, the couple will end up making payments for a total of 35 years on the house.

In those first 5 years, the couple paid a total of  $\$876.94(12)(5) = \$52,616.40$ , and reduced the loan balance by  $(\$208,000 - \$184,926) = \$23,074$ , meaning they paid  $\$29,542.40$  in interest during the first 5 years.

In the remaining 30 years, the couple will pay  $\$882.87(12)(30) = \$317,833.20$ , and reduce the loan balance by  $\$184,926$ , meaning they will pay  $\$132,907.20$  in interest during these 30 years.

In total, the couple will pay  $\$132,907.20 + \$29,542.40 = \$162,449.60$  in interest over the 35 years.

### Important Topics of this Section

- Recognize the type of finance problem given a situation
- Breaking complex finance problems into smaller, simpler pieces

### Try it Now Answers

1.
  - a. This is a payout annuity problem. She can pull out  $\$1833.60$  a quarter.
  - b. This is a savings annuity problem. He will have saved up  $\$7,524.11$ .
  - c. This is compound interest problem. She would need to deposit  $\$22,386.46$ .
  - d. This is a loans problem. She can buy  $\$4,609.33$  of new equipment.
  - e. This is a savings annuity problem. You would need to save  $\$200.46$  each month



## Chapter 7: Sets

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

Section 7.1: Sets .....	253
Section 7.2: Venn Diagrams and Cardinality .....	258

### Section 7.1: Sets

#### Basics

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

#### Set

A **set** is a collection of distinct objects, called **elements** of the set

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

#### Example 1

Some examples of sets defined by describing the contents:

- The set of all even numbers
- The set of all books written about travel to Chile

Some examples of sets defined by listing the elements of the set:

- {1, 3, 9, 12}
- {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

#### Notation

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

The symbol  $\in$  means “is an element of”.

A set that contains no elements,  $\{ \}$ , is called the **empty set** and is notated  $\emptyset$

## Example 2

Let  $A = \{1, 2, 3, 4\}$

To notate that 2 is element of the set, we'd write  $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

## Subset

A **subset** of a set  $A$  is another set that contains only elements from the set  $A$ , but may not contain all the elements of  $A$ .

If  $B$  is a subset of  $A$ , we write  $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

If  $B$  is a proper subset of  $A$ , we write  $B \subset A$

## Example 3

Consider these three sets

$A =$  the set of all even numbers      $B = \{2, 4, 6\}$       $C = \{2, 3, 4, 6\}$

Here  $B \subset A$  since every element of  $B$  is also an even number, so is an element of  $A$ .

More formally, we could say  $B \subset A$  since if  $x \in B$ , then  $x \in A$ .

It is also true that  $B \subset C$ .

$C$  is not a subset of  $A$ , since  $C$  contains an element, 3, that is not contained in  $A$

## Example 4

Suppose a set contains the plays “Much Ado About Nothing”, “MacBeth”, and “A Midsummer's Night Dream”. What is a larger set this might be a subset of?

There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

## Try it Now

1. The set  $A = \{1, 3, 5\}$ . What is a larger set this might be a subset of?

## Union, Intersection, and Complement

Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

### Union, Intersection, and Complement

The **union** of two sets contains all the elements contained in either set (or both sets).

The union is notated  $A \cup B$ .

More formally,  $x \in A \cup B$  if  $x \in A$  or  $x \in B$  (or both)

The **intersection** of two sets contains only the elements that are in both sets.

The intersection is notated  $A \cap B$ .

More formally,  $x \in A \cap B$  if  $x \in A$  and  $x \in B$

The **complement** of a set  $A$  contains everything that is *not* in the set  $A$ .

The complement is notated  $A'$ , or  $A^c$ , or sometimes  $\sim A$ .

### Example 5

Consider the sets:  $A = \{\text{red, green, blue}\}$        $B = \{\text{red, yellow, orange}\}$   
 $C = \{\text{red, orange, yellow, green, blue, purple}\}$

a) Find  $A \cup B$

The union contains all the elements in either set:  $A \cup B = \{\text{red, green, blue, yellow, orange}\}$

Notice we only list red once.

b) Find  $A \cap B$

The intersection contains all the elements in both sets:  $A \cap B = \{\text{red}\}$

c) Find  $A^c \cap C$

Here we're looking for all the elements that are *not* in set  $A$  and are also in  $C$ .

$A^c \cap C = \{\text{orange, yellow, purple}\}$

### Try it Now

2. Using the sets from the previous example, find  $A \cup C$  and  $B^c \cap A$

Notice that in the example above, it would be hard to just ask for  $A^c$ , since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

### Universal Set

A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

A complement is relative to the universal set, so  $A^c$  contains all the elements in the universal set that are not in  $A$ .

### Example 6

- If we were discussing searching for books, the universal set might be all the books in the library.
- If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
- If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

### Example 7

Suppose the universal set is  $U =$  all whole numbers from 1 to 9. If  $A = \{1, 2, 4\}$ , then  $A^c = \{3, 5, 6, 7, 8, 9\}$ .

As we saw earlier with the expression  $A^c \cap C$ , set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

### Example 8

Suppose  $H = \{\text{cat, dog, rabbit, mouse}\}$ ,  $F = \{\text{dog, cow, duck, pig, rabbit}\}$   
 $W = \{\text{duck, rabbit, deer, frog, mouse}\}$

- a) Find  $(H \cap F) \cup W$

We start with the intersection:  $H \cap F = \{\text{dog, rabbit}\}$

Now we union that result with  $W$ :  $(H \cap F) \cup W = \{\text{dog, duck, rabbit, deer, frog, mouse}\}$

- b) Find  $H \cap (F \cup W)$

We start with the union:  $F \cup W = \{\text{dog, cow, rabbit, duck, pig, deer, frog, mouse}\}$

Now we intersect that result with  $H$ :  $H \cap (F \cup W) = \{\text{dog, rabbit, mouse}\}$

- c) Find  $(H \cap F)^c \cap W$

We start with the intersection:  $H \cap F = \{\text{dog, rabbit}\}$

Now we want to find the elements of  $W$  that are *not* in  $H \cap F$

$(H \cap F)^c \cap W = \{\text{duck, deer, frog, mouse}\}$

**Important Topics of this Section**

Sets, elements of sets, subsets  
Union, intersection, and complement of sets  
Universal set, empty set

**Try it Now Answers**

1. There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

2.  $A \cup C = \{\text{red, orange, yellow, green, blue purple}\}$

$B^c \cap A = \{\text{green, blue}\}$

## Section 7.2: Venn Diagrams and Cardinality

To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18<sup>th</sup> century. These illustrations now called **Venn Diagrams**.

### Venn Diagram

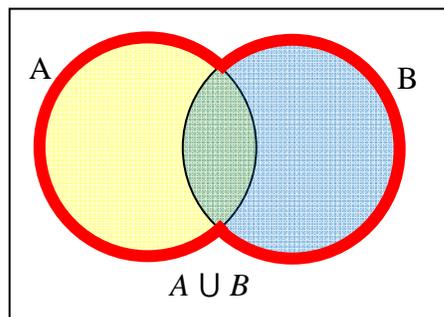
A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

Basic Venn diagrams can illustrate the interaction of two or three sets.

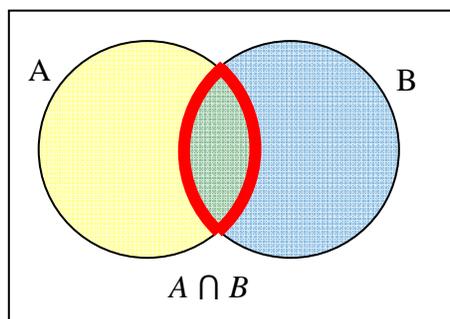
### Example 1

Create Venn diagrams to illustrate  $A \cup B$ ,  $A \cap B$ , and  $A^c \cap B$

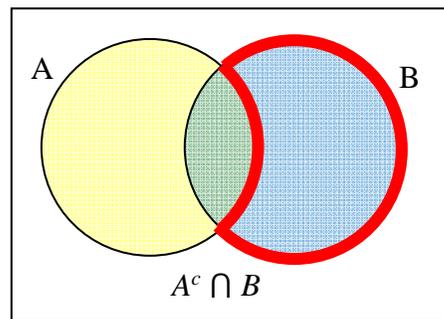
$A \cup B$  contains all elements in *either* set.



$A \cap B$  contains only those elements in both sets – in the overlap of the circles.



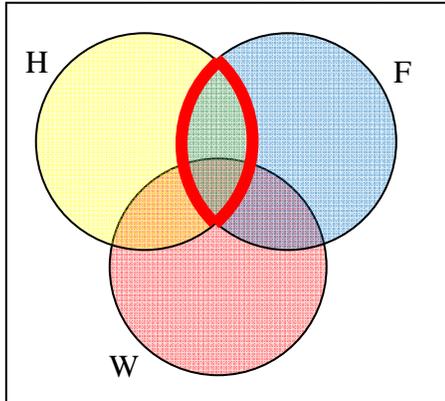
$A^c$  will contain all elements *not* in the set A.  $A^c \cap B$  will contain the elements in set B that are not in set A.



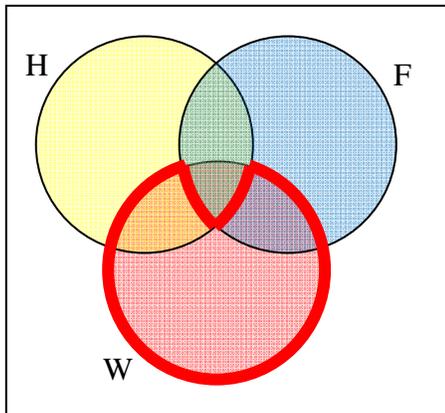
## Example 2

Use a Venn diagram to illustrate  $(H \cap F)^c \cap W$

We'll start by identifying everything in the set  $H \cap F$



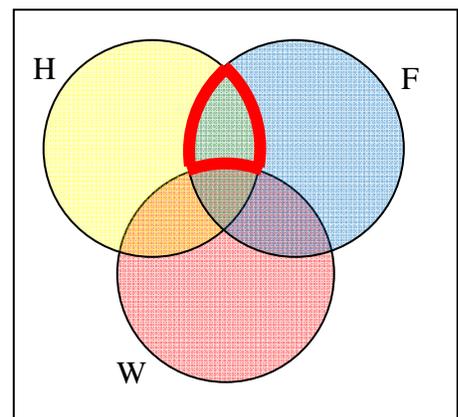
Now,  $(H \cap F)^c \cap W$  will contain everything *not* in the set identified above that is also in set  $W$ .



## Example 3

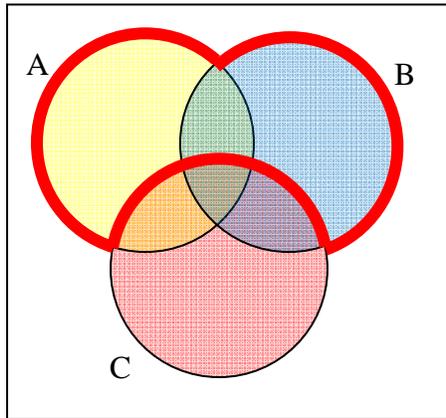
Create an expression to represent the outlined part of the Venn diagram shown.

The elements in the outlined set *are* in sets  $H$  and  $F$ , but are not in set  $W$ . So we could represent this set as  $H \cap F \cap W^c$



## Try it Now

1. Create an expression to represent the outlined portion of the Venn diagram shown

**Cardinality**

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

**Cardinality**

The number of elements in a set is the cardinality of that set.

The cardinality of the set  $A$  is often notated as  $|A|$  or  $n(A)$

**Example 4**

Let  $A = \{1, 2, 3, 4, 5, 6\}$  and  $B = \{2, 4, 6, 8\}$ .

What is the cardinality of  $B$ ?  $A \cup B$ ,  $A \cap B$ ?

The cardinality of  $B$  is 4, since there are 4 elements in the set.

The cardinality of  $A \cup B$  is 7, since  $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$ , which contains 7 elements.

The cardinality of  $A \cap B$  is 3, since  $A \cap B = \{2, 4, 6\}$ , which contains 3 elements.

**Example 5**

What is the cardinality of  $P =$  the set of English names for the months of the year?

The cardinality of this set is 12, since there are 12 months in the year.

Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

## Example 6

A survey asks 200 people “What beverage do you drink in the morning”, and offers choices:

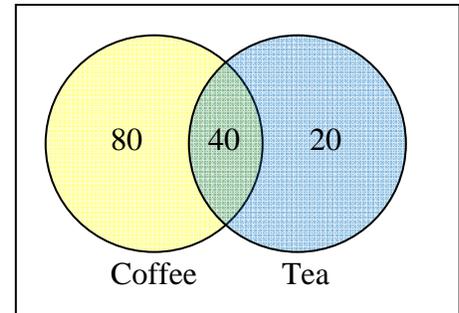
- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

$200 - 20 - 80 - 40 = 60$  people who drink neither.



## Example 7

A survey asks: Which online services have you used in the last month:

- Twitter
- Facebook
- Have used both

The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

Let  $T$  be the set of all people who have used Twitter, and  $F$  be the set of all people who have used Facebook. Notice that while the cardinality of  $F$  is 70% and the cardinality of  $T$  is 40%, the cardinality of  $F \cup T$  is not simply  $70\% + 40\%$ , since that would count those who use both services twice. To find the cardinality of  $F \cup T$ , we can add the cardinality of  $F$  and the cardinality of  $T$ , then subtract those in intersection that we've counted twice.

In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$

$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of  $(F \cup T)^c$ . Since the universal set contains 100% of people and the cardinality of  $F \cup T = 90\%$ , the cardinality of  $(F \cup T)^c$  must be the other 10%.

The previous example illustrated two important properties

## Cardinality properties

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A^c) = n(U) - n(A)$$

Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

## Example 8

Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course

26 were taking a HM course

19 were taking a NS course

9 were taking SS and HM

7 were taking SS and NS

10 were taking HM and NS

3 were taking all three

7 were taking none

How many students are only taking a SS course?

It might help to look at a Venn diagram.

From the given data, we know that there are

3 students in region  $e$  and

7 students in region  $h$ .

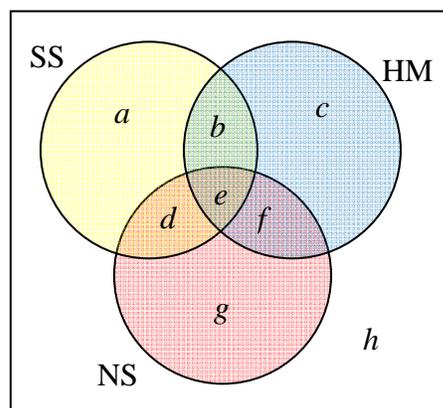
Since 7 students were taking a SS and NS course, we know that  $n(d) + n(e) = 7$ . Since we know there are 3 students in region  $e$ , there must be  $7 - 3 = 4$  students in region  $d$ .

Similarly, since there are 10 students taking HM and NS, which includes regions  $e$  and  $f$ , there must be  $10 - 3 = 7$  students in region  $f$ .

Since 9 students were taking SS and HM, there must be  $9 - 3 = 6$  students in region  $b$ .

Now, we know that 21 students were taking a SS course. This includes students from regions  $a$ ,  $b$ ,  $d$ , and  $e$ . Since we know the number of students in all but region  $a$ , we can determine that  $21 - 6 - 4 - 3 = 8$  students are in region  $a$ .

8 students are taking only a SS course.



## Try it Now

2. One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

43 believed in UFOs

44 believed in ghosts

25 believed in Bigfoot

10 believed in UFOs and ghosts

8 believed in ghosts and Bigfoot

5 believed in UFOs and Bigfoot

2 believed in all three

How many people surveyed believed in at least one of these things?

## Important Topics of this Section

Venn Diagrams

Visualizing the union and intersection of sets using Venn Diagrams

Cardinality of a set

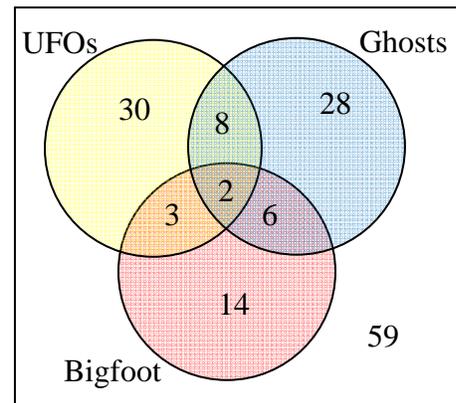
Cardinality properties

Finding cardinality using a Venn diagram

## Try it Now Answers

1.  $A \cup B \cap C^c$

2. Starting with the intersection of all three circles, we work our way out. Since 10 people believe in UFOs and Ghosts, and 2 believe in all three, that leaves 8 that believe in only UFOs and Ghosts. We work our way out, filling in all the regions. Once we have, we can add up all those regions, getting 91 people in the union of all three sets. This leaves  $150 - 91 = 59$  who believe in none.





## Chapter 8: Probability

Section 8.1 Concepts of Probability.....	265
Section 8.2 Conditional Probability and Bayes Theorem.....	275
Section 8.3 Counting.....	283
Section 8.4 Expected Value .....	296

### Section 8.1 Concepts of Probability

#### Introduction

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says “there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with  $n$  equally likely possible outcomes and that  $m$  of those  $n$  outcomes correspond to a particular event; then the **probability** of that event is

defined as  $\frac{m}{n}$ .

#### Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

### Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

### Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events:  $\{1,2,3,4,5,6\}$

Some examples of simple events:

We roll a 1

We roll a 5

Some compound events:

We roll a number bigger than 4

We roll an even number



Two dice

One die

### Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event  $E$  using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event } E}{\text{Total number of equally - likely outcomes}}$$

### Example 2

If we roll a 6-sided die, calculate

a)  $P(\text{rolling a 1})$

b)  $P(\text{rolling a number bigger than 4})$

Recall that the sample space is  $\{1,2,3,4,5,6\}$

a) There is one outcome corresponding to “rolling a 1”, so the probability is  $\frac{1}{6}$

b) There are two outcomes bigger than a 4, so the probability is  $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

## Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is  $\frac{14}{20} = \frac{7}{10}$ .

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

## Try it Now

1. At some random moment, you look at your clock and note the minutes reading.
  - a. What is probability the minutes reading is 15?
  - b. What is the probability the minutes reading is 15 or less?

## Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

## Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

There are 52 cards in the deck and 4 Aces so  $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

### Certain and Impossible events

An impossible event has a probability of 0.

A certain event has a probability of 1.

The probability of any event must be  $0 \leq P(E) \leq 1$

As you're working through this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

### Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is  $P(\text{six}) = 1/6$ . Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is  $P(\text{not a six}) = \frac{5}{6}$ . Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with  $n$  possible outcomes and an event  $E$  that corresponds to  $m$  of these outcomes. Then the remaining  $n - m$  outcomes correspond to  $E$  not happening, thus

$$P(\text{not } E) = \frac{n - m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

### Complement of an Event

The **complement** of an event is the event “ $E$  doesn't happen”

The notation  $\bar{E}$  is used for the complement of event  $E$ .

We can compute the probability of the complement using  $P(\bar{E}) = 1 - P(E)$

Notice also that  $P(E) = 1 - P(\bar{E})$

### Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

There are 13 hearts in the deck, so  $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$ .

The probability of *not* drawing a heart is the complement:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Sometimes you will see probabilities expressed as **odds**.

### Odds

Odds of an event are typically expressed in the form  $A:B$

(Number of outcomes corresponding to the event  $E$ ): (Number of outcomes corresponding to  $\bar{E}$ )

### Example 6

If you pull a random card from a deck of playing cards, what are the odds is an Ace?

There are 4 Aces in the deck, and 48 cards that are not Aces, so the odds would be: 4:48, or 1:12

Notice how this is different than a probability – with probabilities we use the number of successes out of the total number of possible outcomes, while with odds we use the number of successes compared to the number of failures. As another example, if we flipped a coin, the odds of getting a heads would be 1:1.

### Probability of two independent events

### Example 7

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

We could list all possible outcomes:  $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$ .

Notice there are  $2 \cdot 6 = 12$  total outcomes. Out of these, only 1 is the desired outcome, so the probability is  $\frac{1}{12}$ .

The prior example was looking at two independent events.

### Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

## Example 8

Are these events independent?

- a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
- b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
- c) You draw a card from a deck, then draw a second card without replacing the first.
- a) The probability that a head comes up on the second toss is  $1/2$  regardless of whether or not a head came up on the first toss, so these events are independent.
- b) These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
- c) The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

 $P(A \text{ and } B)$  for independent events

If events  $A$  and  $B$  are independent, then the probability of both  $A$  and  $B$  occurring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

where  $P(A \text{ and } B)$  is the probability of events  $A$  and  $B$  both occurring,  $P(A)$  is the probability of event  $A$  occurring, and  $P(B)$  is the probability of event  $B$  occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

## Example 9

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

The probability of choosing a white pair of socks is  $\frac{6}{10}$ .

The probability of choosing a white tee shirt is  $\frac{3}{7}$ .

The probability of both being white is  $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

## Example 10

The manufacturing process for a certain product has a 0.2% defect rate, meaning 2 products out of 1000 is defective on average. If two items are pulled randomly off the assembly line, what's the probability both are defective?

The probability of each being defective is independent, so the probability of both defective

is  $\frac{2}{1000} \cdot \frac{2}{1000} = \frac{4}{1,000,000} = \frac{1}{250,000}$

## Try it Now

2. A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

## Example 11

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Here, there are still 12 possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6,

T6), so the probability is  $\frac{7}{12}$ . How could we have found this from the individual probabilities?

As we would expect,  $\frac{1}{2}$  of these outcomes have a head, and  $\frac{1}{6}$  of these outcomes have a 6

on the die. If we add these,  $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$ , which is not the correct probability.

Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is  $\frac{1}{12}$ .

If we subtract out this double count, we have the correct probability:  $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$ .

### $P(A \text{ or } B)$

The probability of either  $A$  or  $B$  occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

### Example 12

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so

$P(\text{King and Queen}) = 0$ . Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

In the last example, the events were **mutually exclusive**, so  $P(A \text{ or } B) = P(A) + P(B)$ .

### Example 13

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Half the cards are red, so  $P(\text{red}) = \frac{26}{52}$

There are four kings, so  $P(\text{King}) = \frac{4}{52}$

There are two red kings, so  $P(\text{Red and King}) = \frac{2}{52}$

We can then calculate

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

### Try it Now

3. In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

### Example 14

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

- Has a red car *and* got a speeding ticket
- Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is  $\frac{15}{665} \approx 0.0226$ .

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is  $\frac{195}{665} \approx 0.2932$ .

We also could have found this probability by:

$$\begin{aligned} & P(\text{had a red car}) + P(\text{got a speeding ticket}) - P(\text{had a red car and got a speeding ticket}) \\ &= \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665} \end{aligned}$$

**Important Topics of this Section**

Experimental, subjective, and theoretic probability  
Events, sample space  
Basic probability  
Complementary events  
Odds  
Probability for independent events  
Computing an “or” probability

**Try it Now Answers**

1. There are 60 possible readings, from 00 to 59. a.  $\frac{1}{60}$  b.  $\frac{16}{60}$  (counting 00 through 15)

2. Since the second draw is made after replacing the first card, these events are independent. The probability of an ace on each draw is  $\frac{4}{52} = \frac{1}{13}$ , so the probability of an

Ace on both draws is  $\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$

3.  $P(\text{white sock or white tee}) = \frac{6}{10} + \frac{3}{7} - \frac{9}{35} = \frac{27}{35}$

## Section 8.2 Conditional Probability and Bayes Theorem

Often it is required to compute the probability of an event given that another event has occurred. We call that **conditional probability**.

### Conditional Probability

The probability the event  $B$  occurs, given that event  $A$  has happened, is represented as  $P(B | A)$

This is read as “the probability of  $B$  given  $A$ ”

### Example 1

What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

It might seem that you could use the formula for the probability of two independent events and simply multiply  $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$ . This would be incorrect, however, because the two events are not independent. If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case the "condition" is that the first card is an ace. Symbolically, we write this as:

$P(\text{ace on second draw} | \text{an ace on the first draw})$ .

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw."

What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace

has already been drawn is  $\frac{3}{51} = \frac{1}{17}$ .

Thus, the probability of both cards being aces is  $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$ .

### Example 2

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

These are two independent events, so the probability of the die rolling a 6 is  $\frac{1}{6}$ , regardless of the result of the coin flip.

## Example 3

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

- a) Has a speeding ticket *given* they have a red car  
 b) Has a red car *given* they have a speeding ticket

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

- a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1$$

- b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25.$$

Notice from the last example that  $P(B \mid A)$  is **not** equal to  $P(A \mid B)$ .

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

## Conditional Probability Formula

If Events  $A$  and  $B$  are not independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

## Example 4

If you pull 2 cards out of a deck, what is the probability that both are spades?

The probability that the first card is a spade is  $\frac{13}{52}$ .

The probability that the second card is a spade, given the first was a spade, is  $\frac{12}{51}$ , since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is  $\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

## Example 5

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A. The probability that the first card is the Ace of Diamonds is  $\frac{1}{52}$ . The probability that the second card is black given that the first card is

the Ace of Diamonds is  $\frac{26}{51}$  because 26 of the remaining 51 cards are black. The

probability is therefore  $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$ .

Now for Case B: the probability that the first card is black is  $\frac{26}{52} = \frac{1}{2}$ . The probability that

the second card is the Ace of Diamonds given that the first card is black is  $\frac{1}{51}$ . The

probability of Case B is therefore  $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$ , the same as the probability of Case 1.

Recall that the probability of A or B is  $P(A) + P(B) - P(A \text{ and } B)$ . In this problem,  $P(A \text{ and } B) = 0$  since the first card cannot be the Ace of Diamonds and be a black card.

Therefore, the probability of Case A or Case B is  $\frac{1}{102} + \frac{1}{102} = \frac{2}{102} = \frac{1}{51}$ . The probability

that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is  $\frac{1}{51}$ .

## Try it Now

1. In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

## Example 6

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a)  $P(\text{not pregnant} \mid \text{positive test result})$

b)  $P(\text{positive test result} \mid \text{not pregnant})$

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

- a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant.  $P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067$ .
- b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.  $P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263$

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

### Bayes Theorem

Bayes Theorem is a formulaic approach to complex conditional probability problems like the last example. However, using the formula is itself complicated, so we will focus on a more intuitive approach.

#### Example 7

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem. We will discuss this theorem a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

*Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population).* The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: 1/1000. This tells us that about 1 in every 1000 people has the disease. (If we wanted we could write  $P(\text{disease})=0.001$ .)

*A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say  $P(\text{positive} | \text{disease})=1$ .)*

*The false positive rate is 5% (that is, about 5% of people who take the test will test positive, even though they do not have the disease). This is even more straightforward. Another way of looking at it is that of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that  $P(\text{positive} | \text{no disease})=0.05$ .)*

*Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute  $P(\text{disease} | \text{positive})$ . We already know that  $P(\text{positive} | \text{disease})=1$ , but remember that conditional probabilities are not equal if the conditions are switched.*

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about 1/1000 of all people are afflicted with the disease, 1/1000 of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect  $(0.05)(999)=49.95$  (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing  $P(\text{disease} | \text{positive})$ . There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

$$P(\text{disease} | \text{positive}) \approx \frac{1}{51} \approx 0.0196$$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease*.

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and  $(0.05)(99,900)=4995$  test positive but do not have the disease, so the exact probability of having the disease if you test positive is

$$P(\text{disease} | \text{positive}) \approx \frac{100}{5095} \approx 0.0196$$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease.* If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

#### Bayes' Theorem

$$P(A | B) = \frac{P(A)P(B | A)}{P(A)P(B | A) + P(\bar{A})P(B | \bar{A})}$$

In our earlier example, this translates to

$$P(\text{disease} | \text{positive}) = \frac{P(\text{disease})P(\text{positive} | \text{disease})}{P(\text{disease})P(\text{positive} | \text{disease}) + P(\text{no disease})P(\text{positive} | \text{no disease})}$$

Plugging in the numbers gives

$$P(\text{disease} | \text{positive}) = \frac{(0.001)(1)}{(0.001)(1) + (0.999)(0.05)} \approx 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place

of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

### Example 8

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 1% of them, or 98, will test positive.

	Positive test	Negative test	Total
Have disease	180	20	200
Do not have disease	98	9702	9,800
Total	278	9822	10,000

So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{disease} \mid \text{positive}) = \frac{180}{278} \approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{disease} \mid \text{positive}) = \frac{(0.02)(0.90)}{(0.02)(0.90) + (0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

### Example 9

A company has found that 80% of its new management hires are meeting expectations, while 20% are not. Of the satisfactory hires, 75% had sales experience, while of the unsatisfactory hires, 55% had sales experience. What is the probability that a new hire with sales experience will meet expectations?

We can imagine 100 new hires. Of them, 80%, or 80, will meet expectations, and 20 will not. Of the 80 who meet expectations, 75%, or 60, had sales experience, and 20 did not. Of the 20 who did not meet expectations, 55%, or 11, had sales experience, and 9 did not.

Summarizing that in a table:

	Sales Experience	No Sales Experience	Total
Meeting expectations	60	20	80
Not Meeting expectations	11	9	20
Total	71	29	100

Now we can answer the question.

$$P(\text{meet expectations} \mid \text{sales experience}) = \frac{60}{71} \approx 0.845$$

So about 84.5% of new hires with sales experience will meet expectations.

### Try it Now

2. A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

### Important Topics of this Section

Conditional probability  
Probability of “and” for conditional events  
Bayes Theorem

### Try it Now Answers

$$1. \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

2. Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

$$P(\text{disease} \mid \text{positive}) = \frac{500}{500 + 2985} = \frac{500}{3485} \approx 14.3\%$$

## Section 8.3 Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

### Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

#### Example 1

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

**Solution 1:** One way to solve this problem would be to systematically list each possible meal:

soup + hamburger	soup + sandwich	soup + quiche
soup + fajita	soup + pizza	salad + hamburger
salad + sandwich	salad + quiche	salad + fajita
salad + pizza	breadsticks + hamburger	breadsticks + sandwich
breadsticks + quiche	breadsticks + fajita	breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

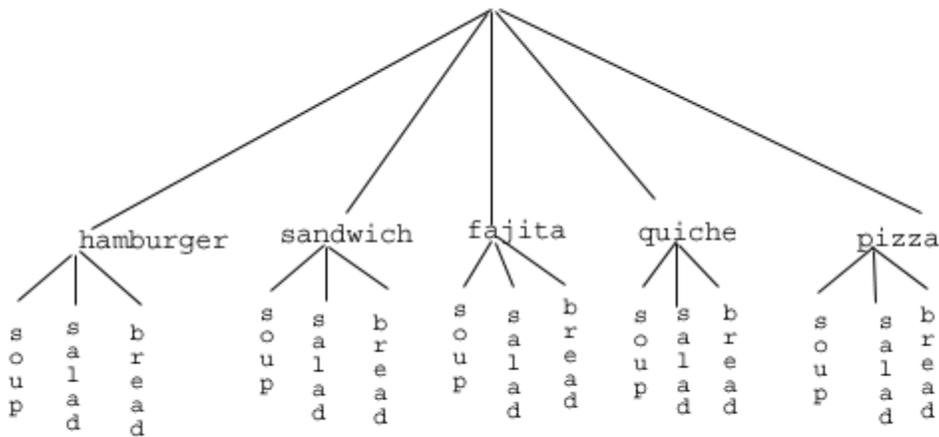
**Solution 2:** Another way to solve this problem would be to list all the possibilities in a table:

	<b>hamburger</b>	<b>sandwich</b>	<b>quiche</b>	<b>fajita</b>	<b>pizza</b>
<b>soup</b>	soup+burger				
<b>salad</b>	salad+burger				
<b>bread</b>	<i>etc.</i>				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

**Solution 3:** We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose from instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15.

Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

### Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are  $m$  items in the first category and  $n$  items in the second category, then the total number of available choices is  $m \cdot n$ .

This is sometimes called the multiplication rule for probabilities.

### Example 2

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

There are 21 choices from the first category and 18 for the second, so there are  $21 \cdot 18 = 378$  possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

### Example 3

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are  $3 \cdot 5 \cdot 2 = 30$  possibilities.

### Example 4

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in  $2 \cdot 2 \cdot 2 = 8$  different ways. Recall that another way to write  $2 \cdot 2 \cdot 2$  is  $2^3$ , which is much more compact.

## Try it Now

1. Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

## Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

## Example 5

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are  $4 \cdot 3 \cdot 2 \cdot 1 = 24$  ways to spell a code word with the letters MATH.

In this example, we needed to calculate  $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$ . This calculation shows up often in mathematics, and is called the **factorial**, and is notated  $n!$

## Factorial

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$$

## Example 6

How many ways can five different door prizes be distributed among five people?

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$  ways.

Now we will consider some slightly different examples.

## Example 7

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of \$100, the second is worth \$25 and the third is worth \$10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Using the Basic Counting Rule, there are 25 choices for the person who receives the \$100 certificate, 24 remaining choices for the \$25 certificate and 23 choices for the \$10 certificate, so there are  $25 \cdot 24 \cdot 23 = 13,800$  ways in which the prizes can be awarded.

## Example 8

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are  $8 \cdot 7 \cdot 6 = 336$  ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the \$100 prize, or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a \$10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order  $r$  items out of  $n$  possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

If you don't see why  $(n - r + 1)$  is the right number to use for the last factor, just think back to the first example in this section, where we calculated  $25 \cdot 24 \cdot 23$  to get 13,800. In this case  $n = 25$  and  $r = 3$ , so  $n - r + 1 = 25 - 3 + 1 = 23$ , which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing  $r$  items out of  $n$  possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

### Permutations

$${}_n P_r = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$

We say that there are  ${}_n P_r$  **permutations** of size  $r$  that may be selected from among  $n$  choices *without replacement* when *order matters*.

It turns out that we can express this result more simply using factorials.

$${}_n P_r = \frac{n!}{(n - r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

### Example 9

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are  ${}_9 P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$  permutations.

### Example 10

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is  ${}_{16} P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680$ .

### Try it Now

2. How many 5 character passwords can be made using the letters A through Z
  - a. if repeats are allowed
  - b. if no repeats are allowed

### Combinations

In the previous section we considered the situation where we chose  $r$  items out of  $n$  possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

#### Example 11

A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are  $25 \cdot 24 \cdot 23 = 13,800$  ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get \$50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC   ACB   BAC   BCA   CAB   CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are  $3 \cdot 2 \cdot 1 = 6$  ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be  $13,800/6 = 2300$ .

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is not important*. If we are choosing  $r$  items out of  $n$  possibilities (instead of 3 out of 25 as in the previous

examples), the number of possible choices will be given by  $\frac{{}^n P_r}{r!}$ , and we could use this

formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing  $r$  items out of  $n$  possibilities *without replacement* where the *order of selection is not important*.

## Combinations

$${}_n C_r = \frac{{}_n P_r}{{}_r P_r}$$

We say that there are  ${}_n C_r$  **combinations** of size  $r$  that may be selected from among  $n$  choices *without replacement* where *order doesn't matter*.

We can also write the combinations formula in terms of factorials:

$${}_n C_r = \frac{n!}{(n-r)!r!}$$

### Example 12

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Since we are choosing 4 people out of 35 *without replacement* where the *order of selection* is **not important** there are  ${}_{35} C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360$  combinations.

### Try it Now

3. The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

### Example 13

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are  ${}_{15} C_{10} = 3003$  ways to choose the 10 Republicans and  ${}_{14} C_9 = 2002$  ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be  $3003 \cdot 2002 = 6,012,006$  possible ways of selecting the members of the Defense Subcommittee.

### Probability using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions

#### Example 14

A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are  $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$  total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute  $10 \cdot 9 \cdot 8 \cdot 7$ , or notice that this is the same as the permutation  ${}_{10}P_4 = 5040$ .

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is

$$\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$$

#### Example 15

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is  ${}_{48}C_6 = 12,271,512$ . Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

## Example 16

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of \$1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

As above, the number of possible outcomes of the lottery drawing is  ${}_{48}C_6 = 12,271,512$ . In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by  ${}_6C_5 = 6$  and the number of ways to choose 1 out of the 42 losing numbers is given by  ${}_{42}C_1 = 42$ . Thus the number of favorable outcomes is then given by the Basic Counting Rule:  ${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$ . So the probability of winning the second prize is.

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

## Try it Now

4. A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

## Example 17

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands,  ${}_{52}C_5$ . This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be  ${}_4C_1$  ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be  ${}_{48}C_4$  ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be  ${}_4C_1 \cdot {}_{48}C_4$  ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

## Example 18

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{{}_4C_2({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

## Try it Now

5. Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

**Birthday Problem**

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

## Example 19

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's

ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is  $364/365$ . Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is  $363/365$ .

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared birthday}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

#### Example 20

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared birthday}) = 1 - \frac{{}^{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

#### Example 21

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Here we can calculate

$$P(\text{shared birthday}) = 1 - \frac{{}^{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: <http://statweb.stanford.edu/~susan/surprise/Birthday.html>, and once the applet has loaded, select 30 birthdays and then keep clicking Start and Reset. If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

### Try it Now

6. Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

### Important Topics of this Section

- Basic counting techniques
- Basic counting rule
- Factorial
- Permutations
- Combinations
- Probability using Permutations and Combinations

### Try it Now Answers

1.  $8 \cdot 11 \cdot 5 = 440$  menu combinations

2. There are 26 characters. a.  $26^5 = 11,881,376$ . b.  ${}_{26}P_5 = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600$

3. Order does not matter.  ${}_{29}C_{19} = 20,030,010$  possible subcommittees

4. There are  $5^{10} = 9,765,625$  different ways the exam can be answered. There are 9 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 36 ways the test could be answered with one wrong answer.

$P(9 \text{ answers correct}) = \frac{36}{5^{10}} \approx 0.0000037$  chance

5.  $P(\text{three Aces and two Kings}) = \frac{{}_4C_3 {}_4C_2}{{}_{52}C_5} = \frac{24}{2598960} \approx 0.0000092$

6.  $P(\text{shared birthday}) = 1 - \frac{{}_{365}P_{10}}{365^{10}} \approx 0.117$

### Section 8.4 Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

#### Example 1

<sup>1</sup>In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?



Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won"  $-\$2/\$38 \approx -\$0.053$ . In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is  $\frac{1}{38}$ . The complement, the probability of losing, is  $\frac{37}{38}$ .

Summarizing these along with the values, we get this table:

Outcome	Probability of outcome
\$35	$\frac{1}{38}$
-\$1	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get  $\$35 \cdot \frac{1}{38} = 0.9211$  and  $-\$1 \cdot \frac{37}{38} = -0.9737$ , and if we add these numbers we get  $0.9211 + (-0.9737) \approx -0.053$ , which is the expected value we computed above.

<sup>1</sup> Photo CC-BY-SA <http://www.flickr.com/photos/stoneflower/>

### Expected Value

**Expected Value** is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

### Try it Now

1. You purchase a raffle ticket to help out a charity. The raffle ticket costs \$5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth \$4000. Compute the expected value for this raffle.

### Example 2

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. If they match 5 numbers, then win \$1,000. It costs \$1 to buy a ticket. Find the expected value.

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all 6 numbers,}$$

$$\frac{{}_6C_5({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ for 5 numbers.}$$

Our probabilities and outcome values are:

Outcome	Probability of outcome
\$999,999	$\frac{1}{12271512}$
\$999	$\frac{252}{12271512}$
-\$1	$1 - \frac{253}{12271512} = \frac{12271259}{12271512}$

The expected value, then is:

$$(\$999,999) \cdot \frac{1}{12271512} + (\$999) \cdot \frac{252}{12271512} + (-\$1) \cdot \frac{12271259}{12271512} \approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose \$1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

### Try it Now

2. A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him \$1. If any two dice match values, you get \$2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

### Example 3

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year<sup>2</sup>. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

The probabilities and outcomes are

Outcome	Probability of outcome
\$100,000 - \$275 = \$99,725	0.00242
-\$275	$1 - 0.00242 = 0.99758$

The expected value is  $(\$99,725)(0.00242) + (-\$275)(0.99758) = -\$33$ .

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

<sup>2</sup> According to the estimator at <http://www.numericaexample.com/index.php?view=article&id=91>

Expected value is also used by businesses for decision making.

#### Example 4

A company is considering two acquisitions. They have evaluated the potential future potential of each. Acquisition A has a 30% probability of increasing profit by \$20MM, a 60% probability of increasing profit by \$5MM, and a 10% probability of decreasing profit by \$5MM. Acquisition B has a 10% probability of increasing profit by \$40MM, a 50% probability of increasing profit by \$10MM, and a 40% probability of decreasing profit by \$4MM. Which acquisition is more prudent?

To compare these, we can look at the expected value:

Acquisition A:  $(20)(0.30) + (5)(0.60) + (-5)(0.10) = \$8.5\text{MM}$

Acquisition B:  $(40)(0.10) + (10)(0.50) + (-4)(0.40) = \$7.4\text{MM}$

Since Acquisition A has a higher expected value, it is the more prudent acquisition.

#### Important Topics of this Section

- Expected value
- Fair game

#### Try it Now Answers

1.  $(\$3,995) \cdot \frac{1}{2000} + (-\$5) \cdot \frac{1999}{2000} \approx -\$3.00$

