

Section 5: Additional Integration Techniques

Integration By Parts

Integration by parts is an integration method which enables us to find antiderivatives of some new functions such as $\ln(x)$ as well as antiderivatives of products of functions such as $x^2 \ln(x)$ and xe^x .

If the function we're trying to integrate can be written as a product of two functions, u , and dv , then integration by parts lets us trade out a complicated integral for hopefully simpler one.

INTEGRATION BY PARTS FORMULA

$$\int u dv = uv - \int v du$$

For definite integrals,

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Example 1

Integrate $\int xe^x dx$

To use the By Parts method, we break apart the product into two parts:

$$u = x \quad \text{and} \quad dv = e^x dx$$

We now calculate du , the derivative of u , and v , the integral of dv .

$$du = \left(\frac{d}{dx} x \right) dx = 1 dx \quad \text{and} \quad v = \int e^x dx = e^x.$$

Using the By Parts formula,

$$\int xe^x dx = uv - \int v du = xe^x - \int e^x dx$$

Notice the remaining integral is simpler than the original, and one we can easily evaluate.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

We could have chosen either x or e^x as our u in the last example, but had we chosen e^x , the second integral would have become messier, rather than simpler.

RULE OF THUMB

When selecting the u for By Parts, select a logarithmic expression if one is present. If not, select an algebraic expression (like x or dx).

Example 2

Integrate $\int_1^4 6x^2 \ln x \, dx$

Since this contains a logarithmic expression, we'll use it for our u .

$$u = \ln x \quad \text{and} \quad dv = 6x^2 dx$$

We now calculate du and v .

$$du = \frac{1}{x} dx \quad \text{and} \quad v = \int 6x^2 dx = 6 \frac{x^3}{3} = 2x^3.$$

Using the By Parts formula,

$$\int_1^4 6x^2 \ln x \, dx = 2x^3 \ln x \Big|_1^4 - \int_1^4 6x^2 \frac{1}{x} dx$$

We can simplify the expression in the integral on the right:

$$\int_1^4 6x^2 \ln x \, dx = 2x^3 \ln x \Big|_1^4 - \int_1^4 6x dx$$

The remaining integral is a basic one we can now evaluate.

$$\int_1^4 6x^2 \ln x \, dx = 2x^3 \ln x \Big|_1^4 - 3x^2 \Big|_1^4$$

Finally, we can evaluate the expressions

$$\int_1^4 6x^2 \ln x \, dx = \left[(2 \cdot 4^3 \ln 4) - (2 \cdot 1^3 \ln 1) \right] - \left[(3 \cdot 4^2) - (3 \cdot 1^2) \right] = 128 \ln(4) - 45 \approx 132.446$$

Integration Using Tables of Integrals

There are many techniques of integration we will not be studying. Many of them lead to general formulas which can be compiled into a Table of Integrals - a type of cheat-sheet for integration.

For example, here are two entries you might find in a table of integrals:

TABLE OF INTEGRAL EXAMPLES

$$\int \frac{1}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} = \ln \left| x + \sqrt{x^2 + a^2} \right| + C$$

Example 3

Integrate $\int \frac{5}{x^2 - 9} dx$

This integral looks very similar to the form of the first integral in the examples table. By employing the rule that allows us to pull out constants, and by rewriting 9 as 3^2 , we can better see the match.

$$\int \frac{5}{x^2 - 9} dx = 5 \int \frac{1}{x^2 - 3^2} dx$$

Now we simply use the formula from the table, with $a = 3$.

$$\int \frac{5}{x^2 - 9} dx = 5 \int \frac{1}{x^2 - 3^2} dx = 5 \left(\frac{1}{2 \cdot 3} \ln \left| \frac{x-3}{x+3} \right| \right) = \frac{5}{6} \ln \left| \frac{x-3}{x+3} \right| + C$$

Sometimes we have to combine the table with other techniques we've learned, like substitution.

Example 4

Integrate $\int \frac{x^2}{\sqrt{x^6 + 16}} dx$

This integral looks somewhat like the second integral in the example table, but the power of x is incorrect, and there is an x^2 in the numerator which does not match. Trying to utilize this rule, we can try to rewrite the denominator to look like (something)². Luckily, $x^6 = (x^3)^2$

$$\int \frac{x^2}{\sqrt{x^6 + 16}} dx = \int \frac{x^2}{\sqrt{(x^3)^2 + 16}} dx$$

Now we can use substitution, letting $u = x^3$, so $du = 3x^2 dx$.

Making the substitution,

$$\int \frac{x^2}{\sqrt{(x^3)^2 + 16}} dx = \int \frac{1}{\sqrt{u^2 + 16}} \frac{du}{3} = \frac{1}{3} \int \frac{1}{\sqrt{u^2 + 16}} du$$

Now we can use the table entry.

$$\frac{1}{3} \int \frac{1}{\sqrt{u^2 + 16}} du = \frac{1}{3} \ln|u + \sqrt{u^2 + 16}| + C$$

Undoing the substitution,

$$\int \frac{x^2}{\sqrt{x^6 + 16}} dx = \frac{1}{3} \ln|x^3 + \sqrt{x^6 + 16}| + C$$

3.5 Exercises

In problems 1–4, a function u or dv is given. Find the piece u or dv which is not given, calculate du and v , and apply the Integration by Parts Formula.

$$1. \int 12x \cdot \ln(x) dx \quad u = \ln(x) \quad 2. \int x \cdot e^{-x} dx \quad u = x$$

$$3. \int x^4 \ln(x) dx \quad dv = x^4 dx \quad 4. \int x \cdot (5x + 1)^{19} dx \quad u = x$$

In problems 5 - 10 evaluate the integrals

$$5. \int_0^1 \frac{x}{e^{3x}} dx \quad 6. \int_0^1 10x \cdot e^{3x} dx \quad 7. \int_1^3 \ln(2x + 5) dx$$

$$8. \int x^3 \ln(5x) dx \quad 9. \int x \ln(x + 1) dx \quad 10. \int_1^2 \frac{\ln(x)}{x^2} dx$$

For problems 11 - 14 integrate each function.

$$11. \int \frac{1}{4 - x^2} \quad 12. \int \frac{2}{9 - x^2} \quad 13. \int \sqrt{4 + x^2} \quad 14. \int \sqrt{9 + x^2}$$