4.2 Proportions

Learning Objective(s)

1 Determine whether a proportion is true or false.
2 Find an unknown in a proportion.
3 Solve application problems using proportions.
4 Solve application problems using similar triangles.

Introduction

A true **proportion** is an equation that states that two **ratios** are equal. If you know one ratio in a proportion, you can use that information to find values in the other equivalent ratio. Using proportions can help you solve problems such as increasing a recipe to feed a larger crowd of people, creating a design with certain consistent features, or enlarging or reducing an image to scale.

For example, imagine you want to enlarge a 5-inch by 8-inch photograph to fit a wood frame that you purchased. If you want the shorter edge of the enlarged photo to measure 10 inches, how long does the photo have to be for the image to scale correctly? You can set up a proportion to determine the length of the enlarged photo.

Determining Whether a Proportion Is True or False

Objective 1

A proportion is usually written as two equivalent fractions. For example:

$$\frac{12\text{ inches}}{1\text{ foot}}=\frac{36\text{ inches}}{3\text{ feet}}$$

Notice that the equation has a ratio on each side of the equal sign. Each ratio compares the same units, inches and feet, and the ratios are equivalent because the units are consistent, and $\frac{12}{1}$ is equivalent to $\frac{36}{3}$.

Proportions might also compare two ratios with the same units. For example, Juanita has two different-sized containers of lemonade mix. She wants to compare them. She could set up a proportion to compare the number of ounces in each container to the number of servings of lemonade that can be made from each container.

$$\frac{40\text{ ounces}}{84\text{ ounces}}=\frac{10\text{ servings}}{21\text{ servings}}$$

Since the units for each ratio are the same, you can express the proportion without the units:

$$\frac{40}{84}=\frac{10}{21}$$

When using this type of proportion, it is important that the numerators represent the same situation – in the example above, 40 ounces for 10 servings – and the denominators represent the same situation, 84 ounces for 21 servings.

Juanita could also have set up the proportion to compare the ratios of the container sizes to the number of servings of each container.

$$\frac{40 \text{ ounces}}{10 \text{ servings}} = \frac{84 \text{ ounces}}{21 \text{ servings}}$$

Sometimes you will need to figure out whether two ratios are, in fact, a true or false proportion. Below is an example that shows the steps of determining whether a proportion is true or false.

Example		
Problem	**Is the proportion true or false?** $\frac{100 \text{ miles}}{4 \text{ gallons}} = \frac{50 \text{ miles}}{2 \text{ gallons}}$	
	miles	The units are consistent across the numerators.
	gallons	The units are consistent across the denominators.
	$\frac{100 \div 4}{4 \div 4} = \frac{25}{1}$	Write each ratio in simplest form.
	$\frac{50 \div 2}{2 \div 2} = \frac{25}{1}$	
	$\frac{25}{1} = \frac{25}{1}$	Since the simplified fractions are equivalent, the proportion is true.
Answer	The proportion is true.	

Identifying True Proportions

To determine if a proportion compares equal ratios or not, you can follow these steps.

1. Check to make sure that the units in the individual ratios are consistent either vertically or horizontally. For example, $\frac{\text{miles}}{\text{hour}} = \frac{\text{miles}}{\text{hour}}$ or $\frac{\text{miles}}{\text{miles}} = \frac{\text{hour}}{\text{hour}}$ are valid setups for a proportion.
2. Express each ratio as a simplified fraction.
3. If the simplified fractions are the same, the proportion is *true*; if the fractions are different, the proportion is *false*.

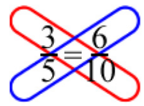
Sometimes you need to create a proportion before determining whether it is true or not. An example is shown below.

Example		
Problem	**One office has 3 printers for 18 computers. Another office has 20 printers for 105 computers. Is the ratio of printers to computers the same in these two offices?**	
	$\frac{\text{printers}}{\text{computers}} = \frac{\text{printers}}{\text{computers}}$	Identify the relationship.
	$\frac{3\text{ printers}}{18\text{ computers}} = \frac{20\text{ printers}}{105\text{ computers}}$	Write ratios that describe each situation, and set them equal to each other.
	printers	Check that the units in the numerators match.
	computers	Check that the units in the denominators match.
	$\frac{3 \div 3}{18 \div 3} = \frac{1}{6}$ $\frac{20 \div 5}{105 \div 5} = \frac{4}{21}$	Simplify each fraction and determine if they are equivalent.
	$\frac{1}{6} \neq \frac{4}{21}$	Since the simplified fractions are not equal (designated by the $\neq$ sign), the proportion is not true.
Answer	The ratio of printers to computers is **not** the same in these two offices.	

There is another way to determine whether a proportion is true or false. This method is called "finding the cross product" or "cross multiplying".

To cross multiply, you multiply the numerator of the first ratio in the proportion by the denominator of the other ratio. Then multiply the denominator of the first ratio by the numerator of the second ratio in the proportion. If these products are equal, the proportion is true; if these products are not equal, the proportion is not true.

This strategy for determining whether a proportion is true is called cross-multiplying because the pattern of the multiplication looks like an "x" or a criss-cross. Below is an example of finding a cross product, or cross multiplying.



In this example, you multiply 3 • 10 = 30, and then multiply 5 • 6 = 30. Both products are equal, so the proportion is true.

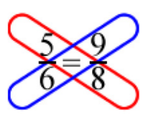
To see why this works, let's start with a true proportion: $\frac{4}{8}=\frac{5}{10}$. If we multiplied both sides by 10, we'd get $10\bullet\frac{4}{8}=\frac{5}{10}\bullet 10$. The right side of this equation would simplify to 5, leaving $10\bullet\frac{4}{8}=5$. Now if we multiplied both sides by 8, we'd get $10\bullet\frac{4}{8}\bullet 8=5\bullet 8$, and the left side would simplify to $10\bullet 4=5\bullet 8$. Notice this is the same equation we would get by cross-multiplying, so cross-multiplying is just a quick way to do these operations.

Below is another example of determining if a proportion is true or false by using cross products.

Example

Problem **Is the proportion true or false?**

$$\frac{5}{6}=\frac{9}{8}$$



Identify the cross product relationship.

5 • 8 = 40
6 • 9 = 54

Use cross products to determine if the proportion is true or false.

$40 \neq 54$

Since the products are not equal, the proportion is false.

Answer The proportion is false.

Self Check A

Is the proportion $\frac{3}{5}=\frac{24}{40}$ true or false?

Finding an Unknown Quantity in a Proportion

Objective 2

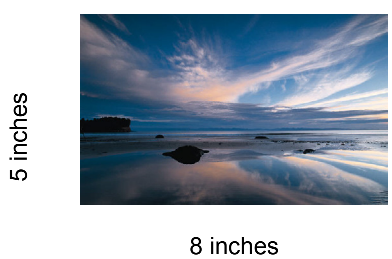
If you know that the relationship between quantities is proportional, you can use proportions to find missing quantities. Below is an example.

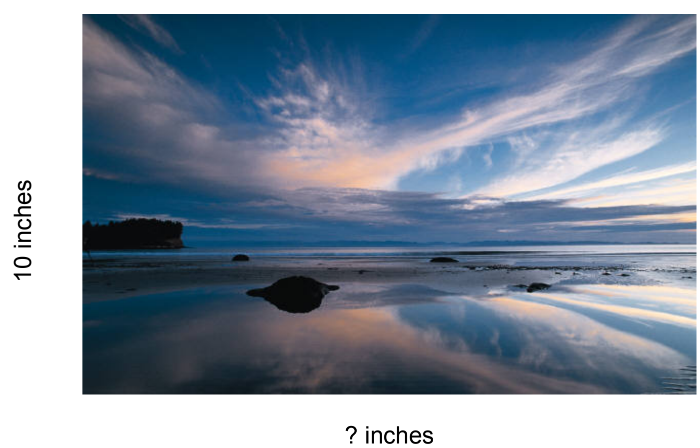
Example	
Problem **Solve for the unknown quantity, *n*.** $$\frac{n}{4} = \frac{25}{20}$$	
$20 \cdot n = 4 \cdot 25$	Cross multiply.
$20n = 100$	You are looking for a number that when you multiply it by 20 you get 100.
$20\overline{)100}$ = 5	You can find this value by dividing 100 by 20.
$n = 5$	
Answer $n = 5$	

Self Check B

Solve for the unknown quantity, *x*. $\frac{15}{x} = \frac{6}{10}$

Now back to the original example. Imagine you want to enlarge a 5-inch by 8-inch photograph to make the length 10 inches and keep the proportion of the width to length the same. You can set up a proportion to determine the width of the enlarged photo.





Example	
Problem	**Find the length of a photograph whose width is 10 inches and whose proportions are the same as a 5- inch by 8-inch photograph.**
$\frac{\text{width}}{\text{length}}$	Determine the relationship.
Original photo: $\frac{5\text{ inches wide}}{8\text{ inches long}}$	Write a ratio that compares the length to the width of each photograph.
Enlarged photo: $\frac{10\text{ inches wide}}{n\text{ inches long}}$	Use a letter to represent the quantity that is not known (the width of the enlarged photo).
$\frac{5}{8}=\frac{10}{n}$	Write a proportion that states that the two ratios are equal.
$5 \bullet n = 8 \bullet 10$	Cross multiply.
$5n = 80$	You are looking for a number that when it is multiplied by 5 will give you 80.
$\frac{5n}{5}=\frac{80}{5}$ $n=\frac{80}{5}$	Divide both sides by 5 to isolate the variable. $5\overline{)80}$ = 16
$n = 16$	
Answer	The length of the enlarged photograph is 16 inches.

Solving Application Problems Using Proportions

Objective **3**

Setting up and solving a proportion is a helpful strategy for solving a variety of proportional reasoning problems. In these problems, it is always important to determine what the unknown value is, and then identify a proportional relationship that you can use to solve for the unknown value. Below are some examples.

Example	
Problem	**Among a species of tropical birds, 30 out of every 50 birds are female. If a certain bird sanctuary has a population of 1,150 of these birds, how many of them would you expect to be female?**
Let x = the number of female birds in the sanctuary.	Determine the unknown item: the number of female birds in the sanctuary. Assign a letter to this unknown quantity.
$\frac{30 \text{ female birds}}{50 \text{ birds}} = \frac{x \text{ female birds in sanctuary}}{1{,}150 \text{ birds in sanctuary}}$	Set up a proportion setting the ratios equal.
$\frac{30 \div 10}{50 \div 10} = \frac{3}{5}$	Simplify the ratio on the left to make the upcoming cross multiplication easier.
$\frac{3}{5} = \frac{x}{1{,}150}$	
$3 \cdot 1{,}150 = 5 \cdot x$ $3{,}450 = 5x$	Cross multiply.
$5\overline{)3{,}450}$ = 690 x = 690 birds	What number when multiplied by 5 gives a product of 3,450? You can find this value by dividing 3,450 by 5.
Answer	You would expect 690 birds in the sanctuary to be female.

Example	
Problem	**It takes Sandra 1 hour to word process 4 pages. At this rate, how long will she take to complete 27 pages?**
$\frac{4 \text{ pages}}{1 \text{ hour}} = \frac{27 \text{ pages}}{x \text{ hours}}$	Set up a proportion comparing the pages she types and the time it takes to type them.
$4 \cdot x = 1 \cdot 27$	Cross multiply.
$4x = 27$	You are looking for a number that when it is multiplied by 4 will give you 27.
$4\overline{)27.00}$ = 6.75 x = 6.75 hours	You can find this value by dividing 27 by 4.
Answer	It will take Sandra 6.75 hours to complete 27 pages.

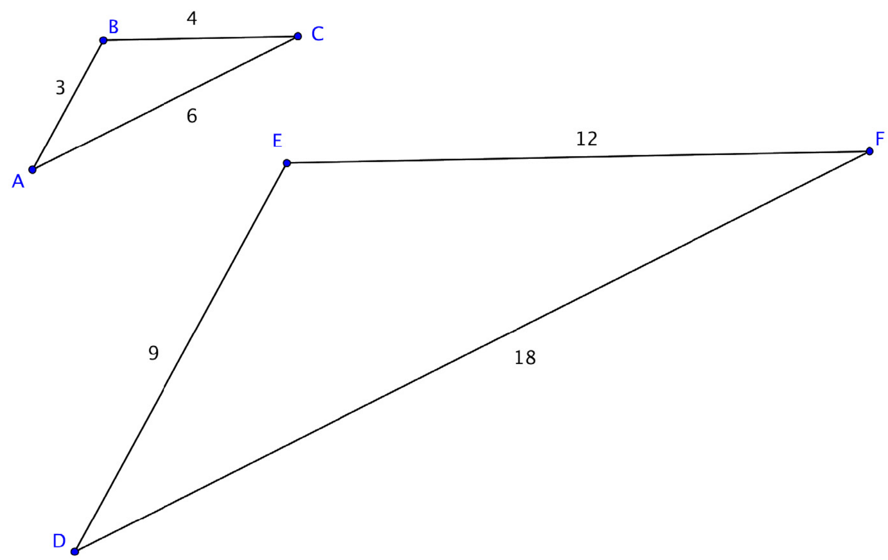
Self Check C

A map uses a scale where 2 inches represents 5 miles. If the distance between two cities is shown on a map as 20 inches, how many miles apart are the two cities?

Solving Application Problems Using Similar Triangles

Objective **4**

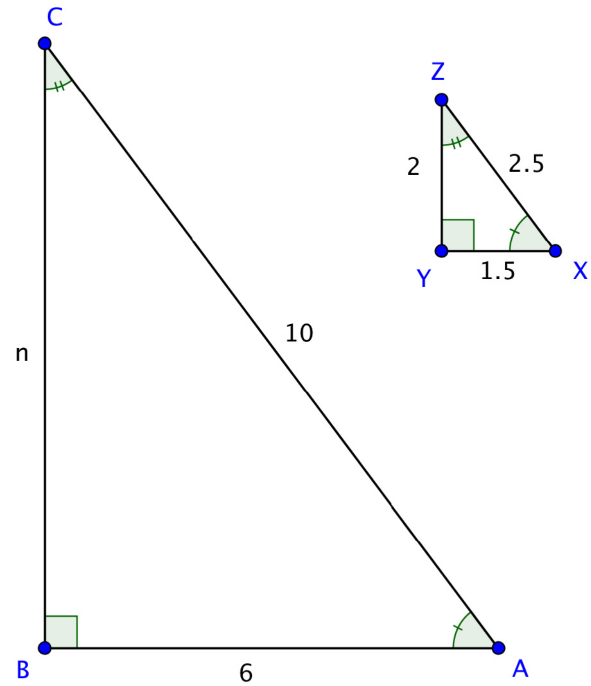
In the photograph problem from earlier, we created an enlargement of the picture, and both the width and height scaled proportionally. We would call the two rectangles **similar**. With triangles, we say two triangles are **similar triangles** if the ratios of the pairs of corresponding sides are equal sides. Consider the two triangles below.



We see that side AB corresponds with side DE and so on, and we can see that each of the ratios of corresponding sides are equal: $\frac{9}{3}=\frac{12}{4}=\frac{18}{6}$, so these triangles are similar. If two triangles have the same angles, then they will also be similar.

You can find the missing measurements in a triangle if you know some measurements of a similar triangle. Let's look at an example.

Example	
Problem	**ΔABC and ΔXYZ are similar triangles. What is the length of side *BC*?**



$$\frac{BC}{YZ} = \frac{AB}{XY}$$

In similar triangles, the ratios of corresponding sides are proportional. Set up a proportion of two ratios, one that includes the missing side.

$$\frac{n}{2} = \frac{6}{1.5}$$

Substitute in the known side lengths for the side names in the ratio. Let the unknown side length be *n.*

$$2 \bullet 6 = 1.5 \bullet n$$

Solve for *n* using cross multiplication.

$$12 = 1.5n$$

$$8 = n$$

Answer The missing length of side *BC* is 8 units.

This process is fairly straightforward—but be careful that your ratios represent corresponding sides, recalling that corresponding sides are opposite corresponding angles.

Applying knowledge of triangles, similarity, and congruence can be very useful for solving problems in real life. Just as you can solve for missing lengths of a triangle drawn on a page, you can use triangles to find unknown distances between locations or objects.

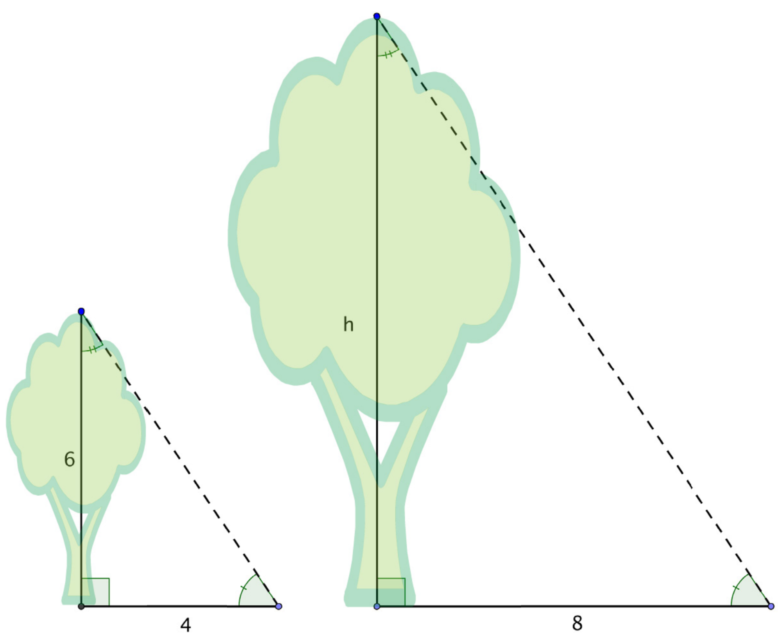
Let’s consider the example of two trees and their shadows. Suppose the sun is shining down on two trees, one that is 6 feet tall and the other whose height is unknown. By measuring the length of each shadow on the ground, you can use triangle similarity to find the unknown height of the second tree.

First, let’s figure out where the triangles are in this situation! The trees themselves create one pair of corresponding sides. The shadows cast on the ground are another pair of corresponding sides. The third side of these imaginary similar triangles runs from the top of each tree to the tip of its shadow on the ground. This is the hypotenuse of the triangle.

If you know that the trees and their shadows form similar triangles, you can set up a proportion to find the height of the tree.

Example

Problem **When the sun is at a certain angle in the sky, a 6-foot tree will cast a 4-foot shadow. How tall is a tree that casts an 8-foot shadow?**



$$\frac{\text{Tree 1}}{\text{Tree 2}} = \frac{\text{Shadow 1}}{\text{Shadow 2}}$$

The angle measurements are the same, so the triangles are similar triangles. Since they are similar triangles, you can use proportions to find the size of the missing side.

Set up a proportion comparing the heights of the trees and the lengths of their shadows.

$$\frac{6}{h} = \frac{4}{8}$$

Substitute in the known lengths. Call the missing tree height *h.*

$$6 \bullet 8 = 4h$$ Solve for h using cross-multiplication.

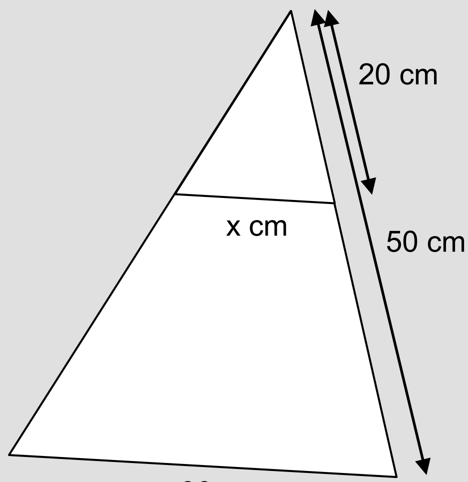
$$48 = 4h$$

$$12 = h$$

Answer The tree is 12 feet tall.

Self Check D

Find the unknown side.



Summary

A proportion is an equation comparing two ratios. If the ratios are equivalent, the proportion is true. If not, the proportion is false. Finding a cross product is another method for determining whether a proportion is true or false. Cross multiplying is also helpful for finding an unknown quantity in a proportional relationship. Setting up and solving proportions is a skill that is useful for solving a variety of problems.

4.2 Self Check Solutions

Self Check A

Is the proportion $\frac{3}{5} = \frac{24}{40}$ true or false?

True

Using cross products, you find that 3 • 40 = 120 and 5 • 24 = 120, so the cross products are equal and the proportion is true.

Self Check B

Solve for the unknown quantity, x. $\frac{15}{x} = \frac{6}{10}$

Cross-multiplying, you get the equation $6x = 150$. Dividing, you find $x = 25$.

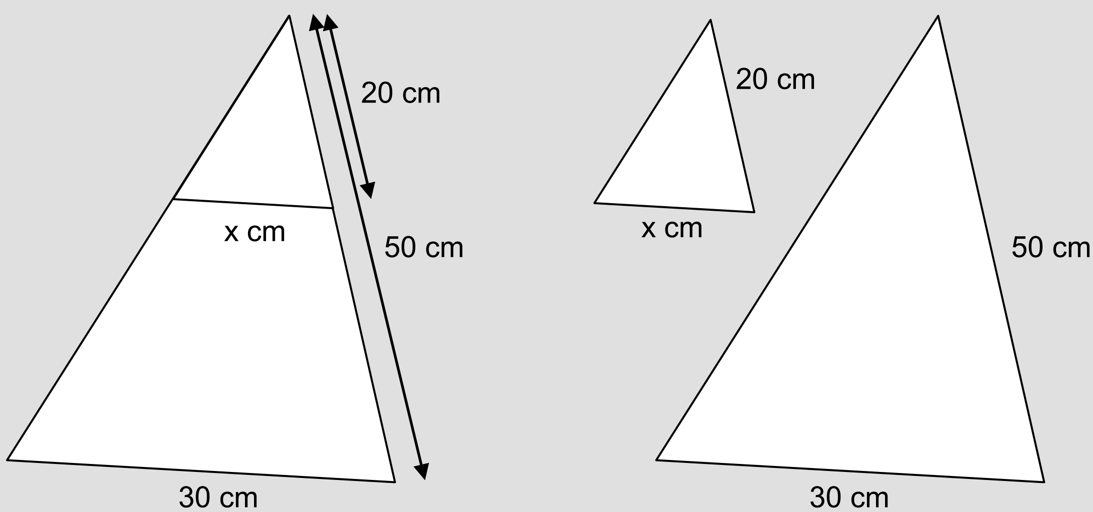
Self Check C

A map uses a scale where 2 inches represents 5 miles. If the distance between two cities is shown on a map as 20 inches, how many miles apart are the two cities?

Setting up the proportion $\frac{\text{2 inches}}{\text{5 miles}} = \frac{\text{20 inches}}{x}$, you find that x = 50 miles.

Self Check D

Find the unknown side.



To see the similar triangles, it may be helpful to split apart the picture, as shown to the right above. Setting up the proportion $\frac{\text{20 cm}}{50\ cm} = \frac{x\text{ cm}}{\text{30 cm}}$, you find x = 12 cm.