Section 3: Optimization

The partial derivatives tell us something about where a surface has local maxima and minima. Remember that even in the one-variable cases, there were critical points which were neither maxima nor minima – this is also true for functions of many variables. In fact, as you might expect, the situation is even more complicated.

Second Derivatives

When you find a partial derivative of a function of two variables, you get another function of two variables – you can take its partial derivatives, too. We've done this before, in the one-variable setting. In the one-variable setting, the second derivative gave information about how the graph was curved. In the two-variable setting, the second partial derivatives give some information about how the surface is curved, as you travel on cross-sections – but that's not very complete information about the entire surface.

Imagine that you have a surface that's ruffled around a point, like what happens near a button on an overstuffed sofa, or a pinched piece of fabric, or the wrinkly skin near your thumb when you make a fist. Right at that point, every direction you move, something different will happen – it might increase, decrease, curve up, curve down … A simple phrase like "concave up" or "concave down" can't describe all the things that can happen on a surface.

Surprisingly enough, though, there is still a second derivative test that can help you decide if a point is a local max or min or neither, so we still do want to find second derivatives.

Second Partial Derivatives

Suppose $f(x, y)$ is a function of two variables. Then it has four **second partial derivatives**:

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = (f_x)_x; \qquad f_{xy} = \frac{\partial}{\partial y}(f_x) = (f_x)_y;$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = (f_y)_x; \qquad f_{yy} = \frac{\partial}{\partial y}(f_y) = (f_y)_y$$

f_{xy} and f_{yx} are called the **mixed (second) partial derivatives of *f***

Leibniz notation for the second partial derivatives is a bit confusing, and we won't use it as often:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}; \qquad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x};$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}; \qquad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Notice that the order of the variables for the mixed partials goes from right to left in the Leibniz notation instead of left to right.

Example 1

Find all four partial derivatives of $f(x,y)=x^2-4xy+4y^2$

We have to start by finding the (first) partial derivatives:

$f_x(x,y)=2x-4y$

$f_y(x,y)=-4x+8y$

Now we're ready to take the second partial derivatives:

$$f_{xx}(x,y)=\frac{\partial}{\partial x}(2x-4y)=2$$

$$f_{xy}(x,y)=\frac{\partial}{\partial y}(2x-4y)=-4$$

$$f_{yx}(x,y)=\frac{\partial}{\partial x}(-4x+8y)=-4$$

$$f_{yy}(x,y)=\frac{\partial}{\partial y}(-4x+8y)=8$$

You might have noticed that the two mixed partial derivatives were equal in this last example. It turns out that it's not a coincidence – it's a theorem.

Mixed Partial Derivative Theorem

If f , f_x, f_y, f_{xy}, and f_{yx} are all continuous (no breaks in their graph)
Then $f_{xy} = f_{yx}$.

In fact, as long as *f* and all its appropriate partial derivatives are continuous, the mixed partials are equal even if they are of higher order, and even if the function has more than two variables.

This theorem means that the confusing Leibniz notation for second derivatives is not a big problem – in almost every situation, the mixed partials are equal, so it doesn't matter in which order we compute them.

Example 2

Find $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x,y)=\frac{e^{x+y}}{y^3+y}+y(\ln y)$

We already found the first partial derivatives in an earlier example:

$$\frac{\partial f}{\partial x}=\frac{1}{y^3+y}e^{x+y}$$

$$\frac{\partial f}{\partial y} = \frac{\left(e^{x+y}(1)\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2} + (1)(\ln y) + (y)\left(\frac{1}{y}\right)$$

Now we need to find the mixed partial derivative – the Theorem says it doesn't matter whether we find the partial derivative of $\frac{\partial f}{\partial x} = \frac{1}{y^3+y}e^{x+y}$ with respect to y or the partial derivative of $\frac{\partial f}{\partial y} = \frac{\left(e^{x+y}(1)\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2} + (1)(\ln y) + (y)\left(\frac{1}{y}\right)$ with respect to x. Which would you rather do?

It looks like it will be easier to compute the mixed partial by finding the partial derivative of $\frac{\partial f}{\partial x} = \frac{1}{y^3+y}e^{x+y}$ with respect to y – it still looks messy, but it looks less messy:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{1}{y^3+y}e^{x+y}\right) = \frac{\left(e^{x+y}\right)\left(y^3+y\right) - \left(e^{x+y}\right)\left(3y^2+1\right)}{\left(y^3+y\right)^2}$$

If you'd decided to do this the other way, you'd end up in the same place. Eventually.

Local Maxima, Minima, and Saddle Points

Let's briefly review max-min problems in one variable.

A local max is a point on a curve that is higher than all the nearby points. A local min is lower than all the nearby points. We know that local max or min can only occur at critical points, where the derivative is zero or undefined. But we also know that not all critical points are max or min, so we also need to test them, with the First Derivative or Second Derivative Test.

The situation with a function of two variables is much the same. Just as in the one-variable case, the first step is to find critical points, places where both the partial derivatives are either zero or undefined.

Definition:
f has a **local maximum** at (a, b) if f(a, b) ≥ f(x, y) for all points (x, y) near (a, b)
f has a **local minimum** at (a, b) if f(a, b) ≤ f(x, y) for all points (x, y) near (a, b)

A **critical point** for a function f(x, y) is a point (x, y) (or (x, y, f(x, y)) where **both** the following are true:

$f_x = 0$ **or is undefined** **and** $f_y = 0$ **or is undefined**

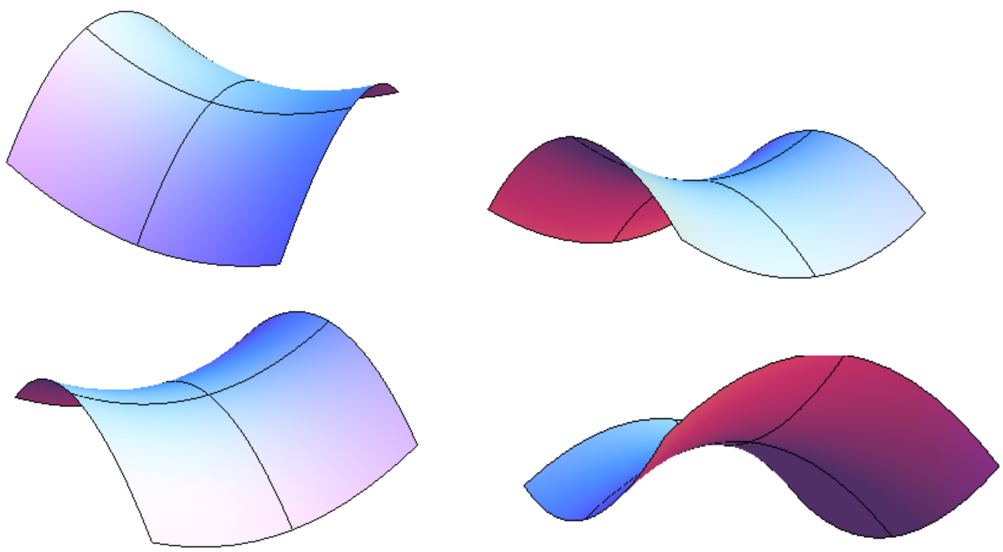
Just as in the one-variable case, a local max or min of f can only occur at a critical point.

And then, just as in the one-variable setting, not all critical points are local max or min. For a function of two variables, the critical point could be a local max, local min, or a saddle point.

A point on a surface is a local maximum if it's higher than all the points nearby; a point is a local minimum if it's lower than all the points nearby.

A saddle point is a point on a surface that is a minimum along some paths and a maximum along some others. It's called this because it's shaped a bit like a saddle you might use to ride a horse. You can see a saddle point by making a fist – between the knuckles of your index and middle fingers, you can see a place that is a minimum as you go across your knuckles, but a maximum as you go along your hand toward your fingers.

Here is a picture of a saddle point from a few different angles. This is the surface $f(x, y) = 5x^2 - 3y^2 + 10$, and there is a saddle point above the origin. The lines show what the surface looks like above the *x*- and *y*-axes. Notice how the point above the origin, where the lines cross, is a local minimum in one direction, but a local maximum in the other direction.



Second Derivative Test

Just as in the one-variable case, we'll need a way to test critical points to see whether they are local max or min. There is a second derivative test for functions of two variables that can help – but, just as in the one-variable case, it won't always give an answer.

The Second Derivative Test for Functions of Two Variables:
Find all critical points of f(x,y).
Compute $D = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx})$, and evaluate it at each critical point.

(a) If D > 0, then f has a local max or min at the critical point. To see which, look at the sign of f_{xx}:
  If $f_{xx} > 0$, then f has a local minimum at the critical point.
  If $f_{xx} < 0$, then f has a local maximum at the critical point.

(b) If D < 0 then f has a saddle point at the critical point.

(c) If D = 0, there could be a local max, local min, or neither.

Example 3

Find all local maxima, minima, and saddle points for the function
$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

First we need the partial derivatives:
$f_x = 3x^2 + 6x$ and $f_y = 3y^2 - 6y$

Critical points are the places where both of these are zero (neither is ever undefined):
$f_x = 3x^2 + 6x = 3x(x+2) = 0$ when $x = 0$ or when $x = -2$.
$f_y = 3y^2 - 6y = 3y(y-2) = 0$ when $y = 0$ or when $y = 2$.
Putting these together, we get four critical points: (0, 0), (−2, 0), (0, 2), and (−2, 2).

Now to classify them, we'll use the Second Derivative Test. We'll need all the second partial derivatives:
$f_{xx} = 6x + 6$, $f_{yy} = 6y - 6$, $f_{xy} = 0 = f_{yx}$

Then $D = (6x+6)(6y-6) - (0)(0) = (6x+6)(6y-6)$.

Now look at each critical point in turn:
At (0, 0): $D = (6(0)+6)(6(0)-6) = (6)(-6) = -36 < 0$; there is a saddle point at (0, 0).

At (−2, 0): $D = (6(-2)+6)(6(0)-6) = (-6)(-6) = 36 > 0$, and $f_{xx} = 6(-2)+6 = -6 < 0$; there is a local maximum at (−2, 0).

At (0, 2): $D = (6)(6) > 0$ and $f_{xx} = 6 > 0$; there is a local minimum at (0, 2).

At (−2, 2): $D = (-6)(6) < 0$; there is another saddle point at (−2, 2).

Example 4

Find all local maxima, minima, and saddle points for the function

$$z = 9x^3 + \frac{y^3}{3} - 4xy.$$

Solution: We'll need all the partial derivatives and second partial derivatives, so let's compute them all first:

$z_x = 27x^2 - 4y; \quad z_y = y^2 - 4x;$

$z_{xx} = 54x; \; z_{yy} = 2y; \; z_{xy} = -4 = z_{yx}$

Now to find the critical points: We need both z_x and z_y to be zero (neither is ever undefined), so we need to solve this set of equations simultaneously:

$z_x = 27x^2 - 4y = 0$

$z_y = y^2 - 4x = 0$

Perhaps it's been a while since you solved systems of equations. Just remember the substitution method – solve one equation for one variable and substitute into the other equation:

$27x^2 - 4y = 0$
$y^2 - 4x = 0$
→ solve $y^2 - 4x = 0$ for $x = \frac{y^2}{4}$, then substitute into the other equation

$27\left(\frac{y^2}{4}\right)^2 - 4y = 0$

$\frac{27}{16}y^4 - 4y = 0$

Now we have just one equation in one variable to solve. Factoring out a y gives

$y\left(\frac{27}{16}y^3 - 4\right) = 0$, so $y = 0$ or $\frac{27}{16}y^3 - 4 = 0$, giving $y = \frac{4}{3}$

Plugging back in to the equation $x = \frac{y^2}{4}$ to find x gives us the two critical points:

(0, 0) and $\left(\frac{4}{9}, \frac{4}{3}\right)$.

Now to test them. Compute $D = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx}) = (54x)(2y) - (-4)(-4) = 108x - 16$. Evaluate it at the two critical points, and see:

At (0,0): D = −16 < 0, so there is a saddle point at (0, 0).

At $\left(\frac{4}{9}, \frac{4}{3}\right)$: D = 48 > 0, and $f_{xx} > 0$, so there is a local minimum at $\left(\frac{4}{9}, \frac{4}{3}\right)$.

Applied Optimization

Example 5

A company makes two products. The demand equations for the two products are given below. p_1, p_2, q_1,and q_2 are the prices and quantities for products 1 and 2.

$$q_1 = 200 - 3p_1 - p_2$$

$$q_2 = 150 - p_1 - 2p_2$$

Find the price the company should charge for each product in order to maximize total revenue. What is that maximum revenue?

Revenue is still price × quantity. If we're selling two products, the total revenue will be the sum of the revenues from the two products:

$$R(p_1, p_2) = p_1q_1 + p_2q_2 = p_1(200 - 3p_1 - p_2) + p_2(150 - p_1 - 2p_2)$$

$$R(p_1, p_2) = 200p_1 - 3{p_1}^2 - 2p_1p_2 + 150p_2 - 2{p_2}^2$$

This is a function of two variables, the two prices, and we need to optimize it – just as in the previous examples. First we find critical points. The notation here gets a bit hard to look at, but hang in there – this is the same stuff we've done before.

$R_{p_1} = 200 - 6p_1 - 2p_2$ and $R_{p_2} = 150 - 2p_1 - 4p_2$

Solving these simultaneously gives the one critical point $(p_1, p_2) = (25{,}25)$.

To confirm that this gives maximum revenue, we need to use the Second Derivative Test. Find all the second derivatives:

$R_{p_1p_1} = -6$, $R_{p_2p_2} = -4$, and $R_{p_1p_2} = -2 = R_{p_2p_{12}}$

So $D = (-6)(-4) - (-2)(-2) > 0$ and $R_{p_1p_1} < 0$, so this really is a local maximum.

To maximize revenue, the company should charge \$25 per unit for both products. This will yield a maximum revenue of \$4375.

4.3 Exercises

For problems 1 through 6, find f_{xx}, f_{yy}, f_{xy} and f_{yx} for the function given. Confirm that $f_{xy} = f_{yx}$.

1. $f(x, y) = x^2 - 5y^2$
2. $f(x, y) = x^4 + 4x^3y - 6x^2y^2 - 4xy^3 + y^4$
3. $f(x, y) = 5x^2y^2$
4. $f(x, y) = e^{x+6y}$
5. $f(x, y) = \ln(xy + 2x - 6y)$
6. $f(x, y) = \dfrac{x^2}{y^4 - 5}$

7. Find the critical points of $f(x, y) = y^3 - x^3 + 15x^2 - 12y + 12$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

8. Find the critical points of $f(x, y) = 2xy - x^2 - 2y^2 + 6x + 4$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

9. Find the critical points of $f(x, y) = y^2 - 4\ln(x) + 4x$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

10. Find the critical points of $f(x, y) = xy - 6x^2 + 3x - y + 2$ and use the Second Derivative Test to classify them. If the test fails, say "the test fails."

11. The origin is a critical point for the function $f(x, y) = x^3 + y^3$, and D = 0 there. That is, the Second Derivative Test fails. Use what you know about shapes of functions to decide if there is a local minimum, local maximum, or saddle point for this function at (0, 0).

12. The origin is a critical point for the function $f(x, y) = 15 - x^2y^2$, and D = 0 there. That is, the Second Derivative Test fails. Use what you know about shapes of functions to decide if there is a local minimum, local maximum, or saddle point for this function at (0, 0).

For problems 13 through 18, find all local maxima, minima, and saddle points for the function.

13. $f(x,y) = xy - 5x^2 - 5y^2 + 33y$

14. $f(x,y) = 10xy - x^2 - y^2 + 3x$

15. $f(x,y) = x^3 + y^3 - 3xy$

16. $f(x,y) = 5x^2 - 4xy + 2y^2 + 4x - 4y + 10$

17. $f(x,y) = y^2 e^x + x^2$

18. $f(x,y) = xy + 2x - \ln(x^2 y)$, for $x > 0$ and $y > 0$.

19. The demand functions for two products are given below. p_1, p_2, q_1,and q_2 are the prices (in dollars) and quantities for products 1 and 2.

$$q_1 = 200 - 3p_1 + p_2$$
$$q_2 = 150 + p_1 - 2p_2$$

a. Are these two products complementary goods or substitute goods?
b. What is the quantity demanded for each when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item?
c. Write a function $R(p_1, p_2)$ that expresses the total revenue from these two products.
d. Find the price and quantity for each product that maximizes the total revenue.

20. The demand functions for two products are given below. p_1, p_2, q_1,and q_2 are the prices (in dollars) and quantities for products 1 and 2.

$$q_1 = 350 + p_1 + 2p_2$$
$$q_2 = 225 + p_1 + p_2$$

a. Are these two products complementary goods or substitute goods?
b. What is the quantity demanded for each when the price for product 1 is \$20 per item and the price for product 2 is \$30 per item?
c. Write a function $R(p_1, p_2)$ that expresses the total revenue from these two products.
d. Find the price and quantity for each product that maximizes the total revenue.

21. Suppose the demand functions for two products are $q_1 = f(p_1, p_2)$ and $q_2 = g(p_1, p_2)$, where p_1, p_2, q_1, and q_2 are the prices (in dollars) and quantities for products 1 and 2. Consider the four partial derivatives $\frac{\partial q_1}{\partial p_1}, \frac{\partial q_1}{\partial p_2}, \frac{\partial q_2}{\partial p_1}$, and $\frac{\partial q_2}{\partial p_2}$. Tell the sign of each of these partial derivatives if

a. the products are complementary goods.
b. the products are substitute goods.