2.1.1 Introduction to Fractions and Mixed Numbers

Learning Objective(s)
1. Identify the numerator and denominator of a fraction.
2. Represent a fraction as part of a whole or part of a set.

Introduction

Many problems in mathematics deal with whole numbers, which are used to count whole units of things. For example, you can count students in a classroom and the number of dollar bills. You need other kinds of numbers to describe units that are not whole. For example, an aquarium might be partly full. A group may have a meeting, but only some of the members are present.

Fractions are numbers used to refer to a part of a whole. This includes measurements that cannot be written as whole numbers. For example, the width of a piece of notebook paper is more than 8 inches but less than 9 inches. The part longer than 8 inches is written as a fraction. Here, you will investigate how fractions can be written and used to represent quantities that are parts of the whole.

Identifying Numerators and Denominators

A whole can be divided into parts of equal size. In the example below, a rectangle has been divided into eight equal squares. Four of these eight squares are shaded.

![Diagram of a rectangle divided into eight equal squares, with four of them shaded.]

The shaded area can be represented by a fraction. A fraction is written vertically as two numbers with a line between them. The denominator (the bottom number) represents the number of equal parts that make up the whole. The numerator (the top number) describes the number of parts that you are describing. So returning to the example above, the rectangle has been divided into 8 equal parts, and 4 of them have been shaded. You can use the fraction $\frac{4}{8}$ to describe the shaded part of the whole.

$\frac{4}{8}$ ← The numerator tells how many parts are shaded.

$\frac{8}{8}$ ← The denominator tells how many parts are required to make up the whole.
Parts of a Set

The rectangle model above provides a good, basic introduction to fractions. However, what do you do with situations that cannot be as easily modeled by shading part of a figure? For example, think about the following situation:

Marc works as a Quality Assurance Manager at an automotive plant. Every hour he inspects 10 cars; \( \frac{4}{5} \) of those pass inspection.

In this case, 10 cars make up the whole group. Each car can be represented as a circle, as shown below.

To show \( \frac{4}{5} \) of the whole group, you first need to divide the whole group into 5 equal parts. (You know this because the fraction has a denominator of 5.)

To show \( \frac{4}{5} \), circle 4 of the equal parts.

Here is another example. Imagine that Aneesh is putting together a puzzle made of 12 pieces. At the beginning, none of the pieces have been put into the puzzle. This means that \( \frac{0}{12} \) of the puzzle is complete. Aneesh then puts four pieces together. The puzzle is \( \frac{4}{12} \) complete. Soon, he adds four more pieces. Eight out of twelve pieces are now connected. This fraction can be written as \( \frac{8}{12} \). Finally, Aneesh adds four more pieces. The puzzle is whole, using all 12 pieces. The fraction can be written as \( \frac{12}{12} \).
Note that the number in the denominator cannot be zero. The denominator tells how many parts make up the whole. So if this number is 0, then there are no parts and therefore there can be no whole.

The numerator can be zero, as it tells how many parts you are describing. Notice that in the puzzle example above, you can use the fraction \( \frac{0}{12} \) to represent the state of the puzzle when 0 pieces have been placed.

Fractions can also be used to analyze data. In the data table below, 3 out of 5 tosses of a coin came up heads, and 2 out of five tosses came up tails. Out of the total number of coin tosses, the portion that was heads can be written as \( \frac{3}{5} \). The portion that was tails can be written as \( \frac{2}{5} \).

<table>
<thead>
<tr>
<th>Coin Toss</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Heads</td>
</tr>
<tr>
<td>2</td>
<td>Tails</td>
</tr>
<tr>
<td>3</td>
<td>Heads</td>
</tr>
<tr>
<td>4</td>
<td>Heads</td>
</tr>
<tr>
<td>5</td>
<td>Tails</td>
</tr>
</tbody>
</table>

**Self Check A**
Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

**Parts of a Whole**

The “parts of a whole” concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now, \( \frac{1}{4} \) of the pizza is gone and \( \frac{3}{4} \) remains. Note that both of these fractions have a denominator of 4, which refers to the number of slices the whole pizza has been cut into.
### Example

**Problem**  
Joaquim bakes a blueberry pie for a potluck dinner. The total pie is cut into 6 equal slices. After everybody eats dessert, only one slice of the pie remains. What fraction of the pie remains?


dfrac{?}{6}  
The pie was cut into six equal slices, so the denominator of the fraction will be 6.

dfrac{1}{6}  
Only 1 slice remains, so the numerator of the fraction will be 1.

**Answer**  
\dfrac{1}{6} of the pie remains.

---

### Example

**Problem**  
Write a fraction to represent the portion of the octagon that is not shaded.

![Octagon](image)

\dfrac{?}{8}  
The octagon has eight equal sections, so the denominator of the fraction will be 8.

\dfrac{5}{8}  
Five sections are not shaded, so the numerator of the fraction will be 5.

**Answer**  
\dfrac{5}{8} of the octagon is not shaded.
Measurement Contexts

You can use a fraction to represent the quantity in a container. This measuring cup is \( \frac{3}{4} \) filled with a liquid. Note that if the cup were \( \frac{4}{4} \) full, it would be a whole cup.

You can also use fractions in measuring the length, width, or height of something that is not a full unit. Using a 12-inch ruler, you measure a shell that is 6 inches long. You know that 12 inches equals one foot. So, the length of this shell is \( \frac{6}{12} \) of a foot; the 12-inch ruler is the "whole", and the length of the shell is the "part."

Self Check B

Which fraction represents the portion of the shape that is shaded?
Summary

Fractions are used to represent parts of a whole. You can use fractions when describing substances, quantities, or diagrams that are not complete. You also use fractions to describe numbers of people or objects that do not make up a complete group. Fractions are written with a numerator and denominator. The numerator (above the fraction bar) tells the number of parts being described, and the denominator (below the fraction bar) tells the number of parts that make up the whole.

2.1.1 Self Check Solutions

**Self Check A**

Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

The entire race is 10 miles long, and Daphne will run 5 miles. This means she will run \( \frac{5}{10} \) of the race.

**Self Check B**

Which fraction represents the portion of the shape that is shaded?

\( \frac{7}{8} \)

The total number of parts that make up the whole, 8, is the denominator (below the fraction bar). The number of parts that are shaded, 7, is the numerator (above the fraction bar).
Mathematicians use three categories to describe fractions: proper, improper, and mixed. Fractions that are greater than 0 but less than 1 are called **proper fractions**. In proper fractions, the **numerator** is less than the **denominator**. When a fraction has a numerator that is greater than or equal to the denominator, the fraction is an **improper fraction**. An improper fraction is always 1 or greater than 1. And, finally, a **mixed number** is a combination of a whole number and a proper fraction.

**Identify Proper and Improper Fractions**

In a proper fraction, the numerator is always less than the denominator. Examples of proper fractions include \(\frac{1}{2}, \frac{9}{13}\) and \(\frac{1}{1001}\).

In an improper fraction, the numerator is always greater than or equal to the denominator. Examples of improper fractions include \(\frac{5}{2}, \frac{9}{9}\) and \(\frac{25}{20}\).

**Self Check A**

Identify \(\frac{5}{3}\) as a proper or improper fraction.

**Changing Improper Fractions to Mixed Numbers**

An improper fraction can also be written as a **mixed number**. Mixed numbers contain both a whole number and a proper fraction. Examples of mixed numbers include \(8 \frac{1}{10}, 1 \frac{19}{20}\) and \(2 \frac{1}{2}\).

Let’s look at a quick example.
Below are three whole pizzas that are each cut into four pieces. A fourth pizza is there as well, but someone has taken one piece, leaving only three pieces.

You can use fractions to compare the number of pieces you have to the number of pieces that make up a whole. In this picture, the denominator is the total number of pieces that make up one whole pizza, which is 4. The total number of all pieces of pizza, which is 15, represents the numerator.

You can use the improper fraction $\frac{15}{4}$ to represent the total amount of pizza here. Think: “Each whole pizza is cut into 4 equal pieces, and there are 15 pieces total. So, the total amount of whole pizzas is $\frac{15}{4}$.”

As you looked at the image of the pizzas, however, you probably noticed right away that there were 3 full pizzas and one pizza with a piece missing. While you can use the improper fraction $\frac{15}{4}$ to represent the total amount of pizza, it makes more sense here to use a mixed number – a fraction that includes both a whole number and a fractional part. For this pizza scenario, you can use the fraction $3\frac{3}{4}$.

The mixed number $3\frac{3}{4}$ can be easier to understand than the improper fraction $\frac{15}{4}$. However, both forms are legitimate ways to represent the number of pizzas.

Rewriting an improper fraction as a mixed number can be helpful, because it helps you see more easily about how many whole items you have.

Let’s look again at the pizzas above.
The improper fraction $\frac{15}{4}$ means there are 15 total pieces, and 4 pieces makes a whole pizza. If you didn’t have the picture, you could change $\frac{15}{4}$ into a mixed fraction by determining:

- How many groups of 4 pieces are there in 15 pieces? Since $15 \div 4 = 3$ with a remainder, there are 3 whole pizzas.
- What is the remainder? The remainder is 3. So, there are 3 pieces of the last pizza left, out of the 4 that would make a whole pizza. So, $\frac{3}{4}$ of a pizza is left.

Now, put the number of whole pizzas with the fraction of a pizza that is left over. The mixed number is $3 \frac{3}{4}$.

### Writing Improper Fractions as Mixed Numbers

Step 1: Divide the denominator into the numerator.

Step 2: The quotient is the whole number part of the mixed number.

Step 3: The remainder is the numerator of the fractional part of the mixed number.

Step 4: The divisor is the denominator of the fractional part of the mixed number.

#### Example

Problem

Write the improper fraction $\frac{47}{7}$ as a mixed number.

$47 \div 7 = 6$, remainder 5

Divide the denominator into the numerator.

The quotient, 6, becomes the whole number.

The remainder, 5, becomes the numerator.

The denominator, which is also used as the divisor, remains as 7.

Answer

$\frac{47}{7} = 6 \frac{5}{7}$
Changing Mixed Numbers to Improper Fractions

Mixed numbers can also be changed to improper fractions. This is sometimes helpful when doing calculations with mixed numbers, especially multiplication.

Let’s start by considering the idea of one whole as an improper fraction. If you divide a cake into five equal slices, and keep all the slices, the one whole cake is equal to the 5 slices. So, 1 cake is the same as \( \frac{5}{5} \) cake.

\[
\begin{array}{c}
5 \\
5
\end{array}
\quad \text{The number of parts present…}
\]

\[
\begin{array}{c}
\frac{5}{5} \\
\end{array}
\quad \text{… is equal to the number of parts that make up the whole.}
\]

Had you cut the cake into 4 pieces or 3 pieces, as shown below, you could have used the fractions \( \frac{4}{4} \) or \( \frac{3}{3} \) to represent the whole cake. The fractions may change depending on the number of cuts you make to the cake, but you are still dealing with only one cake.

Let’s explore how to write a simple mixed number, \( 2\frac{1}{3} \), as an improper fraction. The mixed number is represented below. Each full circle represents one whole.
To write an improper fraction, you need to know how many equal sized pieces make one whole. You also need to know how many of those pieces you have. Since you have $\frac{1}{3}$, you should divide up all of the circles into 3 pieces.

Each whole circle has 3 pieces. You can multiply the number of whole circles, 2, by 3 to find how many one-third pieces are in the two whole circles. Then you add 1 for the one-third piece in the final, incomplete circle. As you can see from the diagram, there are 7 individual one-third pieces. The improper fraction for $2\frac{1}{3}$ is $\frac{7}{3}$.

**Writing Mixed Numbers as Improper Fractions**

**Step 1.** Multiply the denominator of the fraction by the whole number.

**Step 2.** Add this product to the numerator of the fraction.

**Step 3.** The sum is the numerator of the improper fraction.

**Step 4.** The denominator of the improper fraction is the same as the denominator of the fractional part of the mixed number.

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**Example**

*Problem*  
Write $\frac{3}{4}$ as an improper fraction.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Multiply the denominator by the whole number.</th>
<th>Add this result to the numerator of the fraction.</th>
<th>This answer becomes the numerator of the improper fraction.</th>
<th>Notice that the denominator of the improper fraction is the same as the denominator that was in the fractional part of the mixed number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{4}{3}$</td>
<td>$4 \times 4 = 16$</td>
<td>$16 + 3 = 19$</td>
<td>$\frac{19}{4}$</td>
<td></td>
</tr>
</tbody>
</table>

*Answer:* $\frac{4}{3} = \frac{19}{4}$
**Self Check C**

Change \( \frac{5}{6} \) from a mixed number to an improper fraction.

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**Summary**

A fraction can be identified as proper or improper by comparing the numerator and the denominator. Fractions that are less than one are known as proper fractions, and the numerator (the top number) is less than the denominator (the bottom number). A fraction with a numerator that is greater than or equal to the denominator is known as an improper fraction. It represents a number greater than or equal to one. Numbers that are not whole numbers, but are greater than one, can be written as improper fractions or mixed numbers. A mixed number has a whole number part and a fraction part.

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**2.1.2 Self Check Solutions**

**Self Check A**

Identify \( \frac{5}{3} \) as a proper or improper fraction.

The fraction is greater than 1, and the numerator is greater than the denominator, so \( \frac{5}{3} \) is an improper fraction.

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**Self Check B**

Change \( \frac{12}{5} \) from an improper fraction to a mixed number.

The improper fraction \( \frac{12}{5} \) can be thought of as \( 12 \div 5 = 2 \), with a remainder of 2. So, \( 2 \frac{2}{5} \) is the correct answer.

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**Self Check C**

Change \( 3 \frac{5}{6} \) from a mixed number to an improper fraction.

\[ 3 \frac{5}{6} = (6 \cdot 3) + 5 = 23 \] The denominator stays the same, so \( \frac{23}{6} \) is the improper form.
2.2.1 Factors and Primes

Learning Objective(s)
1. Recognize (by using the divisibility rule) if a number is divisible by 2, 3, 4, 5, 6, 9, or 10.
2. Find the factors of a number.
3. Determine whether a number is prime, composite, or neither.
4. Find the prime factorization of a number.

Introduction

Natural numbers, also called counting numbers (1, 2, 3, and so on), can be expressed as a product of their factors. When working with a fraction, you often need to make the fraction as simple as possible. This means that the numerator and the denominator have no common factors other than 1. It will help to find factors, so that later you can simplify and compare fractions.

Tests of Divisibility

When a natural number is expressed as a product of two other natural numbers, those other numbers are factors of the original number. For example, two factors of 12 are 3 and 4, because $3 \times 4 = 12$.

When one number can be divided by another number with no remainder, we say the first number is divisible by the other number. For example, 20 is divisible by 4 ($20 \div 4 = 5$). If a number is divisible by another number, it is also a multiple of that number. For example, 20 is divisible by 4, so 20 is a multiple of 4.

Divisibility tests are rules that let you quickly tell if one number is divisible by another. There are many divisibility tests. Here are some of the most useful and easy to remember:

- A number is divisible by 2 if the last (ones) digit is divisible by 2. That is, the last digit is 0, 2, 4, 6, or 8. (We then say the number is an even number.) For example, in the number 236, the last digit is 6. Since 6 is divisible by 2 ($6 \div 2 = 3$), 236 is divisible by 2.

- A number is divisible by 3 if the sum of all the digits is divisible by 3. For example, the sum of the digits of 411 is $4 + 1 + 1 = 6$. Since 6 is divisible by 3 ($6 \div 3 = 2$), 411 is divisible by 3.

- A number is divisible by 5 if the last digit is 0 or 5. For example, 275 and 1,340 are divisible by 5 because the last digits are 5 and 0.

- A number is divisible by 10 if the last digit is 0. For example, 520 is divisible by 10 (last digit is 0).
Other useful divisibility tests:

**4:** A number is divisible by 4 if the last two digits are divisible by 4.

**6:** A number is divisible by 6 if it is divisible by both 2 and 3.

**9:** A number is divisible by 9 if the sum of its digits is divisible by 9.

Here is a summary of the most commonly used divisibility rules.

<table>
<thead>
<tr>
<th>A number is divisible by</th>
<th>Condition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>The last digit is even (0, 2, 4, 6, 8).</td>
<td>426 yes 273 no</td>
</tr>
<tr>
<td>3</td>
<td>The sum of the digits is divisible by 3.</td>
<td>642 yes (6 + 4 + 2 = 12, 12 is divisible by 3) 721 no (7 + 2 + 1 = 10, 10 is not divisible by 3)</td>
</tr>
<tr>
<td>4</td>
<td>The last two digits form a number that is divisible by 4.</td>
<td>164 yes (64 is divisible by 4) 135 no (35 is not divisible by 4)</td>
</tr>
<tr>
<td>5</td>
<td>The last digit is 0 or 5.</td>
<td>685 yes 432 no</td>
</tr>
<tr>
<td>6</td>
<td>The number is divisible by 2 and 3.</td>
<td>324 yes (it is even and 3 + 2 + 4 = 9) 411 no (although divisible by 3, it is not even)</td>
</tr>
<tr>
<td>9</td>
<td>The sum of the digits is divisible by 9.</td>
<td>279 yes (2 + 7 + 9 = 18) 512 no (5 + 1 + 2 = 8)</td>
</tr>
<tr>
<td>10</td>
<td>The last digit is a 0.</td>
<td>620 yes 238 no</td>
</tr>
</tbody>
</table>

If you need to check for divisibility of a number without a rule, divide (either using a calculator or by hand). If the result is a number without any fractional part or remainder, then the number is divisible by the divisor. If you forget a rule, you can also use this strategy.

**Self Check A**
Determine whether 522 is divisible by 2, 3, 4, 5, 6, 9, or 10.
Factoring Numbers

To find all the factors of a number, you need to find all numbers that can divide into the original number without a remainder. The divisibility rules from above will be extremely useful!

Suppose you need to find the factors of 30. Since 30 is a number you are familiar with, and small enough, you should know many of the factors without applying any rules. You can start by listing the factors as they come to mind:

2 • 15
3 • 10
5 • 6

Is that it? Not quite. All natural numbers except 1 also have 1 and the number itself as factors:

1 • 30

The factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30.

When you find one factor of a number, you can easily find another factor—it is the quotient using that first factor as the divisor. For example, once you know 2 is a factor of 30, then 30 ÷ 2 is another factor. A pair of factors whose product is a given number is a factor pair of the original number. So, 2 and 15 are a factor pair for 30.

What do you do if you need to factor a greater number and you can’t easily see its factors? That’s where the divisibility rules will come in quite handy. Here is a general set of steps that you may follow:

1. Begin with 1 and check the numbers sequentially, using divisibility rules or division.
2. When you find a factor, find the other number in the factor pair.
3. Keep checking sequentially, until you reach the second number in the last factor pair you found, or until the result of dividing gives a number less than the divisor.

Note that you can stop checking when the result of dividing is less than the number you’re checking. This means that you have already found all factor pairs, and continuing the process would find pairs that have been previously found.
## Example

<table>
<thead>
<tr>
<th>Problem</th>
<th><strong>Find factors of 165.</strong></th>
<th>Factors</th>
<th>Explanation</th>
<th>Divisible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>divisible by 1?</td>
<td>1 • 165 = 165</td>
<td>All numbers are divisible by 1.</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>divisible by 2?</td>
<td>The last digit, 5, is not even, so 165 is not divisible by 2.</td>
<td>No</td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 3?</td>
<td>1 + 6 + 5 = 12, which is divisible by 3, so 165 is divisible by 3.</td>
<td>Yes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>165 ÷ 3 = 55</td>
<td>3 • 55 = 165</td>
<td>Use division to find the other factor.</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>divisible by 4?</td>
<td>Since 165 is not an even number, it will not be divisible by any even number. The divisibility test for 4 also applies: 65 is not divisible by 4, so 165 is not divisible by 4.</td>
<td>No</td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 5?</td>
<td>Since the last digit is 5, 165 is divisible by 5.</td>
<td>Yes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>165 ÷ 5 = 33</td>
<td>5 • 33 = 165</td>
<td>Use division to find the other factor.</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>divisible by 6?</td>
<td>Since 165 is not divisible by 2, it is not divisible by 6.</td>
<td>No</td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 7?</td>
<td>There is no divisibility test for 7, so you have to divide. 165 ÷ 7 is not a whole number, so it is not divisible by 7.</td>
<td>No</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question</td>
<td>Answer</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------------------------</td>
<td>--------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 8?</td>
<td>No</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 9?</td>
<td>No</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 10?</td>
<td>No</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 11?</td>
<td>Yes</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 12?</td>
<td>No</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>divisible by 13?</td>
<td>No</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Done checking numbers

Since the result of 165 ÷ 13 is less than 13, you can stop. Any factor greater than 13 would already have been found as the pair of a factor less than 13.

Answer  The factors of 165 are 1, 3, 5, 11, 15, 33, 55, 165.
If a number has exactly two factors, 1 and itself, the number is a **prime number**. A number that has more factors than itself and 1 is called a **composite number**. The number 1 is considered neither prime nor composite, as its only factor is 1. To determine whether a number is prime, composite, or neither, check factors. Here are some examples.

<table>
<thead>
<tr>
<th>Number</th>
<th>Composite, Prime, or Neither?</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Neither</td>
<td>1 does not have two different factors, so it is not prime.</td>
</tr>
<tr>
<td>2</td>
<td>Prime</td>
<td>2 has only the factors 2 and 1.</td>
</tr>
<tr>
<td>3</td>
<td>Prime</td>
<td>3 has only the factors 3 and 1.</td>
</tr>
<tr>
<td>4</td>
<td>Composite</td>
<td>4 has more than two factors: 1, 2, and 4, so it is composite.</td>
</tr>
<tr>
<td>5, 7, 11, 13</td>
<td>Prime</td>
<td>Each number has only two factors: 1 and itself.</td>
</tr>
<tr>
<td>6, 8, 9, 10, 50, 63</td>
<td>Composite</td>
<td>Each number has more than two factors.</td>
</tr>
</tbody>
</table>

**Example**

**Problem**  Find all the factors of 48.

**Answer:** 1, 2, 3, 4, 6, 8, 12, 16, 24, 48

**Prime Factorization**

A composite number written as a product of only prime numbers is called the **prime factorization** of the number. One way to find the prime factorization of a number is to begin with the prime numbers 2, 3, 5, 7, 11 and so on, and determine whether the number is divisible by the primes.

For example, if you want to find the prime factorization of 20, start by checking if 20 is divisible by 2. Yes, $2 \times 10 = 20$.

Then factor 10, which is also divisible by 2 ($2 \times 5 = 10$).

Both of those factors are prime, so you can stop. The prime factorization of 20 is $2 \times 2 \times 5$, which you can write using **exponential notation** as $2^2 \times 5$.

One way to find the prime factorization of a number is to use successive divisions.
Divide 20 by 2 to get 10. 2 is being used because it is a prime number and a factor of 20. You could also have started with 5.

Then divide 10 by 2 to get 5.

Multiplying these divisors forms the prime factorization of 20.

To help you organize the factoring process, you can create a factor tree. This is a diagram that shows a factor pair for a composite number. Then, each factor that isn’t prime is also shown as a factor pair. You can continue showing factor pairs for composite factors, until you have only prime factors. When a prime number is found as a factor, circle it so you can find it more easily later.

Written using exponential notation, the prime factorization of 20 is again $2^2 \cdot 5$.

Notice that you don’t have to start checking the number using divisibility of prime numbers. You can factor 20 to $4 \cdot 5$, and then factor 4 to $2 \cdot 2$, giving the same prime factorization: $2 \cdot 2 \cdot 5$.

Now look at a more complicated factorization.

Notice that there are two different trees, but they both produce the same result: five 2s and one 3. Every number will only have one, unique prime factorization. You can use any sets of factor pairs you wish, as long as you keep factoring composite numbers.

When you rewrite the prime factorization of 96 ($2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$) in exponential notation, the five 2s can be written as $2^5$. So, $96 = 2^5 \cdot 3$. 

2.19
Self Check B
When finding the prime factorization of 72, Marie began a tree diagram using the two factors 9 and 8. Which of the following statements are true?

1. Marie started the diagram incorrectly and should have started the tree diagram using the factors 2 and 36.
2. Marie’s next set of factor pairs could be 3, 3 and 2, 4.
3. Marie’s next set of factor pairs could be 3, 3 and 9, 8.
4. Marie didn’t have to use a tree diagram.

2, 3, 6, and 9
Marie’s next set of factor pairs could read 3, 3, and 2, 4, as 3 \cdot 3 is a factorization of 9 and 2 \cdot 4 is a factorization of 8. Marie could also find the prime factorization by using successive divisions.

Summary
Finding the factors of a natural number means that you find all the possible numbers that will divide into the given number without a remainder. There are many rules of divisibility to help you to find factors more quickly. A prime number is a number that has exactly two factors. A composite number is a number that has more than two factors. The prime factorization of a number is the product of the number’s prime factors.

2.2.1 Self Check Solutions

Self Check A
Determine whether 522 is divisible by 2, 3, 4, 5, 6, 9, or 10.

A) 2 and 3 only
B) 4 only
C) 2, 3, 6, and 9 only
D) 4, 5, and 10 only

2, 3, 6, and 9
522 is divisible by 2 (the last digit is even) and 3 (5 + 2 + 2 = 9, which is a multiple of 3). Since it is divisible by 2 and 3, it is also divisible by 6. Also, the sum of the digits is divisible by 9, so 522 is divisible by 9. Since the last digit is not 0 or 5, 522 is not divisible by 5 or 10. The number formed by the last two digits, 22, is not divisible by 4, so 522 is not divisible by 4.
2.2.2 Simplifying Fractions

Learning Objective(s)
1. Find an equivalent fraction with a given denominator.
2. Simplify a fraction to lowest terms.

Introduction

Fractions are used to represent a part of a whole. Fractions that represent the same part of a whole are called equivalent fractions. Factoring, multiplication, and division are all helpful tools for working with equivalent fractions.

Equivalent Fractions

We use equivalent fractions every day. Fifty cents can be 2 quarters, and we have \( \frac{2}{4} \) of a dollar, because there are 4 quarters in a dollar. Fifty cents is also 50 pennies out of 100 pennies, or \( \frac{50}{100} \) of a dollar. Both of these fractions are the same amount of money, but written with a different numerator and denominator.

Think about a box of crackers that contains 3 packets of crackers. Two of these packets are \( \frac{2}{3} \) of the box. Suppose each packet has 30 crackers in it. Two packets are also 60 \((30 \times 2)\) crackers out of 90 \((30 \times 3)\) crackers. This is \( \frac{60}{90} \) of the box. The fractions \( \frac{2}{3} \) and \( \frac{60}{90} \) both represent two packets of crackers, so they are equivalent fractions.

Equivalent fractions represent the same part of a whole, even if the numerator and denominator are different. For example, \( \frac{1}{4} = \frac{5}{20} \). In these diagrams, both fractions represent one of four rows in the rectangle.
Since \( \frac{1}{4} \) and \( \frac{5}{20} \) are naming the same part of a whole, they are equivalent.

There are many ways to name the same part of a whole using equivalent fractions.

Let’s look at an example where you need to find an equivalent fraction.

### Example

**Problem**
John is making cookies for a bake sale. He made 20 large cookies, but he wants to give away only \( \frac{3}{4} \) of them for the bake sale. What fraction of the cookies does he give away, using 20 as the denominator?

**Answer**
He gives away \( \frac{15}{20} \) of the cookies.

When you regroup and reconsider the parts and whole, you are multiplying the numerator and denominator by the same number. In the above example, you multiply 4 by 5 to get the needed denominator of 20, so you also need to multiply the numerator 3 by 5, giving the new numerator of 15.
Finding Equivalent Fractions

To find equivalent fractions, multiply or divide both the numerator and the denominator by the same number.

Examples:

\[
\frac{20}{25} = \frac{20 \div 5}{25 \div 5} = \frac{4}{5}
\]

\[
\frac{2}{7} = \frac{2 \cdot 6}{7 \cdot 6} = \frac{12}{42}
\]

Self Check A
Write an equivalent fraction to \(\frac{2}{3}\) that has a denominator of 27.

Simplifying Fractions

A fraction is in its simplest form, or lowest terms, when it has the least numerator and the least denominator possible for naming this part of a whole. The numerator and denominator have no common factor other than 1.

Here are 10 blocks, 4 of which are green. So, the fraction that is green is \(\frac{4}{10}\). To simplify, you find a common factor and then regroup the blocks by that factor.
**Example**

**Problem**

Simplify \( \frac{4}{10} \).

We start with 4 green blocks out of 10 total blocks.

The fraction can be simplified by finding a common factor. In this case, we can group the blocks in twos, since 2 is a common factor. You have 2 groups of green blocks and a total of 5 groups, each group containing 2 blocks.

\[
\frac{\text{green blocks}}{\text{blocks}} = \frac{2(2)}{5(2)} = \frac{4}{10} = \frac{2}{5}
\]

Now, consider the groups as the part and you have 2 green groups out of 5 total groups.

\[
\frac{\text{green blocks}}{\text{blocks}} = \frac{2(2)}{5(2)} = \frac{2}{5}
\]

**Answer**

\[
\frac{4}{10} = \frac{2}{5}
\]

The simplified fraction is \( \frac{2}{5} \).

Once you have determined a common factor, you can divide the blocks into the groups by dividing both the numerator and denominator to determine the number of groups that you have.
For example, to simplify \( \frac{6}{9} \) you find a common factor of 3, which will divide evenly into both 6 and 9. So, you divide 6 and 9 into groups of 3 to determine how many groups of 3 they contain. This gives \( \frac{6 \div 3}{9 \div 3} = \frac{2}{3} \), which means 2 out of 3 groups, and \( \frac{2}{3} \) is equivalent to \( \frac{6}{9} \).

It may be necessary to group more than one time. Each time, determine a common factor for the numerator and denominator using the tests of divisibility, when possible. If both numbers are even numbers, start with 2. For example:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Simplify ( \frac{32}{48} ).</th>
</tr>
</thead>
</table>
|         | \[
|         | \frac{32}{48} = \frac{32 \div 2}{48 \div 2} = \frac{16}{24} \]
|         | \[
|         | \frac{16}{24} = \frac{16 \div 2}{24 \div 2} = \frac{8}{12} \]
|         | \[
|         | \frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3} \]
| Answer  | \[
|         | \frac{32}{48} = \frac{2}{3} \]

\( \frac{2}{3} \) is the simplified fraction equivalent to \( \frac{32}{48} \).

In the example above, 16 is a factor of both 32 and 48, so you could have shortened the solution.

\[
\frac{32}{48} = \frac{2 \cdot 16}{3 \cdot 16} = \frac{2}{3}
\]
You can also use prime factorization to help regroup the numerator and denominator.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
</tbody>
</table>

\[
\frac{54}{72} = \frac{2\cdot3\cdot3\cdot3}{2\cdot2\cdot3\cdot3}
\]

The prime factorization of 54 is $2\cdot3\cdot3\cdot3$. The prime factorization of 72 is $2\cdot2\cdot3\cdot3$.

\[
\frac{3\cdot(2\cdot3\cdot3)}{2\cdot2\cdot(2\cdot3\cdot3)}
\]

Rewrite, finding common factors.

\[
\frac{3}{2\cdot2} \cdot \frac{2\cdot3\cdot3}{2\cdot3\cdot3} = 1
\]

Multiply: $2\cdot2$.

\[
\frac{3}{4}
\]

**Answer**

\[
\frac{54}{72} = \frac{3}{4}
\]

$\frac{3}{4}$ is the simplified fraction equivalent to $\frac{54}{72}$.

Notice that when you simplify a fraction, you divide the numerator and denominator by the same number, in the same way you multiply by the same number to find an equivalent fraction with a greater denominator. In the example above, you could have divided the numerator and denominator by 9, a common factor of 54 and 72.

\[
\frac{54 \div 9}{72 \div 9} = \frac{6}{8}
\]

Since the numerator (6) and the denominator (8) still have a common factor, the fraction is not yet in lowest terms. So, again divide by the common factor 2.

\[
\frac{6 \div 2}{8 \div 2} = \frac{3}{4}
\]

Repeat this process of dividing by a common factor until the only common factor is 1.

**Simplifying Fractions to Lowest Terms**

To simplify a fraction to lowest terms, divide both the numerator and the denominator by their common factors. Repeat as needed until the only common factor is 1.
Summary

Multiplication of binomials and polynomials requires use of the distributive property and integer operations. Whether the polynomials are monomials, binomials, or trinomials, carefully multiply each term in one polynomial by each term in the other polynomial. Be careful to watch the addition and subtraction signs and negative coefficients. A product is written in simplified if all of its like terms have been combined.

2.2.2 Self Check Solutions

Self Check A

Write an equivalent fraction to \( \frac{2}{3} \) that has a denominator of 27.

\[
\frac{18}{27}
\]

The multiplying factor is 9, so the denominator is \( 3 \times 9 = 27 \) and the numerator is \( 2 \times 9 = 18 \).

Self Check B

Simplify \( \frac{36}{72} \).

\[
\frac{36}{72} = \frac{36 \div 36}{72 \div 36} = \frac{1}{2}
\]

This is in lowest terms since 1 is the only common factor of 1 and 2.
**2.2.3 Comparing Fractions**

**Learning Objective(s)**
1. Determine whether two fractions are equivalent.
2. Use > or < to compare fractions.

**Introduction**

You often need to know when one fraction is greater or less than another fraction. Since a fraction is a part of a whole, to find the greater fraction you need to find the fraction that contains more of the whole. If the two fractions simplify to fractions with a common denominator, you can then compare numerators. If the denominators are different, you can find a common denominator first and then compare the numerators.

**Determining Equivalent Fractions**

Two fractions are equivalent fractions when they represent the same part of a whole. Since equivalent fractions do not always have the same numerator and denominator, one way to determine if two fractions are equivalent is to find a common denominator and rewrite each fraction with that denominator. Once the two fractions have the same denominator, you can check to see if the numerators are equal. If they are equal, then the two fractions are equal as well.

One way to find a common denominator is to check to see if one denominator is a factor of the other denominator. If so, the greater denominator can be used as the common denominator.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Are $\frac{2}{6}$ and $\frac{8}{18}$ equivalent fractions?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Does $\frac{2}{6} = \frac{8}{18}$?</td>
<td>To solve this problem, find a common denominator for the two fractions. This will help you compare the two fractions. Since 6 is a factor of 18, you can write both fractions with 18 as the denominator.</td>
</tr>
<tr>
<td>$\frac{2\cdot3}{6\cdot3} = \frac{6}{18}$</td>
<td>Start with the fraction $\frac{2}{6}$. Multiply the denominator, 6, by 3 to get a new denominator of 18. Since you multiply the denominator by 3, you must also multiply the numerator by 3.</td>
</tr>
</tbody>
</table>
The fraction \( \frac{8}{18} \) already has a denominator of 18, so you can leave it as is.

\[ \frac{6}{18} \text{ does not equal } \frac{8}{18} \]

Compare the fractions. Now that both fractions have the same denominator, 18, you can compare numerators.

**Answer** \( \frac{2}{6} \) and \( \frac{8}{18} \) are not equivalent fractions.

When one denominator is not a factor of the other denominator, you can find a common denominator by multiplying the denominators together.

---

**Example**

**Problem**

Determine whether \( \frac{3}{6} \) and \( \frac{5}{10} \) are equivalent fractions.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
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</tr>
<tr>
<td><strong>Problem</strong></td>
<td><strong>Determine whether</strong> ( \frac{3}{6} ) <strong>and</strong> ( \frac{5}{10} ) <strong>are equivalent fractions.</strong></td>
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<tr>
<td><strong>Example</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Problem</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Solution</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 6 \times 10 = 60 )</td>
<td>Use 60 as a common denominator.</td>
<td></td>
</tr>
<tr>
<td>( \frac{3}{6} = \frac{3 \times 10}{6 \times 10} = \frac{30}{60} )</td>
<td>Multiply the numerator and denominator of ( \frac{3}{6} ) by 10 to get 60 in the denominator.</td>
<td></td>
</tr>
<tr>
<td>( \frac{5}{10} = \frac{5 \times 6}{10 \times 6} = \frac{30}{60} )</td>
<td>Multiply numerator and denominator of ( \frac{5}{10} ) by 6.</td>
<td></td>
</tr>
<tr>
<td>( 30 \div 60 = \frac{30}{60} )</td>
<td>Now that the denominators are the same, compare the numerators.</td>
<td></td>
</tr>
<tr>
<td><strong>Answer</strong></td>
<td>Yes, ( \frac{3}{6} ) and ( \frac{5}{10} ) are equivalent fractions.</td>
<td></td>
</tr>
<tr>
<td><strong>Solution</strong></td>
<td>Since 30 is the value of the numerator for both fractions, the two fractions are equal.</td>
<td></td>
</tr>
</tbody>
</table>

Notice in the above example you can use 30 as the least common denominator since both 6 and 10 are factors of 30. Any common denominator will work.
In some cases you can simplify one or both of the fractions, which can result in a common denominator.

### Example

#### Problem
Determine whether \( \frac{2}{3} \) and \( \frac{40}{60} \) are equivalent fractions.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tbody>
</table>

\[
\frac{40}{60} = \frac{40 \div 10}{60 \div 10} = \frac{4}{6}
\]

Simplify \( \frac{40}{60} \). Divide the numerator and denominator by the common factor 10.

\[
\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}
\]

\( \frac{2}{3} \) is still not in lowest terms, so divide the numerator and the denominator again, this time by the common factor 2.

\[
\frac{2}{3} = \frac{2}{3}
\]

Compare the fractions. The numerators and denominators are the same.

**Answer** Yes, \( \frac{2}{3} \) and \( \frac{40}{60} \) are equivalent fractions.

Note: In the example above you could have used the common factor of 20 to simplify \( \frac{40}{60} \) directly to \( \frac{2}{3} \).

### Determining Equivalent Fractions

To determine whether or not two fractions are equivalent:

1. **Step 1:** Rewrite one or both of the fractions so that they have common denominators.
2. **Step 2:** Compare the numerators to see if they have the same value. If so, then the fractions are equivalent.
**Self Check A**
Which of the following fraction pairs are equivalent?

A) \( \frac{5}{7} \) and \( \frac{7}{5} \)

B) \( \frac{12}{30} \) and \( \frac{6}{10} \)

C) \( \frac{4}{20} \) and \( \frac{1}{5} \)

D) \( \frac{8}{11} \) and \( \frac{8}{22} \)

**Comparing Fractions Using < and >**

When given two or more fractions, it is often useful to know which fraction is greater than or less than the other. For example, if the discount in one store is \( \frac{1}{3} \) off the original price and the discount in another store is \( \frac{1}{4} \) off the original price, which store is offering a better deal? To answer this question, and others like it, you can compare fractions.

To determine which fraction is greater, you need to find a common denominator. You can then compare the fractions directly. Since 3 and 4 are both factors of 12, you will divide the whole into 12 parts, create equivalent fractions for \( \frac{1}{3} \) and \( \frac{1}{4} \), and then compare.
Now you see that $\frac{1}{3}$ contains 4 parts of 12, and $\frac{1}{4}$ contains 3 parts of 12. So, $\frac{1}{3}$ is greater than $\frac{1}{4}$.

As long as the denominators are the same, the fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction as it contains fewer parts of the whole.

Recall that the symbol $<$ means “less than”, and the symbol $>$ means “greater than”. These symbols are inequality symbols. So, the true statement $3 < 8$ is read as “3 is less than 8” and the statement $5 > 3$ is read as “5 is greater than 3”. One way to help you remember the distinction between the two symbols is to think that the smaller end of the symbol points to the lesser number.

As with comparing whole numbers, the inequality symbols are used to show when one fraction is “greater than” or “less than” another fraction.

**Comparing Fractions**

To compare two fractions:

1. **Step 1:** Compare denominators. If they are different, rewrite one or both fractions with a common denominator.
2. **Step 2:** Check the numerators. If the denominators are the same, then the fraction with the greater numerator is the greater fraction. The fraction with the lesser numerator is the lesser fraction. And, as noted above, if the numerators are equal, the fractions are equivalent.
Example

Problem

Use < or > to compare the two fractions $\frac{4}{5}$ and $\frac{14}{20}$.

| Is $\frac{4}{5} > \frac{14}{20}$, or is $\frac{4}{5} < \frac{14}{20}$? | You cannot compare the fractions directly because they have different denominators. You need to find a common denominator for the two fractions.
|---|---|
| $\frac{4}{5} = \frac{?}{20}$ | Since 5 is a factor of 20, you can use 20 as the common denominator.
| $\frac{4 \cdot 4}{5 \cdot 4} = \frac{16}{20}$ | Multiply the numerator and denominator by 4 to create an equivalent fraction with a denominator of 20.
| $\frac{16}{20} > \frac{14}{20}$ | Compare the two fractions. $\frac{16}{20}$ is greater than $\frac{14}{20}$.

Answer

$\frac{4}{5} > \frac{14}{20}$

If $\frac{16}{20} > \frac{14}{20}$, then $\frac{4}{5} > \frac{14}{20}$, since $\frac{4}{5} = \frac{16}{20}$.

Self Check B

Which of the following is a true statement?

A) $\frac{5}{6} < \frac{24}{30}$

B) $\frac{25}{100} > \frac{9}{12}$

C) $\frac{4}{16} > \frac{1}{3}$

D) $\frac{3}{8} < \frac{20}{40}$
Summary

You can compare two fractions with like denominators by comparing their numerators. The fraction with the greater numerator is the greater fraction, as it contains more parts of the whole. The fraction with the lesser numerator is the lesser fraction as it contains fewer parts of the whole. If two fractions have the same denominator, then equal numerators indicate equivalent fractions.

2.2.3 Self Check Solutions

Self Check A
Which of the following fraction pairs are equivalent?

A) \( \frac{5}{7} \) and \( \frac{7}{5} \)  
B) \( \frac{12}{30} \) and \( \frac{6}{10} \)  
C) \( \frac{4}{20} \) and \( \frac{1}{5} \)  
D) \( \frac{8}{11} \) and \( \frac{8}{22} \)

\( \frac{4}{20} \) and \( \frac{1}{5} \)

Take the fraction \( \frac{1}{5} \) and multiply both the numerator and denominator by 4. You are left with the fraction \( \frac{4}{20} \). This means that the two fractions are equivalent.

Self Check B
Which of the following is a true statement?

A) \( \frac{5}{6} < \frac{24}{30} \)  
B) \( \frac{25}{100} > \frac{9}{12} \)  
C) \( \frac{4}{16} > \frac{1}{3} \)  
D) \( \frac{3}{8} < \frac{20}{40} \)

\( \frac{3}{8} < \frac{20}{40} \)

Simplifying \( \frac{20}{40} \), you get the equivalent fraction \( \frac{1}{2} \). Since you still don’t have a common denominator, write \( \frac{1}{2} \) as an equivalent fraction with a denominator of 8: \( \frac{1}{2} = \frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8} \).

You find that \( \frac{3}{8} < \frac{4}{8} \), so \( \frac{3}{8} < \frac{20}{40} \) as well.
2.3 Multiplying Fractions and Mixed Numbers

Learning Objective(s)
1. Multiply two or more fractions.
2. Multiply a fraction by a whole number.
3. Multiply two or more mixed numbers.
4. Solve application problems that require multiplication of fractions or mixed numbers.
5. Find the area of triangles.

Introduction

Just as you add, subtract, multiply, and divide when working with whole numbers, you also use these operations when working with fractions. There are many times when it is necessary to multiply fractions and **mixed numbers**. For example, this recipe will make 4 crumb piecrusts:

- \(5 \text{ cups graham crackers}\)
- \(8 \text{T. sugar}\)
- \(1\frac{1}{2} \text{ cups melted butter}\)
- \(1\frac{1}{4} \text{ tsp. vanilla}\)

Suppose you only want to make 2 crumb piecrusts. You can multiply all the ingredients by \(\frac{1}{2}\), since only half of the number of piecrusts are needed. After learning how to multiply a fraction by another fraction, a whole number or a mixed number, you should be able to calculate the ingredients needed for 2 piecrusts.

Multiplying Fractions

When you multiply a fraction by a fraction, you are finding a “fraction of a fraction.”

Suppose you have \(\frac{3}{4}\) of a candy bar and you want to find \(\frac{1}{2}\) of the \(\frac{3}{4}\):

\[
\begin{array}{c}
\frac{3}{4} \\
\hline
\frac{6}{8} \\
\hline
\frac{3}{8}
\end{array}
\]

By dividing each fourth in half, you can divide the candy bar into eighths.

Then, choose half of those to get \(\frac{3}{8}\).

\[
\begin{array}{c}
\frac{3}{8} \\
\hline
\end{array}
\]

In both of the above cases, to find the answer, you can multiply the numerators together and the denominators together.
Multiplying Two Fractions

\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \text{product of the numerators} \cdot \text{product of the denominators}
\]

Example:

\[
\frac{3}{4} \cdot \frac{1}{2} = \frac{3 \cdot 1}{4 \cdot 2} = \frac{3}{8}
\]

Multiplying More Than Two Fractions

\[
\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = \frac{a \cdot c \cdot e}{b \cdot d \cdot f}
\]

Example:

\[
\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} = \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5} = \frac{6}{60}
\]

Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>(\frac{2}{3} \cdot \frac{4}{5})</th>
<th>Multiply.</th>
</tr>
</thead>
</table>

Multiply the numerators and multiply the denominators.

\[
\frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}
\]

Simplify, if possible. This fraction is already in lowest terms.

Answer

\[
\frac{8}{15}
\]

If the resulting product needs to be simplified to lowest terms, divide the numerator and denominator by common factors.
### Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>Multiply. Simplify the answer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{2}{3} \cdot \frac{1}{4} ]</td>
<td>[ \frac{2 \cdot 1}{3 \cdot 4} ]</td>
</tr>
</tbody>
</table>

Multiply the numerators and multiply the denominators.

\[ \frac{2 \div 2}{12 \div 2} \]

Simplify by dividing the numerator and denominator by the common factor 2.

\[ \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6} \]

**Answer**

You can also simplify the problem before multiplying, by dividing common factors.

### Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>Multiply. Simplify the answer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{2}{3} \cdot \frac{1}{4} ]</td>
<td>[ \frac{2 \cdot 1}{3 \cdot 4} ]</td>
</tr>
</tbody>
</table>

Reorder the numerators so that you can see a fraction that has a common factor.

\[ \frac{1 \cdot 1}{3 \cdot 2} \]

Simplify.

\[ \frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2} \]

**Answer**

\[ \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6} \]

You do not have to use the “simplify first” shortcut, but it could make your work easier because it keeps the numbers in the numerator and denominator smaller while you are working with them.

### Self Check A

\[ \frac{3}{4} \cdot \frac{1}{3} \]

Multiply. Simplify the answer.
Multiplying a Fraction by a Whole Number

When working with both fractions and whole numbers, it is useful to write the whole number as an improper fraction (a fraction where the numerator is greater than or equal to the denominator). All whole numbers can be written with a “1” in the denominator. For example: \(2 = \frac{2}{1}\), \(5 = \frac{5}{1}\), and \(100 = \frac{100}{1}\). Remember that the denominator tells how many parts there are in one whole, and the numerator tells how many parts you have.

Multiplying a Fraction and a Whole Number

\[
a \cdot \frac{b}{c} = \frac{a \cdot b}{c}
\]

Example:

\[
\frac{2}{3} \cdot \frac{4}{1} = \frac{8}{3}
\]

Often when multiplying a whole number and a fraction the resulting product will be an improper fraction. It is often desirable to write improper fractions as a mixed number for the final answer. You can simplify the fraction before or after rewriting as a mixed number. See the examples below.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
<tr>
<td><strong>Multiply. Simplify the answer and write as a mixed number.</strong></td>
</tr>
<tr>
<td><strong>Answer</strong></td>
</tr>
</tbody>
</table>

Rewrite 7 as the improper fraction \( \frac{7}{1} \).

Multiply the numerators and multiply the denominators.

Rewrite as a mixed number. \( 21 \div 5 = 4 \) with a remainder of 1.
Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>( 4 \cdot \frac{3}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{4}{1} \cdot \frac{3}{4} )</td>
<td>Rewrite 4 as the improper fraction ( \frac{4}{1} ).</td>
</tr>
<tr>
<td>( \frac{4}{1} \cdot \frac{3}{4} )</td>
<td>Multiply the numerators and multiply the denominators.</td>
</tr>
<tr>
<td>( \frac{12}{4} = 3 )</td>
<td>Simplify.</td>
</tr>
<tr>
<td>Answer</td>
<td>( 4 \cdot \frac{3}{4} = 3 )</td>
</tr>
</tbody>
</table>

Self Check B

\( 3 \cdot \frac{5}{6} \) Multiply. Simplify the answer and write it as a mixed number.

Multiplying Mixed Numbers

If you want to multiply two mixed numbers, or a fraction and a mixed number, you can again rewrite any mixed number as an improper fraction.

So, to multiply two mixed numbers, rewrite each as an improper fraction and then multiply as usual. Multiply numerators and multiply denominators and simplify. And, as before, when simplifying, if the answer comes out as an improper fraction, then convert the answer to a mixed number.

Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>( 2 \frac{1}{5} \cdot 4 \frac{1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{11}{5} )</td>
<td>Change ( 2 \frac{1}{5} ) to an improper fraction. ( 5 \cdot 2 + 1 = 11 ), and the denominator is 5.</td>
</tr>
<tr>
<td>( \frac{9}{2} )</td>
<td>Change ( 4 \frac{1}{2} ) to an improper fraction. ( 2 \cdot 4 + 1 = 9 ), and the denominator is 2.</td>
</tr>
</tbody>
</table>
Rewrite the multiplication problem, using the improper fractions.

\[ \frac{11 \cdot 9}{5 \cdot 2} \]

Multiply numerators and multiply denominators.

\[ \frac{11 \cdot 9}{5 \cdot 2} = \frac{99}{10} \]

Write as a mixed number.

\[ \frac{99}{10} = \frac{9}{10} \]

99 \div 10 = 9 with a remainder of 9.

**Answer**

\[ 2 \frac{1}{5} \cdot 4 \frac{1}{2} = \frac{9}{10} \]

### Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>( \frac{1}{2} \cdot \frac{3}{3} )</th>
<th><strong>Multiply. Simplify the answer and write as a mixed number.</strong></th>
</tr>
</thead>
</table>

\[ \frac{3 \cdot 1}{3} = \frac{10}{3} \]

Change \( \frac{3\frac{1}{3}}{3} \) to an improper fraction. \(3 \cdot 3 + 1 = 10\), and the denominator is 3.

\[ \frac{1}{2} \cdot \frac{10}{3} \]

Rewrite the multiplication problem, using the improper fraction in place of the mixed number.

\[ \frac{1 \cdot 10}{2 \cdot 3} = \frac{10}{6} \]

Multiply numerators and multiply denominators.

\[ \frac{10}{6} = 1 \frac{4}{6} \]

Rewrite as a mixed number.

\(10 \div 6 = 1\) with a remainder of 4.

\[ \frac{1\frac{2}{3}}{3} \]

Simplify the fractional part to lowest terms by dividing the numerator and denominator by the common factor 2.

**Answer**

\[ \frac{1 \cdot 3 \cdot \frac{1}{3}}{3} = \frac{2}{3} \]

As you saw earlier, sometimes it’s helpful to look for common factors in the numerator and denominator before you simplify the products.
## Example

**Problem**  
\[
\frac{3}{5} \cdot 2\frac{1}{4}
\]

**Multiply. Simplify the answer and write as a mixed number.**

\[
\frac{3}{5} = \frac{8}{5} \quad \text{Change } \frac{3}{5} \text{ to an improper fraction. } 5 \cdot 1 + 3 = 8, \text{ and the denominator is } 5.
\]

\[
2\frac{1}{4} = \frac{9}{4} \quad \text{Change } 2\frac{1}{4} \text{ to an improper fraction. } 4 \cdot 2 + 1 = 9, \text{ and the denominator is } 4.
\]

\[
\frac{8}{5} \cdot \frac{9}{4} \quad \text{Rewrite the multiplication problem using the improper fractions.}
\]

\[
\frac{8 \cdot 9}{5 \cdot 4} = \frac{9 \cdot 8}{5 \cdot 4} \quad \text{Reorder the numerators so that you can see a fraction that has a common factor.}
\]

\[
\frac{9 \cdot 8}{5 \cdot 4} = \frac{9 \cdot 2}{5 \cdot 1} \quad \text{Simplify. } \frac{8}{4} = \frac{8 \div 4}{4 \div 4} = \frac{2}{1}
\]

\[
\frac{18}{5} \quad \text{Multiply.}
\]

\[
\frac{18}{5} = 3\frac{3}{5} \quad \text{Write as a mixed fraction.}
\]

**Answer**  
\[
\frac{3}{5} \cdot 2\frac{1}{4} = 3\frac{3}{5}
\]

In the last example, the same answer would be found if you multiplied numerators and multiplied denominators without removing the common factor. However, you would get \(\frac{72}{20}\), and then you would need to simplify more to get your final answer.

**Self Check C**  
\[
\frac{3}{5} \cdot \frac{1}{3} \quad \text{Multiply. Simplify the answer and write as a mixed number.}
\]
### Solving Problems by Multiplying Fractions and Mixed Numbers

Now that you know how to multiply a fraction by another fraction, by a whole number, or by a mixed number, you can use this knowledge to solve problems that involve multiplication and fractional amounts. For example, you can now calculate the ingredients needed for the 2 crumb piecrusts.

#### Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>5 cups graham crackers</th>
<th>8 T. sugar</th>
<th>The recipe at left makes 4 piecrusts. Find the ingredients needed to make only 2 piecrusts.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1(\frac{1}{2}) cups melted butter</td>
<td>(\frac{1}{4}) tsp. vanilla</td>
<td></td>
</tr>
</tbody>
</table>

Since the recipe is for 4 piecrusts, you can multiply each of the ingredients by \(\frac{1}{2}\) to find the measurements for just 2 piecrusts.

\[
5 \cdot \frac{1}{2} = \frac{5}{2} \cdot \frac{1}{2} = \frac{5}{2} 
\]

\[
2\frac{1}{2} \text{ cups of graham crackers are needed.}
\]

\[
8 \cdot \frac{1}{2} = \frac{8}{2} \cdot \frac{1}{2} = 4 
\]

\[
8 \text{ T. sugar is needed.}
\]

\[
1\frac{1}{2} \text{ cups melted butter: You need to multiply a mixed number by a fraction. So, first rewrite } 1\frac{1}{2} \text{ as the improper fraction } \frac{3}{2} : 2 \cdot 1 + 1, \text{ and the denominator is } 2. \text{ Then, rewrite the multiplication problem, using the improper fraction in place of the mixed number. Multiply.}
\]

\[
\frac{3}{4} \text{ cup melted butter is needed.}
\]
\[
\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}
\]

\[
\frac{1}{4} \text{ tsp. vanilla: Here, you multiply a fraction by a fraction.}
\]

\[
\frac{1}{8} \text{ tsp. vanilla is needed.}
\]

**Answer**
The ingredients needed for 2 pie crusts are:
- \(2 \frac{1}{2}\) cups graham crackers
- 4 T. sugar
- \(\frac{3}{4}\) cup melted butter
- \(\frac{1}{8}\) tsp. vanilla

Often, a problem indicates that multiplication by a fraction is needed by using phrases like “half of,” “a third of,” or “\(\frac{3}{4}\) of.”

### Example

**Problem**
The cost of a vacation is $4,500 and you are required to pay \(\frac{1}{5}\) of that amount when you reserve the trip. How much will you have to pay when you reserve the trip?

\[
\begin{align*}
4,500 \cdot \frac{1}{5} & \quad \text{You need to find } \frac{1}{5} \text{ of } 4,500. \text{ “Of” tells you to multiply.} \\
\frac{4,500}{1} \cdot \frac{1}{5} & \quad \text{Change } 4,500 \text{ to an improper fraction by rewriting it with } 1 \text{ as the denominator.} \\
\frac{4,500}{5} & \quad \text{Divide.} \\
900 & \quad \text{Simplify.}
\end{align*}
\]

**Answer**
You will need to pay $900 when you reserve the trip.
The pie chart at left represents the fractional part of daily activities. Given a 24-hour day, how many hours are spent sleeping? Attending school? Eating? Use the pie chart to determine your answers.

Sleeping is \( \frac{1}{3} \) of the pie, so the number of hours spent sleeping is \( \frac{1}{3} \) of 24.

Rewrite 24 as an improper fraction with a denominator of 1.

Multiply numerators and multiply denominators. Simplify \( \frac{24}{3} \) to 8.

Attending school is \( \frac{1}{6} \) of the pie, so the number of hours spent attending school is \( \frac{1}{6} \) of 24.

Rewrite 24 as an improper fraction with a denominator of 1.

Multiply numerators and multiply denominators. Simplify \( \frac{24}{6} \) to 4.
\[ \frac{1}{12} \cdot 24 = \text{number of hours spent eating} \]

Eating is \( \frac{1}{12} \) of the pie, so the number of hours spent eating is \( \frac{1}{12} \) of 24.

\[ \frac{1}{12} \cdot \frac{24}{1} = \frac{24}{12} = 2 \]

2 hours spent eating

\textbf{Answer}

Hours spent:
- sleeping: 8 hours
- attending school: 4 hours
- eating: 2 hours

\textbf{Self Check D}

Neil bought a dozen (12) eggs. He used \( \frac{1}{3} \) of the eggs for breakfast. How many eggs are left?

\textbf{Area of Triangles}

The formula for the area of a triangle can be explained by looking at a right triangle. Look at the image below—a rectangle with the same height and base as the original triangle. The area of the triangle is one half of the rectangle!
Since the area of two congruent triangles is the same as the area of a rectangle, you can come up with the formula \( \text{Area} = \frac{1}{2} b \cdot h \) to find the area of a triangle.

When you use the formula for a triangle to find its area, it is important to identify a base and its corresponding height, which is perpendicular to the base.

**Example**

**Problem**  A triangle has a height of 4 inches and a base of 10 inches. Find the area.

\[ A = \frac{1}{2} bh \]

Start with the formula for the area of a triangle.

\[ A = \frac{1}{2} \cdot 10 \cdot 4 \]

Substitute 10 for the base and 4 for the height.

\[ A = \frac{1}{2} \cdot 40 \]

Multiply.

\[ A = 20 \]

**Answer**  \( A = 20 \text{ in}^2 \)
Summary

You multiply two fractions by multiplying the numerators and multiplying the denominators. Often the resulting product will not be in lowest terms, so you must also simplify. If one or both fractions are whole numbers or mixed numbers, first rewrite each as an improper fraction. Then multiply as usual, and simplify.

2.3 Self Check Solutions

Self Check A

\[
\frac{3}{4} \cdot \frac{1}{3}
\]

Multiply. Simplify the answer.

\[
\frac{3 \cdot 1}{4 \cdot 3} = \frac{3}{12}, \text{ then simplify: } \frac{3}{12} = \frac{1}{4}.
\]

Self Check B

\[
\frac{5}{6}
\]

Multiply. Simplify the answer and write it as a mixed number.

\[
2 \frac{1}{2} \cdot \frac{15}{6}, \text{ and since } 15 \div 6 = 2R3, \text{ the mixed number is } 2 \frac{3}{6}. \text{ The fractional part simplifies to } \frac{1}{2}.
\]

Self Check C

\[
1 \frac{3}{5} \cdot 3 \frac{1}{3}
\]

Multiply. Simplify the answer and write as a mixed number.

\[
5 \frac{1}{3} \cdot \text{ First, rewrite each mixed number as an improper fraction: } \frac{13}{5} = \frac{8}{5} \text{ and } \frac{10}{3} = \frac{10}{3}.
\]

Next, multiply numerators and multiply denominators: \[
\frac{8 \cdot 10}{5 \cdot 3} = \frac{80}{15}. \text{ Then write as a mixed fraction } \frac{80}{15} = \frac{5 \cdot 5}{15}. \text{ Finally, simplify the fractional part by dividing both numerator and denominator by the common factor 5.}
\]

Self Check D

Neil bought a dozen (12) eggs. He used \(\frac{1}{3}\) of the eggs for breakfast. How many eggs are left?

\[
\frac{1}{3} \text{ of } 12 = 4 \left( \frac{1}{3} \cdot \frac{12}{1} = \frac{12}{3} = 4 \right), \text{ so he used } 4 \text{ of the eggs. Because } 12 - 4 = 8, \text{ there are } 8 \text{ eggs left.}
\]
### 2.4 Dividing Fractions and Mixed Numbers

<table>
<thead>
<tr>
<th>Learning Objective(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Find the reciprocal of a number.</td>
</tr>
<tr>
<td>2 Divide two fractions.</td>
</tr>
<tr>
<td>3 Divide two mixed numbers.</td>
</tr>
<tr>
<td>4 Divide fractions, mixed numbers, and whole numbers.</td>
</tr>
<tr>
<td>5 Solve application problems that require division of fractions or mixed numbers.</td>
</tr>
</tbody>
</table>

#### Introduction

There are times when you need to use division to solve a problem. For example, if painting one coat of paint on the walls of a room requires 3 quarts of paint and there are 6 quarts of paint, how many coats of paint can you paint on the walls? You divide 6 by 3 for an answer of 2 coats. There will also be times when you need to divide by a fraction.

Suppose painting a closet with one coat only required $\frac{1}{2}$ quart of paint. How many coats could be painted with the 6 quarts of paint? To find the answer, you need to divide 6 by the fraction, $\frac{1}{2}$.

#### Reciprocals

If the [product] of two numbers is 1, the two numbers are [reciprocals] of each other. Here are some examples:

<table>
<thead>
<tr>
<th>Original number</th>
<th>Reciprocal</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{4}$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{3 \cdot 4}{4 \cdot 3} = \frac{12}{12} = 1$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{2}{1}$</td>
<td>$\frac{1 \cdot 2}{2 \cdot 1} = \frac{2}{2} = 1$</td>
</tr>
<tr>
<td>$\frac{3}{1}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{3 \cdot 1}{1 \cdot 3} = \frac{3}{3} = 1$</td>
</tr>
<tr>
<td>$2\frac{1}{3} = \frac{7}{3}$</td>
<td>$\frac{3}{7}$</td>
<td>$\frac{7 \cdot 3}{3 \cdot 7} = \frac{21}{21} = 1$</td>
</tr>
</tbody>
</table>
In each case, the original number, when multiplied by its reciprocal, equals 1.

To create two numbers that multiply together to give an answer of one, the numerator of one is the denominator of the other. You sometimes say one number is the “flip” of the other number: flip $\frac{2}{5}$ to get the reciprocal $\frac{5}{2}$. In order to find the reciprocal of a mixed number, write it first as an improper fraction so that it can be “flipped.”

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
<tr>
<td>$\frac{5\frac{1}{4}}{4} = \frac{21}{4}$</td>
</tr>
<tr>
<td><strong>Answer</strong></td>
</tr>
</tbody>
</table>

**Self Check A**

What is the reciprocal of $\frac{2}{5}$?

---

**Dividing a Fraction or a Mixed Number by a Whole Number**

When you divide by a whole number, you multiply by the reciprocal of the divisor. In the painting example where you need 3 quarts of paint for a coat and have 6 quarts of paint, you can find the total number of coats that can be painted by dividing 6 by 3, $6 \div 3 = 2$.

You can also multiply 6 by the reciprocal of 3, which is $\frac{1}{3}$, so the multiplication problem becomes $\frac{6}{1} \cdot \frac{1}{3} = \frac{6}{3} = 2$.

The same idea will work when the divisor is a fraction. If you have $\frac{3}{4}$ of a candy bar and need to divide it among 5 people, each person gets $\frac{1}{5}$ of the available candy: $\frac{1}{5}$ of $\frac{3}{4}$ is $\frac{1 \cdot 3}{5 \cdot 4} = \frac{3}{20}$, so each person gets $\frac{3}{20}$ of a whole candy bar.
If you have a recipe that needs to be divided in half, you can divide each ingredient by 2, or you can multiply each ingredient by $\frac{1}{2}$ to find the new amount.

Similarly, with a **mixed number**, you can either divide by the whole number or you can multiply by the reciprocal. Suppose you have $1\frac{1}{2}$ pizzas that you want to divide evenly among 6 people.

Dividing by 6 is the same as multiplying by the reciprocal of 6, which is $\frac{1}{6}$. Cut the available pizza into six equal-sized pieces.

Each person gets one piece, so each person gets $\frac{1}{4}$ of a pizza.
Dividing a fraction by a whole number is the same as multiplying by the reciprocal, so you can always use multiplication of fractions to solve such division problems.

### Example

#### Problem

Find $\frac{2}{3} \div 4$. Write your answer as a mixed number with any fraction part in lowest terms.

- **Rewrite** $\frac{2}{3}$ as an improper fraction. The numerator is $2 \cdot 3 + 2$. The denominator is still 3.

- Dividing by 4 or $\frac{4}{4}$ is the same as multiplying by the reciprocal of 4, which is $\frac{1}{4}$.

- Multiply numerators and multiply denominators.

- Simplify to lowest terms by dividing numerator and denominator by the common factor 4.

#### Answer

$$\frac{2}{3} \div 4 = \frac{2}{3}$$

### Self Check B

Find $\frac{3}{5} \div 2$. Simplify the answer and write as a mixed number.
Dividing by a Fraction

Sometimes you need to solve a problem that requires dividing by a fraction. Suppose you have a pizza that is already cut into 4 slices. How many $\frac{1}{2}$ slices are there?

There are 8 slices. You can see that dividing 4 by $\frac{1}{2}$ gives the same result as multiplying 4 by 2.

What would happen if you needed to divide each slice into thirds?

You would have 12 slices, which is the same as multiplying 4 by 3.

### Dividing with Fractions

1. Find the reciprocal of the number that follows the division symbol.
2. Multiply the first number (the one before the division symbol) by the reciprocal of the second number (the one after the division symbol).

Examples:

$\frac{6}{3} ÷ \frac{2}{3} = 6 \cdot \frac{3}{2}$ and $\frac{2}{5} ÷ \frac{1}{3} = \frac{2}{5} \cdot \frac{3}{1}$
Any easy way to remember how to divide fractions is the phrase “keep, change, flip”. This means to **KEEP** the first number, **CHANGE** the division sign to multiplication, and then **FLIP** (use the reciprocal) of the second number.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
<tr>
<td><strong>Multiply by the reciprocal:</strong></td>
</tr>
<tr>
<td>( \frac{2 \cdot 6}{3 \cdot 1} = \frac{12}{3} )</td>
</tr>
<tr>
<td>( \frac{12}{3} = 4 )</td>
</tr>
<tr>
<td><strong>Answer</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
<tr>
<td><strong>Multiply by the reciprocal:</strong></td>
</tr>
<tr>
<td>( \frac{3 \cdot 3}{5 \cdot 2} = \frac{9}{10} )</td>
</tr>
<tr>
<td>( \frac{3}{5} \div \frac{2}{3} = \frac{9}{10} )</td>
</tr>
</tbody>
</table>

When solving a division problem by multiplying by the reciprocal, remember to write all whole numbers and mixed numbers as improper fractions. The final answer should be simplified and written as a mixed number.
### Example

**Problem**

\[
\frac{2\frac{1}{4}}{\frac{3}{4}}
\]

**Divide.**

\[
\frac{9}{4} \div \frac{3}{4} = 3
\]

Write \(2\frac{1}{4}\) as an improper fraction.

\[
\frac{9}{4} \cdot \frac{4}{3} = \frac{36}{12} = 3
\]

Multiply by the reciprocal:

\[\frac{9}{4} \text{, change } \div \text{ to •, and flip } \frac{3}{4}.\]

\[
\frac{9 \cdot 4}{4 \cdot 3} = \frac{36}{12}
\]

Multiply numerators and multiply denominators.

\[
\frac{36}{12} = 3
\]

Simplify.

\[\frac{2\frac{1}{4}}{\frac{3}{4}} = 3\]

**Answer**

\[\frac{2\frac{1}{4}}{\frac{3}{4}} = 3\]

---

### Example

**Problem**

\[
\frac{3\frac{1}{5}}{2\frac{1}{10}}
\]

**Divide. Simplify the answer and write as a mixed number.**

\[
\frac{16}{5} \div \frac{21}{10} = \frac{32}{21}
\]

Write \(3\frac{1}{5}\) and \(2\frac{1}{10}\) as improper fractions.

\[
\frac{16}{5} \cdot \frac{10}{21} = \frac{16 \cdot 10}{21 \cdot 5} = \frac{16 \cdot 2}{21 \cdot 1}
\]

Multiply by the reciprocal of \(\frac{21}{10}\).

Multiply numerators, multiply denominators, and regroup.

\[
\frac{16 \cdot 2}{21 \cdot 1} = \frac{32}{21} = 1\frac{11}{21}
\]

Simplify: \(\frac{10}{5} = \frac{2}{1}\).

\[
\frac{16 \cdot 2}{21 \cdot 1} = \frac{32}{21}
\]

Multiply.

Rewrite as a mixed number.

\[\frac{3\frac{1}{5}}{2\frac{1}{10}} = 1\frac{11}{21}\]

**Answer**

\[\frac{3\frac{1}{5}}{2\frac{1}{10}} = 1\frac{11}{21}\]
Self Check C
Find \( 5\frac{1}{3} \div \frac{2}{3} \). Simplify the answer and write as a mixed number.

Dividing Fractions or Mixed Numbers to Solve Problems

Using multiplication by the reciprocal instead of division can be very useful to solve problems that require division and fractions.

Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>A cook has ( 18\frac{3}{4} ) pounds of ground beef. How many quarter-pound burgers can he make?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>You need to find how many quarter pounds there are in ( 18\frac{3}{4} ), so use division.</td>
</tr>
<tr>
<td></td>
<td>( \frac{18\frac{3}{4}}{\frac{1}{4}} ) Write ( 18\frac{3}{4} ) as an improper fraction.</td>
</tr>
<tr>
<td></td>
<td>( \frac{75}{4} \div \frac{1}{4} ) Multiply by the reciprocal.</td>
</tr>
<tr>
<td></td>
<td>( \frac{75}{4} \cdot \frac{4}{1} ) Multiply numerators and multiply denominators.</td>
</tr>
<tr>
<td></td>
<td>( \frac{4 \cdot 75}{4 \cdot 1} ) Regroup and simplify ( \frac{4}{4} ), which is 1.</td>
</tr>
</tbody>
</table>

Answer 75 burgers
Example

Problem
A child needs to take \(2\frac{1}{2}\) tablespoons of medicine per day in 4 equal doses. How much medicine is in each dose?

\[
\frac{2\frac{1}{2}}{4} \quad \text{You need to make 4 equal doses, so you can use division.}
\]
\[
\frac{5}{2} \div 4 \quad \text{Write} \ 2\frac{1}{2} \ \text{as an improper fraction.}
\]
\[
\frac{5 \cdot 1}{2 \cdot 4} \quad \text{Multiply by the reciprocal.}
\]
\[
\frac{5 \cdot 1}{2 \cdot 4} = \frac{5}{8} \quad \text{Multiply numerators and multiply denominators. Simplify, if possible.}
\]

Answer
\(\frac{5}{8}\) tablespoon in each dose.

Self Check D
How many \(2\frac{2}{5}\)-cup salt shakers can be filled from 12 cups of salt?

Summary
Division is the same as multiplying by the reciprocal. When working with fractions, this is the easiest way to divide. Whether you divide by a number or multiply by the reciprocal of the number, the result will be the same. You can use these techniques to help you solve problems that involve division, fractions, and/or mixed numbers.

2.4 Self Check Solutions

Self Check A
What is the reciprocal of \(3\frac{2}{5}\)?

\[
\frac{5}{17} \quad \text{First, write} \ 3\frac{2}{5} \ \text{as an improper fraction,} \ \frac{17}{5}. \ \text{The reciprocal of} \ \frac{17}{5} \ \text{is found by interchanging ("flipping") the numerator and denominator.}
\]
Self Check B
Find \( \frac{3}{5} \div 2 \). Simplify the answer and write as a mixed number.

Write \( \frac{3}{5} \) as the improper fraction \( \frac{23}{5} \). Then multiply by \( \frac{1}{2} \), the reciprocal of 2. This gives the improper fraction \( \frac{23}{10} \), and the mixed number is \( 23 \div 10 = 2R \frac{3}{10} \).

Self Check C
Find \( \frac{1}{3} \div \frac{2}{3} \). Simplify the answer and write as a mixed number.

Write \( \frac{1}{3} \) as an improper fraction, \( \frac{16}{3} \). Then multiply by the reciprocal of \( \frac{2}{3} \), which is \( \frac{3}{2} \), giving you \( \frac{16}{3} \cdot \frac{3}{2} = \frac{16}{2} = 8 \).

Self Check D
How many \( \frac{2}{5} \)-cup salt shakers can be filled from 12 cups of salt?

\( 12 \div \frac{2}{5} \) will show how many salt shakers can be filled. Write 12 as \( \frac{12}{1} \) and multiply by the reciprocal (“flip”) of \( \frac{2}{5} \), giving you \( \frac{12}{1} \cdot \frac{5}{2} = \frac{60}{2} = 30 \).
**2.5 Adding and Subtracting Fractions and Mixed Numbers with Like Denominators**

**Learning Objective(s)**
1. Add fractions with like denominators.
2. Subtract fractions with like denominators.
3. Add mixed numbers with like denominators.
4. Subtract mixed numbers with like denominators.
5. Solve application problems that require the addition of fractions or mixed numbers.

**Introduction**

Fractions are used in many areas of everyday life: recipes, woodworking, rainfall, timecards, and measurements, to name just a few. Sometimes you have parts of wholes that you need to combine. Just as you can add whole numbers, you can add fractions and mixed numbers. Consider, for example, how to determine the monthly rainfall if you know the daily rainfall in inches. You have to add fractions. Also, consider several painters who are working to paint a house together with multiple cans of paint. They might add the fractions of what remains in each can to determine if there is enough paint to finish the job or if they need to buy more.

**Adding Fractions with Like Denominators**

When the pieces are the same size, they can easily be added. Consider the pictures below showing the fractions \(\frac{3}{6}\) and \(\frac{2}{6}\).

This picture represents \(\frac{3}{6}\) shaded because 3 out of 6 blocks are shaded.

```
[\Box\Box\Box\Box\Box\Box]
```

This picture represents \(\frac{2}{6}\) shaded because 2 out of 6 blocks are shaded.

```
[\Box\Box\Box\Box\Box\Box]
```

If you add these shaded blocks together, you are adding \(\frac{3}{6} + \frac{2}{6}\).

You can create a new picture showing 5 shaded blocks in a rectangle containing 6 blocks.

```
[\Box\Box\Box\Box\Box\Box]
```

So, \(\frac{3}{6} + \frac{2}{6} = \frac{5}{6}\).
Without drawing rectangles and shading boxes, you can get this answer simply by adding the numerators, 3 + 2, and keeping the denominator, 6, the same. This procedure works for adding any fractions that have the same denominator, called **like denominators**.

### Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>( \frac{3}{5} + \frac{1}{5} )</th>
<th>Add.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \frac{3 + 1}{5} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{4}{5} )</td>
<td></td>
</tr>
<tr>
<td><strong>Answer</strong></td>
<td>( \frac{3}{5} + \frac{1}{5} = \frac{4}{5} )</td>
<td></td>
</tr>
</tbody>
</table>

Since the denominator of each fraction is 5, these fractions have like denominators. So, add the numerators and write the sum over the denominator, 5.

### Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>( \frac{3}{8} + \frac{5}{8} )</th>
<th>Add. Simplify the answer.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \frac{3 + 5}{8} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{8}{8} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{8}{8} = 1 )</td>
<td>Simplify the fraction.</td>
</tr>
<tr>
<td><strong>Answer</strong></td>
<td>( \frac{3}{8} + \frac{5}{8} = 1 )</td>
<td></td>
</tr>
</tbody>
</table>
Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>11/12 + 5/12</th>
<th>Add. Simplify the answer and write as a mixed number.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>11 + 5</td>
<td>The denominators are alike, so add the numerators.</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>= 16</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>12 ÷ 4 = 4/3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Simplify the fraction.</td>
</tr>
<tr>
<td></td>
<td>16 = 4</td>
<td>4 ÷ 3 = 1 with a remainder of 1.</td>
</tr>
<tr>
<td></td>
<td>12 = 3</td>
<td>Write the improper fraction as a mixed number.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Answer</td>
<td>11/12 + 5/12</td>
<td>11 + 5/12 = 1 1/3</td>
</tr>
</tbody>
</table>

In the previous example, the fraction was simplified and then converted to a mixed number. You could just as easily have first converted the improper fraction to a mixed number and then simplified the fraction in the mixed number. Notice that the same answer is reached with both methods.

\[
\frac{16}{12} = 1 \frac{4}{12}
\]

The fraction \(\frac{4}{12}\) can be simplified. \(\frac{4}{12} = \frac{4 \div 4}{12 \div 4} = \frac{1}{3}\)

But, don’t forget about the 1 that is part of the mixed number! The final answer is \(1 \frac{1}{3}\).

**Adding Fractions with Like Denominators**

1. Add the numerators (the number in the top of each fraction).
2. Keep the denominator (the bottom number) the same.
3. Simplify to lowest terms.

**Self Check A**

\[
\frac{7}{10} + \frac{8}{10}
\]

Add. Simplify the answer and write as a mixed number.
Sometimes subtraction, rather than addition, is required to solve problems that involve fractions. Suppose you are making pancakes and need $4\frac{1}{2}$ cups of flour but you only have $2\frac{3}{4}$ cups. How many additional cups will you have to get to make the pancakes? You can solve this problem by subtracting the mixed numbers.

**Subtracting Fractions with Like Denominators**

The most simple fraction subtraction problems are those that have two proper fractions with a **common denominator**. That is, each denominator is the same. The process is just as it is for addition of fractions with **like denominators**, except you subtract! You subtract the second numerator from the first and keep the denominator the same.

Imagine that you have a cake with equal-sized pieces. Some of the cake has already been eaten, so you have a fraction of the cake remaining. You could represent the cake pieces with the picture below.

The cake is cut into 12 equal pieces to start. Two are eaten, so the remaining cake can be represented with the fraction $\frac{10}{12}$. If three more pieces of cake are eaten, what fraction of the cake is left? You can represent that problem with the expression $\frac{10}{12} - \frac{3}{12}$.

If you subtract 3 pieces, you can see below that $\frac{7}{12}$ of the cake remains.

You can solve this problem without the picture by subtracting the numerators and keeping the denominator the same:

$$\frac{10}{12} - \frac{3}{12} = \frac{7}{12}$$
Subtracting Fractions with Like Denominators

If the denominators (bottoms) of the fractions are the same, subtract the numerators (tops) and keep the denominator the same. *Remember to simplify the resulting fraction, if possible.*

<table>
<thead>
<tr>
<th>Problem</th>
<th>Subtract.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{6}{7} - \frac{1}{7} )</td>
<td>( \frac{5}{7} )</td>
</tr>
</tbody>
</table>

Both fractions have a denominator of 7, so subtract the numerators and keep the same denominator.

<table>
<thead>
<tr>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{6}{7} - \frac{1}{7} = \frac{5}{7} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem</th>
<th>Subtract. Simplify the answer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{5}{9} - \frac{2}{9} )</td>
<td>( \frac{3}{9} )</td>
</tr>
</tbody>
</table>

The fractions have a **like denominator**, also known as a common denominator, so subtract the numerators.

| \( \frac{3}{9} \div \frac{3}{9} = \frac{1}{3} \) |

Simplify the fraction.

<table>
<thead>
<tr>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{5}{9} - \frac{2}{9} = \frac{1}{3} )</td>
</tr>
</tbody>
</table>

Self Check B

\( \frac{11}{16} - \frac{7}{16} \) Subtract and simplify the answer.

Adding Mixed Numbers

Just as you can add whole numbers and proper fractions, you can also add mixed numbers. To add mixed numbers, add the whole numbers together and the fraction parts of the mixed numbers together and then recombine to express the value as a mixed number. The steps for adding two mixed numbers are shown in the examples below.

You can keep the whole numbers and the fractions together using a vertical method for adding mixed numbers as shown below.
### Example

**Problem**

\[
\frac{21}{8} + \frac{33}{8}
\]

Add. Simplify the answer and write as a mixed number.

**Solution**

1. Arrange the mixed numbers vertically so the whole numbers align and the fractions align.

\[
\begin{array}{cccc}
& 2 & 1 & 8 \\
+ & 3 & 3 & 8 \\
\hline
& 5 & 4 & 8 \\
\end{array}
\]

2. Add whole numbers. Add fractions.

\[
\frac{54}{8} = \frac{51}{2}
\]

**Answer**

\[
\frac{21}{8} + \frac{33}{8} = \frac{51}{2}
\]

When adding mixed numbers you may need to regroup if the fractional parts add to more than one whole.

### Example

**Problem**

\[
\frac{65}{7} + \frac{84}{7}
\]

Add. Simplify the answer and write as a mixed number.

**Solution**

1. Arrange the mixed numbers vertically so the whole numbers align and the fractions align.

\[
\begin{array}{cccc}
& 6 & 5 & 7 \\
+ & 8 & 4 & 7 \\
\hline
& 14 & 9 & 7 \\
\end{array}
\]

2. Add whole numbers. Add fractions.

\[
\frac{149}{7}
\]
Write the improper fraction as a mixed number.

\[
\frac{9}{7} = 1\frac{2}{7}
\]

Combine whole numbers and fraction to write a mixed number.

\[
14 + 1\frac{2}{7} = 15\frac{2}{7}
\]

\text{Answer}

\[
\frac{6}{7} + \frac{8}{7} = 15\frac{2}{7}
\]

\textbf{Self Check C}

\[
\frac{7}{9} + \frac{4}{9}
\]

Add. Simplify the answer and write as a mixed number.

\textbf{Subtracting Mixed Numbers} \hspace{1cm} \textbf{Objective 4}

Subtracting mixed numbers works much the same way as adding mixed numbers. To subtract mixed numbers, subtract the whole number parts of the mixed numbers and then subtract the fraction parts in the mixed numbers. Finally, combine the whole number answer and the fraction answer to express the answer as a mixed number.

\begin{tabular}{|c|c|}
\hline
\textbf{Example} & \\
\hline
\textbf{Problem} & 6\frac{4}{5} - 3\frac{1}{5} \\
\hline
\textbf{Subtract. Simplify the answer and write as a mixed number.} & \\
\hline
\textbf{Subtract the whole numbers and subtract the fractions.} & 6 - 3 = 3 \\
\hline
\textbf{Combine the fraction and the whole number. Make sure the fraction in the mixed number is simplified.} & \frac{4}{5} - \frac{1}{5} = \frac{3}{5} \\
\hline
\textbf{Answer} & 3\frac{3}{5} \\
\hline
\end{tabular}

Sometimes it might be easier to express the mixed number as an improper fraction first and then solve. Consider the example below.
**Example**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Subtract. Simplify the answer and write as a mixed number.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[8 \frac{1}{3} - 4 \frac{2}{3}]</td>
<td></td>
</tr>
</tbody>
</table>

Write each mixed number as an improper fraction.

\[
\begin{align*}
8 \frac{1}{3} &= \frac{8 \cdot 3 + 1}{3} = \frac{25}{3} \\
4 \frac{2}{3} &= \frac{4 \cdot 3 + 2}{3} = \frac{14}{3}
\end{align*}
\]

Since the fractions have a like denominator, subtract the numerators.

\[
\frac{25}{3} - \frac{14}{3} = \frac{11}{3}
\]

Write the answer as a mixed number. Divide 11 by 3 to get 3 with a remainder of 2.

\[
\frac{11}{3} = 3 \frac{2}{3}
\]

**Answer**

\[
8 \frac{1}{3} - 4 \frac{2}{3} = 3 \frac{2}{3}
\]

Since addition is the inverse operation of subtraction, you can check your answer to a subtraction problem with addition. In the example above, if you add \(4 \frac{2}{3}\) to your answer of \(3 \frac{2}{3}\), you should get \(8 \frac{1}{3}\).

\[
\begin{align*}
4 \frac{2}{3} + 3 \frac{2}{3} &= 7 \frac{4}{3} \\
4 + 3 + \frac{2}{3} + \frac{2}{3} &= 7 + \frac{4}{3} \\
7 + 1 \frac{1}{3} &= 8 \frac{1}{3}
\end{align*}
\]

**Subtracting Mixed Numbers with Regrouping**

Sometimes when subtracting mixed numbers, the fraction part of the second mixed number is larger than the fraction part of the first number. Consider the problem:

\[
2.65
\]
\[ \frac{7}{6} - \frac{3}{6}. \] The standard procedure would be to subtract the fractions, but \( \frac{1}{6} - \frac{5}{6} \) would result in a negative number. You don’t want that!

You can regroup one of the whole numbers from the first number, writing the first mixed number in a different way:

\[
\frac{7}{6} = \frac{6}{6} + \frac{1}{6} = 1 + \frac{1}{6}
\]

\[
6 + \frac{1}{6} = \frac{6}{6} + \frac{7}{6} = \frac{6}{6} + \frac{7}{6} = \frac{6 + 7}{6} = \frac{13}{6}
\]

Now, you can write an equivalent problem to the original:

\[
\frac{6}{6} - \frac{7}{6} = \frac{5}{6}
\]

Then, you just subtract like you normally subtract mixed numbers:

\[
6 - 3 = 3
\]

\[
\frac{7}{6} - \frac{5}{6} = \frac{2}{6} = \frac{1}{3}
\]

So, the answer is \( 3 \frac{1}{3} \).

Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>Subtract. Simplify the answer and write as a mixed number.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{7}{4} - \frac{3}{4} )</td>
<td>( \frac{1}{4} = 6 + \frac{4}{4} + \frac{1}{4} ) Since the second fraction part, ( \frac{3}{4} ), is larger than the first fraction part, ( \frac{1}{4} ), regroup one of the whole numbers and write it as ( \frac{4}{4} ).</td>
</tr>
</tbody>
</table>
Rewrite the subtraction expression using the equivalent fractions.

\[
\begin{align*}
7 \frac{1}{4} - 3 \frac{3}{4} \\
6 \frac{5}{4} - 3 \frac{3}{4} \\
6 - 3 = 3 \\
\frac{5}{4} - \frac{3}{4} = \frac{2}{4} = \frac{1}{2}
\end{align*}
\]

Subtract the whole numbers, subtract the fractions.

\[
\begin{align*}
\frac{2}{4} &= \frac{1}{2} \\
\frac{3}{2} &= \frac{3}{2}
\end{align*}
\]

Combine the whole number and the fraction.

\[
\frac{7}{4} - \frac{3}{4} = \frac{3}{2}
\]

Answer

Sometimes a mixed number is subtracted from a whole number. In this case, you can also rewrite the whole number as a mixed number in order to perform the subtraction. You use an equivalent mixed number that has the same denominator as the fraction in the other mixed number.

Example

Problem

\[
8 - 4 \frac{2}{5}
\]

Subtract. Simplify the answer and write as a mixed number.

\[
\begin{align*}
8 &= 7 + 1 \\
7 + \frac{5}{5} &= 7 \frac{5}{5}
\end{align*}
\]

Regroup one from the whole number and write it as \(\frac{5}{5}\).

\[
\begin{align*}
7 \frac{5}{5} - 4 \frac{2}{5} \\
7 - 4 = 3 \\
\frac{5}{5} - \frac{2}{5} = \frac{3}{5}
\end{align*}
\]

Rewrite the subtraction expression using the equivalent fractions. Subtract the whole numbers, subtract the fractions.

\[
\frac{3}{5}
\]

Combine the whole number and the fraction.

Answer

\[
8 - 4 \frac{2}{5} = 3 \frac{3}{5}
\]
Subtracting Mixed Numbers

If the fractional part of the mixed number being subtracted is larger than the fractional part of the mixed number from which it is being subtracted, or if a mixed number is being subtracted from a whole number, follow these steps:

1. Subtract 1 from the whole number part of the mixed number being subtracted.
2. Add that 1 to the fraction part to make an improper fraction. For example,
   \[ \frac{7}{3} = 6 + \frac{3}{3} + \frac{2}{3} = \frac{65}{3}. \]
3. Then, subtract as with any other mixed numbers.

Alternatively, you can change both numbers to improper fractions and then subtract.

Self Check D

15 – 13\frac{1}{4} \text{ Subtract. Simplify the answer and write as a mixed number.}

Adding and Subtracting Fractions to Solve Problems

Knowing how to add fractions is useful in a variety of situations. When reading problems, look for phrases that help you know you want to add the fractions.

Example

Problem: A stack of pamphlets is placed on top of a book. If the stack of pamphlets is \(3 \frac{1}{4}\) inches thick and the book is \(5 \frac{3}{4}\) inches thick, how high is the pile?

Find the total height of the pile by adding the thicknesses of the stack of pamphlets and the book.

Group the whole numbers and fractions to make adding easier.

Add whole numbers.

Add fractions.

Combine whole number and fraction.

Answer: The pile is 9 inches high.
Knowing how to subtract fractions and mixed numbers is useful in a variety of situations. When reading problems, look for key words that indicate that the problem can be solved using subtraction.

### Example

**Problem**

Sherry loves to quilt, and she frequently buys fabric she likes when she sees it. She purchased 5 yards of blue print fabric and decided to use $2\frac{3}{8}$ yards of it in a quilt. How much of the blue print fabric will she have left over after making the quilt?

**Answer**

Sherry has $2\frac{5}{8}$ yards of blue print fabric left over.

### Summary

Adding and subtracting fractions with like denominators involves adding or subtracting the numerators and keeping the denominator the same. Always simplify the answer. Adding mixed numbers involves adding the fractional parts, adding the whole numbers, and then recombining them as a mixed number.

When subtracting mixed numbers, if the fraction in the second mixed number is larger than the fraction in the first mixed number, rewrite the first mixed number by regrouping one whole as a fraction. Alternatively, rewrite all fractions as improper fractions and then subtract. This process is also used when subtracting a mixed number from a whole number.
2.5 Self Check Solutions

Self Check A

\[ \frac{7}{10} + \frac{8}{10} \]  Add. Simplify the answer and write as a mixed number.

\[ \frac{7+8}{10} = \frac{15}{10} = 1 \frac{5}{10} = 1 \frac{1}{2} . \]

Self Check B

\[ \frac{11}{16} - \frac{7}{16} \]  Subtract and simplify the answer.

\[ \frac{11 - 7}{16} = \frac{4}{16} = \frac{1}{4} . \]

Self Check C

\[ 3 \frac{7}{9} + 1 \frac{4}{9} \]  Add. Simplify the answer and write as a mixed number.

Adding the fractions: \[ \frac{7}{9} + \frac{4}{9} = \frac{11}{9} = 1 \frac{2}{9} . \] Adding the whole numbers, 3+1 = 4. Combining these, \[ 4 + 1 \frac{2}{9} = 5 \frac{2}{9} . \]

Self Check D

\[ 15 - 13 \frac{1}{4} \]  Subtract. Simplify the answer and write as a mixed number.

Regrouping, \[ 15 = 14 + 1 = 14 + \frac{4}{4} = 14 \frac{4}{4} \]

\[ 15 - 13 \frac{1}{4} = 14 \frac{4}{4} - 13 \frac{1}{4} \]

Subtracting the whole numbers, 14-13 = 1. Subtracting fractions, \[ \frac{4}{4} - \frac{1}{4} = \frac{3}{4} \]

\[ 15 - 13 \frac{1}{4} = 1 \frac{3}{4} \]
2.6 Adding and Subtracting Fractions and Mixed Numbers with Unlike Denominators

Learning Objective(s)
1. Find the least common multiple (LCM) of two or more numbers.
2. Find the Least Common Denominator
3. Add fractions with unlike denominators.
4. Add mixed numbers
5. Subtract fractions with unlike denominators.
6. Subtract mixed numbers without regrouping.
7. Subtract mixed numbers with regrouping.
8. Solve application problems that require the subtraction of fractions or mixed numbers.

Finding Least Common Multiples

Sometimes fractions do not have the same denominator. They have unlike denominators. Think about the example of the house painters. If one painter has $\frac{2}{3}$ can of paint and his painting partner has $\frac{1}{2}$ can of paint, how much do they have in total? How can you add these fractions when they do not have like denominators?

The answer is that you can rewrite one or both of the fractions so that they have the same denominator. This is called finding a common denominator. While any common denominator will do, it is helpful to find the least common multiple of the two numbers in the denominator because this will save having to simplify at the end. The least common multiple is the least number that is a multiple of two or more numbers. Least common multiple is sometimes abbreviated LCM.

There are several ways to find common multiples, some of which you used when comparing fractions. To find the least common multiple (LCM), you can list the multiples of each number and determine which multiples they have in common. The least of these numbers will be the least common multiple. Consider the numbers 4 and 6. Some of their multiples are shown below. You can see that they have several common multiples, and the least of these is 12.

<table>
<thead>
<tr>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
<th>28</th>
<th>32</th>
<th>36</th>
<th>40</th>
<th>44</th>
<th>48</th>
<th>52</th>
<th>56</th>
<th>60</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>42</td>
<td>48</td>
<td>54</td>
<td>60</td>
<td>66</td>
<td>68</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.71
Example

Problem  Find the least common multiple of 30 and 50.

<table>
<thead>
<tr>
<th>30, 60, 90, 120, 150, 180, 210, 240</th>
</tr>
</thead>
<tbody>
<tr>
<td>List some multiples of 30.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>50, 100, 150, 200, 250</th>
</tr>
</thead>
<tbody>
<tr>
<td>List some multiples of 50.</td>
</tr>
</tbody>
</table>

150 is found on both lists of multiples.

Look for the least number found on both lists.

Answer  The least common multiple of 30 and 50 is 150.

The other method for finding the least common multiple is to use prime factorization. This is the method you need for working with rational expressions. The following shows how the factor method works with the numeric example, 4 and 6.

Start by finding the prime factorization of each denominator:

\[
4 = 2 \cdot 2 \\
6 = 3 \cdot 2
\]

Identify the greatest number of times any factor appears in either factorization and multiply those factors to get the least common multiple. For 4 and 6, it would be:

\[ 3 \cdot 2 \cdot 2 = 12 \]

Notice that 2 is included twice, because it appears twice in the prime factorization of 4. 12 is the least common multiple of 4 and 6.

The next example also shows how to use prime factorization.

Example

Problem  Find the least common multiple of 28 and 40.

<table>
<thead>
<tr>
<th>28 = 2 \cdot 2 \cdot 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write the prime factorization of 28.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>40= 2 \cdot 2 \cdot 2 \cdot 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Write the prime factorization of 40.</td>
</tr>
</tbody>
</table>

\[ 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 = 280 \]

Write the factors the greatest number of times they appear in either factorization and multiply.

Answer  The least common multiple of 28 and 40 is 280.
Self Check A
Find the least common multiple of 12 and 80.

Finding Least Common Denominators

You can use the least common multiple of two denominators as the least common denominator for those fractions. Then you rewrite each fraction using the same denominator.

The example below shows how to use the least common multiple as the least common denominator.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
<tr>
<td><strong>Rewrite the fractions</strong> (\frac{2}{3}) and (\frac{1}{2}) <strong>as fractions with a least common denominator.</strong></td>
</tr>
</tbody>
</table>
| **Multiples of 3:** 3, 6, 9, 12  
**Multiples of 2:** 2, 4, 6  
6 is the least common denominator. |
| **Find the least common multiple of the denominators. This is the least common denominator.** |
| \(\frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6}\)  
Rewrite \(\frac{2}{3}\) with a denominator of 6. |
| \(\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6}\)  
Rewrite \(\frac{1}{2}\) with a denominator of 6. |
| **Answer** |
| The fraction \(\frac{2}{3}\) can be rewritten as \(\frac{4}{6}\). |
| The fraction \(\frac{1}{2}\) can be rewritten as \(\frac{3}{6}\). |

Self Check B
Find the least common denominator of \(\frac{3}{4}\) and \(\frac{1}{6}\). Then express each fraction using the least common denominator.
To add fractions with unlike denominators, first rewrite them with like denominators. Then, you know what to do! The steps are shown below.

**Adding Fractions with Unlike Denominators**

1. Find a common denominator.
2. Rewrite each fraction using the common denominator.
3. Now that the fractions have a common denominator, you can add the numerators.
4. Simplify to lowest terms, expressing improper fractions as mixed numbers.

You can always find a common denominator by multiplying the two denominators together. See the example below.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
<tr>
<td><strong>Step</strong></td>
</tr>
<tr>
<td>3 ( \cdot ) 5 = 15</td>
</tr>
<tr>
<td>( \frac{2}{3} \cdot \frac{5}{5} = \frac{10}{15} )</td>
</tr>
<tr>
<td>( \frac{1}{3} \cdot \frac{3}{3} = \frac{3}{15} )</td>
</tr>
<tr>
<td>( \frac{10}{15} + \frac{3}{15} = \frac{13}{15} )</td>
</tr>
<tr>
<td><strong>Answer</strong></td>
</tr>
</tbody>
</table>

You can find a common denominator by finding the common multiples of the denominators. The least common multiple is the easiest to use.
### Example

**Problem**
\[
\frac{3}{7} + \frac{2}{21}
\]

#### Add. Simplify the answer.

Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 7 and 21.

- Multiples of 7: 7, 14, 21
- Multiples of 21: 21

\[
\frac{3}{7} \cdot \frac{3}{3} = \frac{9}{21}
\]

Rewrite each fraction with a denominator of 21.

\[
\frac{2}{21}
\]

\[
\frac{9}{21} + \frac{2}{21} = \frac{11}{21}
\]

Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.

**Answer**
\[
\frac{3}{7} + \frac{2}{21} = \frac{11}{21}
\]

You can also add more than two fractions as long as you first find a common denominator for all of them. An example of a sum of three fractions is shown below. In this example, you will use the prime factorization method to find the LCM.

### Example

**Problem**
\[
\frac{3}{4} + \frac{1}{6} + \frac{5}{8}
\]

#### Add. Simplify the answer and write as a mixed number.

- 4 = 2 • 2
- 6 = 3 • 2
- 8 = 2 • 2 • 2

LCM: 2 • 2 • 2 • 3 = 24

Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 4, 6 and 8.

\[
\frac{3}{4} \cdot \frac{6}{6} = \frac{18}{24}
\]

\[
\frac{1}{4} \cdot \frac{4}{4} = \frac{4}{24}
\]

\[
\frac{6}{4} \cdot \frac{3}{3} = \frac{18}{24}
\]

\[
\frac{5}{3} \cdot \frac{5}{5} = \frac{15}{24}
\]

\[
\frac{8}{3} \cdot \frac{4}{4} = \frac{24}{24}
\]

Rewrite each fraction with a denominator of 24.
Add the fractions by adding the numerators and keeping the denominator the same.

\[
\frac{18}{24} + \frac{4}{24} + \frac{15}{24} = \frac{37}{24}
\]

Write the improper fraction as a mixed number and simplify the fraction.

\[
\frac{37}{24} = 1\frac{13}{24}
\]

Answer

\[
\frac{3}{4} + \frac{1}{6} + \frac{5}{8} = 1\frac{13}{24}
\]

Self Check C

\[
\frac{2}{3} + \frac{4}{5} + \frac{1}{12}
\]

Add. Simplify the answer and write as a mixed number.

Adding Mixed Numbers

When adding mixed numbers you may also need to find a common denominator first. Consider the example below.

**Example**

<table>
<thead>
<tr>
<th>Problem</th>
<th>(\frac{8}{6} + \frac{7}{9})</th>
<th>Add. Simplify the answer and write as a mixed number.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\frac{5}{6})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{3}{9})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{4}{2})</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\frac{2}{9})</td>
<td></td>
</tr>
</tbody>
</table>

Multiples of 6: 6, 12, 18
Multiple of 9: 9, 18

Find a least common denominator for the fractions.

\[
\frac{5}{6} \cdot 3 = \frac{15}{18}
\]

Express each fraction with a denominator of 18.

\[
\frac{4}{2} = \frac{8}{18}
\]

Arrange the mixed numbers vertically so the whole numbers align and the fractions align.

\[
\frac{15}{18} + \frac{7}{18} = \frac{22}{18}
\]
Add whole numbers. Add fractions.

\[ \frac{8}{18} + \frac{7}{18} = \frac{15}{18} \]

Write the improper fraction as a mixed number.

\[ \frac{23}{18} = 1\frac{5}{18} \]

Combine whole numbers and fraction to write a mixed number.

\[ 15 + 1 + \frac{5}{18} = 16\frac{5}{18} \]

Answer

\[ \frac{5}{6} + \frac{4}{9} = \frac{16}{18} \]

**Self Check D**

\[ 3\frac{3}{5} + 1\frac{4}{9} \]

Add. Simplify the answer and write as a mixed number.

**Subtracting Fractions with Unlike Denominators**

If the denominators are not the same (they have **unlike denominators**), you must first rewrite the fractions with a common denominator. The **least common denominator**, which is the least common multiple of the denominators, is the most efficient choice, but any common denominator will do. Be sure to check your answer to be sure that it is in simplest form. You can use prime factorization to find the **least common multiple** (LCM), which will be the least common denominator (LCD). See the example below.
### Example

**Problem**
\[
\begin{array}{c}
\frac{1}{5} - \frac{1}{6} \\
\end{array}
\]

**Subtract. Simplify the answer.**

- \(5 \cdot 6 = 30\)
- The fractions have unlike denominators, so you need to find a common denominator. Recall that a common denominator can be found by multiplying the two denominators together.
- \(\frac{1 \cdot 6}{5 \cdot 6} = \frac{6}{30}\)
- Rewrite each fraction as an equivalent fraction with a denominator of 30.
- \(\frac{1 \cdot 5}{6 \cdot 5} = \frac{5}{30}\)
- \(\frac{6}{30} - \frac{5}{30} = \frac{1}{30}\)
- Subtract the numerators. Simplify the answer if needed.

**Answer**
\[
\begin{array}{c}
\frac{1}{5} - \frac{1}{6} = \frac{1}{30} \\
\end{array}
\]

The example below shows using multiples to find the least common multiple, which will be the least common denominator.

### Example

**Problem**
\[
\begin{array}{c}
\frac{5}{6} - \frac{1}{4} \\
\end{array}
\]

**Subtract. Simplify the answer.**

- Multiples of 6: 6, 12, 18, 24
- Multiples of 4: 4, 8, 12, 16, 20
- 12 is the least common multiple of 6 and 4.
- Find the least common multiple of the denominators – this is the least common denominator.
- \(\frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}\)
- Rewrite each fraction with a denominator of 12.
- \(\frac{1 \cdot 3}{4 \cdot 3} = \frac{3}{12}\)
- \(\frac{10}{12} - \frac{3}{12} = \frac{7}{12}\)
- Subtract the fractions. Simplify the answer if needed.

**Answer**
\[
\begin{array}{c}
\frac{5}{6} - \frac{1}{4} = \frac{7}{12} \\
\end{array}
\]
Self Check E
\[
\frac{2}{3} - \frac{1}{6}
\]
Subtract and simplify the answer.

Subtracting Mixed Numbers

Sometimes you have to find a common denominator in order to solve a mixed number subtraction problem.

Example

<table>
<thead>
<tr>
<th>Problem</th>
<th>Subtract. Simplify the answer and write as a mixed number.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7\frac{1}{2} - 2\frac{1}{3}]</td>
<td></td>
</tr>
</tbody>
</table>

\[
2 \cdot 3 = 6
\]
Recall that a common denominator can easily be found by multiplying the denominators together.

\[
1 \frac{3}{2} = \frac{3}{6}
\]
\[
1 \frac{2}{3} = \frac{2}{6}
\]
Rewrite each fraction using the common denominator 6.

\[
\frac{3}{6} - \frac{2}{6} = \frac{1}{6}
\]
Subtract the fractions.

\[
7 - 2 = 5
\]
Subtract the whole numbers.

\[
\frac{5}{6}
\]
Combine the whole number and the fraction.

Answer

\[
7\frac{1}{2} - 2\frac{1}{3} = 5\frac{1}{6}
\]

Self Check F

\[
9\frac{4}{5} - 4\frac{2}{3}
\]
Subtract. Simplify the answer and write it as a mixed number.
The regrouping approach shown in the last section will also work with unlike denominators.

**Example**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Subtract. Simplify the answer and write as a mixed number.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7 \frac{1}{5} - 3 \frac{1}{4})</td>
<td></td>
</tr>
</tbody>
</table>

Multiples of 5: 5, 10, 15, 20, 25
Multiples of 4: 4, 8, 12, 16, 20, 24

Find a least common denominator. 20 is the least common multiple, so use it for the least common denominator.

Rewrite each fraction using the common denominator.

\[\frac{1}{5} \cdot \frac{4}{4} = \frac{4}{20} \quad \text{and} \quad \frac{1}{4} \cdot \frac{5}{5} = \frac{5}{20}\]

Write the expression using the mixed numbers with the like denominator.

\[7 \frac{4}{20} - 3 \frac{5}{20}\]

Since the second fraction part, \(\frac{5}{20}\), is larger than the first fraction part, \(\frac{4}{20}\), regroup one of the whole numbers and write it as \(\frac{20}{20}\).

\[6 + \frac{20}{20} + \frac{4}{20} = \frac{24}{20}\]

Rewrite the subtraction expression using the equivalent fractions.

\[7 \frac{4}{20} - 3 \frac{5}{20}\]

Subtract the whole numbers, subtract the fractions.

\[6 - 3 = 3 \quad \frac{24}{20} - \frac{5}{20} = \frac{19}{20}\]

Combine the whole number and the fraction.

**Answer**

\[7 \frac{1}{5} - 3 \frac{1}{4} = 3 \frac{19}{20}\]
Self Check G

\[ 6 \frac{1}{2} - 2 \frac{5}{6} \]
Subtract. Simplify the answer and write as a mixed number.

---

Adding and Subtracting Fractions to Solve Problems

**Example**

**Problem**

A cake recipe requires \( 2 \frac{1}{4} \) cups of milk and \( 1 \frac{1}{2} \) cups of melted butter. If these are the only liquids, how much liquid is in the recipe?

\[ 2 \frac{1}{4} + 1 \frac{1}{2} \]

Find the total amount of liquid by adding the quantities.

\[ 2 + 1 + \frac{1}{4} + \frac{1}{2} \]

Group the whole numbers and fractions to make adding easier.

\[ 3 + \frac{1}{4} + \frac{1}{2} \]

Add whole numbers.

\[ 3 + \frac{1}{4} + \frac{2}{4} = 3 + \frac{3}{4} \]

Add fractions. Recall that \( \frac{1}{2} = \frac{2}{4} \).

\[ 3 \frac{3}{4} \]
Combine whole number and fraction.

**Answer**

There are \( 3 \frac{3}{4} \) cups of liquid in the recipe.

---

**Self Check H**

What is the total rainfall in a three-day period if it rains \( 3 \frac{1}{4} \) inches the first day, \( \frac{3}{8} \) inch the second day, and \( 2 \frac{1}{2} \) inches on the third day?
Example

Problem Pilar and Farouk are training for a marathon. On a recent Sunday, they both completed a run. Farouk ran $12\frac{7}{8}$ miles and Pilar ran $14\frac{3}{4}$ miles. How many more miles did Pilar run than Farouk?

\[
14\frac{3}{4} - 12\frac{7}{8} \quad \text{Write an expression using subtraction to describe the situation.}
\]

\[
14\frac{6}{8} - 12\frac{7}{8} \quad \text{Rewrite the mixed numbers using the least common denominator.}
\]

\[
14\frac{6}{8} = 13 + 1\frac{6}{8} \quad \text{Since the fraction part of the second mixed number is larger than the fraction part of the first mixed number, regroup one as a fraction and rewrite the first mixed number.}
\]

\[
13\frac{8}{8} + 6\frac{6}{8} \quad \text{8}
\]

\[
13\frac{14}{8} \quad \text{Write the subtraction expression in its new form.}
\]

\[
13\frac{14}{8} - 12\frac{7}{8} \quad \text{Subtract.}
\]

Answer Pilar ran $1\frac{7}{8}$ miles more than Farouk.
Example

**Problem**
Mike and Jose are painting a room. Jose used \( \frac{2}{3} \) of a can of paint and Mike used \( \frac{1}{2} \) of a can of paint. How much more paint did Jose use? Write the answer as a fraction of a can.

Write an expression using subtraction to describe the situation.

\[
\frac{2}{3} - \frac{1}{2}
\]

Rewrite the fractions using a common denominator.

\[
\frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6} \quad \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6}
\]

Subtract. Check that the fraction is simplified.

\[
\frac{4}{6} - \frac{3}{6} = \frac{1}{6}
\]

**Answer**
Jose used \( \frac{1}{6} \) of a can more paint than Mike.

**Self Check I**
Mariah’s sunflower plant grew \( 18\frac{2}{3} \) inches in one week. Her tulip plant grew \( 3\frac{3}{4} \) inches in one week. How many more inches did the sunflower grow in a week than the tulip?

**Summary**

To adding or subtracting fractions with unlike denominators, first find a common denominator. The least common denominator is easiest to use. The least common multiple can be used as the least common denominator.
2.6 Self Check Solutions

**Self Check A**
Find the least common multiple of 12 and 80.

240
2 • 2 • 2 • 2 • 3 • 5 = 240.

**Self Check B**
Find the least common denominator of \( \frac{3}{4} \) and \( \frac{1}{6} \). Then express each fraction using the least common denominator.

LCD: 12
\[
\frac{3}{4} \cdot \frac{3}{12} = \frac{9}{12}, \quad \frac{1}{6} \cdot \frac{2}{12} = \frac{2}{12}
\]

**Self Check C**
\[
\frac{2}{3} + \frac{4}{5} + \frac{1}{12} \quad \text{Add. Simplify the answer and write as a mixed number.}
\]
\[
\frac{40}{60} + \frac{48}{60} + \frac{5}{60} = \frac{93}{60} = 1 \frac{33}{60} = 1 \frac{11}{20}.
\]

**Self Check D**
\[
3 \frac{3}{5} + 1 \frac{4}{9} \quad \text{Add. Simplify the answer and write as a mixed number.}
\]
\[
3 + \frac{3}{5} + \frac{4}{9} + \frac{27}{45} + \frac{20}{45} = \frac{47}{45} = 4 + \frac{2}{45} = 5 \frac{2}{45}.
\]

**Self Check E**
\[
\frac{2}{3} - \frac{1}{6} \quad \text{Subtract and simplify the answer.}
\]
\[
\frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.
\]
Self Check F

9 \frac{4}{5} - 4 \frac{2}{3}

Subtract. Simplify the answer and write it as a mixed number.

9 − 4 = 5; \quad \frac{4}{5} - \frac{2}{3} = \frac{12}{15} - \frac{10}{15} = \frac{2}{15}. \quad \text{Combining them gives } 5 \frac{2}{15}.

Self Check G

6 \frac{1}{2} - 2 \frac{5}{6}

Subtract. Simplify the answer and write it as a mixed number.

6 \frac{1}{2} - 2 \frac{5}{6} = 6 \frac{3}{6} - 2 \frac{5}{6}. \quad \text{Regrouping, } 5 \frac{9}{6} - 2 \frac{5}{6} = 3 \frac{4}{6} = 3 \frac{2}{3}

Self Check H

What is the total rainfall in a three-day period if it rains 3 \frac{1}{4} inches the first day, 3 \frac{3}{8} inch the second day, and 2 \frac{1}{2} inches on the third day?

6 \frac{1}{8} \quad \text{inches}

3 \frac{2}{8} + 3 \frac{3}{8} + 2 \frac{4}{8} = 5 \frac{9}{8} = 6 \frac{1}{8}

Self Check I

Mariah’s sunflower plant grew 18 \frac{2}{3} inches in one week. Her tulip plant grew 3 \frac{3}{4} inches in one week. How many more inches did the sunflower grow in a week than the tulip?

18 \frac{2}{3} - 3 \frac{3}{4} = 18 \frac{8}{12} - 3 \frac{9}{12} = 17 \frac{20}{12} - 3 \frac{9}{12} = 14 \frac{11}{12} \quad \text{inches.}