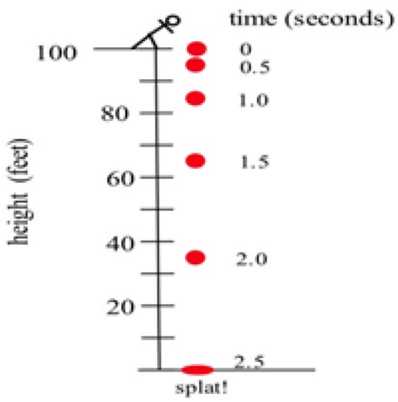
Section 1: Instantaneous Rate of Change and Tangent Lines

Instantaneous Velocity

Suppose we drop a tomato from the top of a 100 foot building and time its fall.



time (s)	height (ft)
0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

Figure 2

Some questions are easy to answer directly from the table:

(a) How long did it take for the tomato n to drop 100 feet? (2.5 seconds)
(b) How far did the tomato fall during the first second? (100 – 84 = 16 feet)
(c) How far did the tomato fall during the last second? (64 – 0 = 64 feet)
(d) How far did the tomato fall between t =.5 and t = 1? (96 – 84 = 12 feet)

Some other questions require a little calculation:

(e) What was the average velocity of the tomato during its fall?

$$\textbf{Average velocity} = \frac{\textbf{distance fallen}}{\textbf{total time}} = \frac{\boldsymbol{\Delta}\textbf{ position}}{\boldsymbol{\Delta}\textbf{ time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \text{ ft/s}.$$

(f) What was the average velocity between t=1 and t=2 seconds?

$$\text{Average velocity} = \frac{\Delta \text{ position}}{\Delta \text{ time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \text{ ft/s}.$$

Some questions are more difficult.

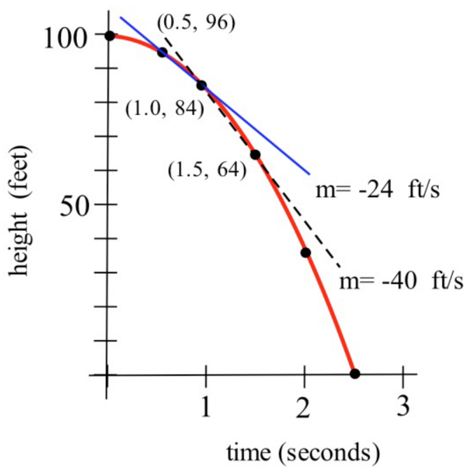
(g) How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, –40 ft/s . But the tomato fell slowly at the beginning and rapidly near the end so the "–40 ft/s" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at t=1 by calculating the average velocities over a short time interval near t = 1 . The average velocity between t = 0.5 and t = 1 is $\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24$ ft/s, and the average velocity between t = 1 and t = 1.5 is $\frac{-20 \text{ feet}}{.5 \text{ s}} = -40$ ft/s so we can be reasonably sure that the instantaneous velocity is between –24 ft/s and –40 ft/s.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is $\frac{\Delta \text{ position}}{\Delta \text{ time}}$, which is the slope of the **secant line** through two points on the graph of height versus time. The instantaneous velocity at a particular time and height is the slope of the **tangent line** to the graph at the point given by that time and height.

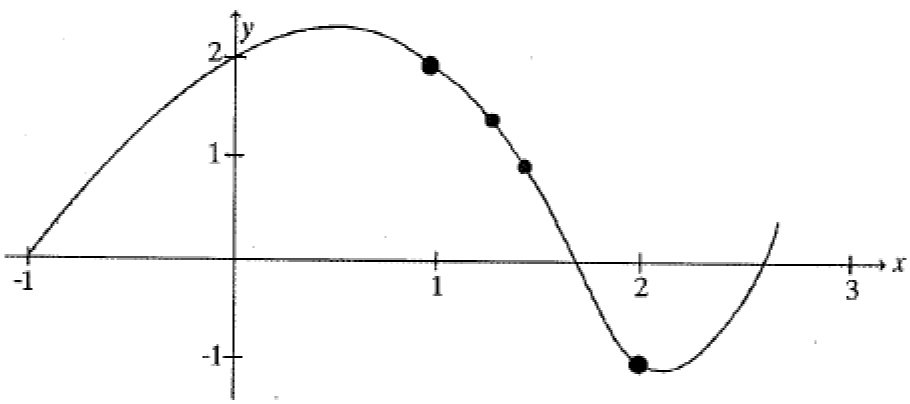


Average velocity $= \frac{\Delta \text{ position}}{\Delta \text{ time}} =$ **slope of the secant line through 2 points.**

Instantaneous velocity = slope of the line tangent to the graph.

Tangent Lines

Do this!
The graph below is the graph of $y = f(x)$. We want to find the slope of the tangent line at the point (1, 2).
First, draw the secant line between (1, 2) and (2, −1) and compute its slope.
Now draw the secant line between (1, 2) and (1.5, 1) and compute its slope.
Compare the two lines you have drawn. Which would be a better approximation of the tangent line to the curve at (1, 2)?
Now draw the secant line between (1, 2) and (1.3, 1.5) and compute its slope. Is this line an even better approximation of the tangent line?
Now draw your best guess for the tangent line and measure its slope. Do you see a pattern in the slopes?

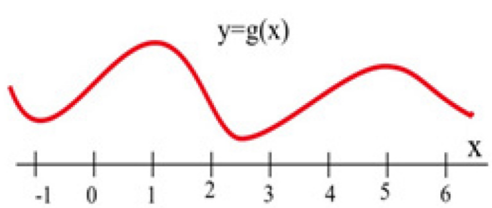


You should have noticed that as the interval got smaller and smaller, the secant line got closer to the tangent line and its slope got closer to the slope of the tangent line. That's good news – we know how to find the slope of a secant line.

In some applications, we need to know where the graph of a function f(x) has horizontal tangent lines (slopes = 0).

Example 1

At right is the graph of y = g(x). At what values of x does the graph of y = g(x) below have horizontal tangent lines?



The tangent lines to the graph of g(x) are horizontal (slope = 0) when $x \approx -1, 1, 2.5$, and 5.

2.1 Exercises

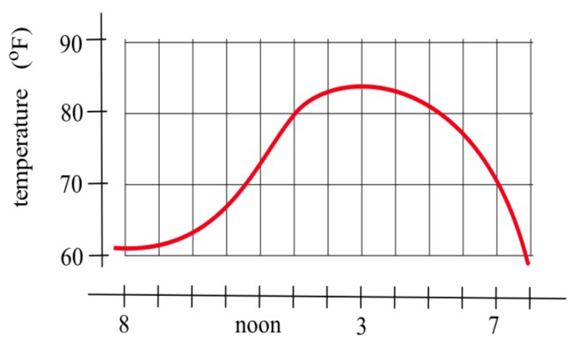
1. What is the slope of the line through (3,9) and (x, y) for $y = x^2$ and x = 2.97? x = 3.001? x = 3+h? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near 3.

2. What is the slope of the line through (–2,4) and (x, y) for $y = x^2$ and x = –1.98? x = –2.03? x = –2+h? What happens to this last slope when h is very small (close to 0)? Sketch the graph of $y = x^2$ for x near –2.

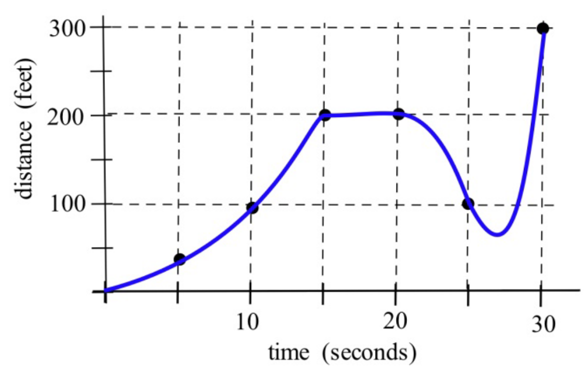
3. What is the slope of the line through (2,4) and (x, y) for $y = x^2 + x - 2$ and x = 1.99? x = 2.004? x = 2+h? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x - 2$ for x near 2.

4. What is the slope of the line through (–1,–2) and (x, y) for $y = x^2 + x - 2$ and x = –.98? x = –1.03? x = –1+h? What happens to this last slope when h is very small? Sketch the graph of $y = x^2 + x - 2$ for x near –1.

5. The graph to the right shows the temperature during a day in Ames.
   (a) What was the average change in temperature from 9 am to 1 pm?
   (b) Estimate how fast the temperature was rising **at** 10 am and **at** 7 pm?

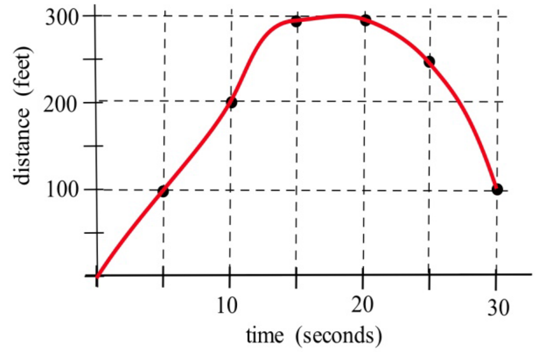


6. The graph shows the distance of a car from a measuring position located on the edge of a straight road.
   (a) What was the average velocity of the car from t = 0 to t = 30 seconds?
   (b) What was the average velocity of the car from t = 10 to t = 30 seconds?
   (c) About how fast was the car traveling **at** t = 10 seconds? **at** t = 20 s ? **at** t = 30 s ?
   (d) What does the horizontal part of the graph between t = 15 and t = 20 seconds mean?
   (e) What does the negative velocity at t = 25 represent?



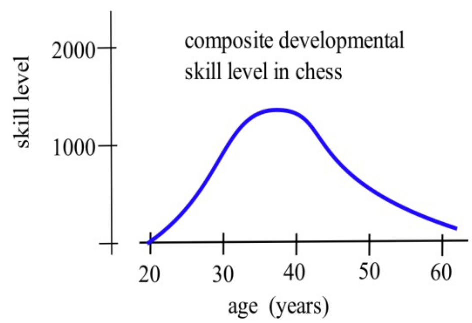
7. The graph shows the distance of a car from a measuring

position located on the edge of a straight road.



(a) What was the average velocity of the car from t = 0 to t = 20 seconds?
(b) What was the average velocity from t = 10 to t = 30 seconds?
(c) About how fast was the car traveling **at** t = 10 seconds? **at** t = 20 s ? **at** t = 30 s ?

8. The graph shows the composite developmental skill level of chessmasters at different ages as determined by their performance against other chessmasters. (From "Rating Systems for Human Abilities", by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)



(a) At what age is the "typical" chessmaster playing the best chess?
(b) At approximately what age is the chessmaster's skill level increasing most rapidly?
(c) Describe the development of the "typical" chessmaster's skill in words.
(d) Sketch graphs which you **think** would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician, and a professional in your major field.