Related Rates

General steps

1. Draw a picture!! (This may not be possible for every problem, but there's usually something you can draw.)

2. Label everything. If a quantity is fixed for the entire problem, write in the number. If it can change, then assign it a variable. There are often multiple ways to draw and label things, but the final answer will be the same irrespective of how you label things.

3. Write down what you know, and what you want to know. Note: When writing down given/known rates of change, make them positive if the variable is getting larger, negative if the variable is getting smaller (this is going to depend on how you labeled your picture).

4. Figure out how everything is related and come up with a formula relating the variables. This can involve using geometric formulas, triangles, similar triangles, etc.

5. Differentiation implicitly with respect to t (or whatever the independent variable is). Remember, in these problems all variables are viewed as functions of t, so you'll pick up $\frac{d}{dt}$ terms from the chain rule as you differentiate.

6. Plug in known values for variables and rates, then solve for the quantity in which you're interested. Warning: DO NOT plug in numbers for quantities that can change until *after* you differentiate!! If a quantity is constant throughout the entire problem and cannot change, then you should have already put in the picture as a number in step 2. (Replacing a variable with an expression involving another variable *is* allowed.)

Note: It's okay to get a negative answer in these problems. A negative answer tells us that that the quantity is decreasing and positive tells us it's increasing. On a test or homework I want to see the correct sign in your answer. In some books all rates are positive in the final answers, even if the quantity is decreasing (the negative is inferred from inclusion of the word "decreasing" in the problem).

Problems

1. A spotlight on the ground shines on a building 36 ft away. A man 6 ft tall walks from the spotlight toward the building at a speed of 10 ft/s. How fast is the length of his shadow on the building changing when he is 24 ft from the building?

2. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

3. A rectangular swimming pool is being filled with water at a rate of 5 m^3/min. The length of the pool is 10 m and the width is 4 m. How fast is the height of the water increasing?

4. Car A is 100 mi. east of Car B at 3:00pm. Car A is moving west at 60 mi/h and Car B is moving south at 70 mi/h. How fast is the distance between the cars changing at 4:00pm?

5. A balloon rises at a rate of 3 m/s from a point on the ground 30 m from an observer. Find the rate of change of the distance between the observer and the balloon when the balloon is 30 m above the ground.

6. A company that manufactures sport supplements calculates that its cost and revenue can be modeled by the equations

$$C = 125{,}000 + 0.75q \qquad \text{and} \qquad R = 250q - \frac{1}{10}q^2$$

   where q is the number of units of sport supplements produced in 1 week. If production in one particular week is 1000 units and is increasing at a rate of 150 units per week, find:

   (a) The rate at which the cost is changing.
   (b) The rate at which the revenue is changing.
   (c) The rate at which the profit is changing.

Answers (See videos for full solutions to problems 1-4)

1. $-2.1\ ft/s$

2. $48\ cm^2/s$

3. $+\frac{1}{8}$ m/min

4. +31 mi/h

5. The picture should form a right triangle with base 30, height y, and hypotenuse x, so the equation relating them is $x^2 = 30^2 + y^2$. We are given that $\frac{dy}{dt} = +3$, and we want to know what $\frac{dx}{dt}$ is when $y = 30$ (note: when $y = 30$, $x = \sqrt{30^2 + 30^2} = \sqrt{1800} \approx 42.43$). Differentiating the equation yields

$$2x\frac{dx}{dt} = 0 + 2y\frac{dy}{dt}$$

Plugging in all the values we know and solving for $\frac{dx}{dt}$ gives us:

$$2 \cdot 42.43 \cdot \frac{dx}{dt} = 2 \cdot 30 \cdot 3$$
$$\frac{dx}{dt} = \frac{90}{42.43} \approx \mathbf{2.12\ m/s}$$

6. For all parts, we're finding the rates for the particular week referenced in the problem, when $q = 1000$. There aren't any pictures to draw for this problem, and the equations are already given to us. We're also given that $\frac{dq}{dt} = +150$. So, all we need to do is differentiate C, R, and

$$P = R - C = 250q - \frac{1}{10}q^2 - (125000 + 0.75q) = -\frac{1}{10}q^2 + 249.25q - 125000$$

and plug in our values for q and $\frac{dq}{dt}$:

(a)

$$\frac{dR}{dt} = 250 \cdot \frac{dq}{dt} - \frac{1}{10} \cdot 2 \cdot q \cdot \frac{dq}{dt} \longrightarrow \frac{dR}{dt} = 250 \cdot 150 - \frac{1}{5} \cdot 1000 \cdot 150 = \mathbf{7500}$$

(b)

$$\frac{dC}{dt} = 0.75 \cdot \frac{dq}{dt} \longrightarrow \frac{dC}{dt} = 0.75 \cdot 150 = \mathbf{112.5}$$

(c)

$$\frac{dP}{dt} = -\frac{1}{10} \cdot 2 \cdot q \cdot \frac{dq}{dt} + 249.25 \cdot \frac{dq}{dt} \longrightarrow \frac{dP}{dt} = -\frac{1}{10} \cdot 2 \cdot 1000 \cdot 150 + 249.25 \cdot 150 = \mathbf{7387.5}$$