

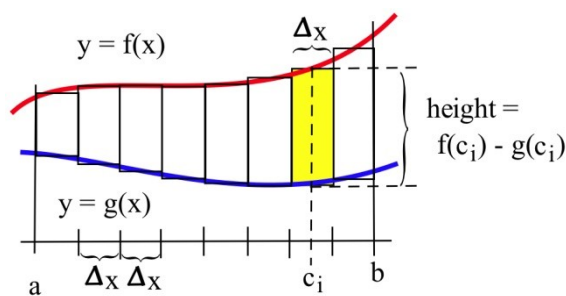
## Section 6: Area, Volume, and Average Value

### Area

We have already used integrals to find the area between the graph of a function and the horizontal axis. Integrals can also be used to find the area between two graphs.

If  $f(x) \geq g(x)$  for all  $x$  in  $[a,b]$ , then we can approximate the area between  $f$  and  $g$  by partitioning the interval  $[a,b]$  and forming a Riemann sum, as shown in the picture. The height of each rectangle is top – bottom,  $f(c_i) - g(c_i)$  so the area of the  $i^{\text{th}}$  rectangle is

(height)•(base) =  $(f(c_i) - g(c_i)) \cdot \Delta x$ . Adding up this rectangles gives an approximation of the total area as  $\sum_{i=1}^n (f(c_i) - g(c_i)) \Delta x$ , a Riemann sum.



The limit of this Riemann sum, as the number of rectangles gets larger and their width gets smaller, is the definite integral  $\int_a^b (f(x) - g(x)) dx$ .

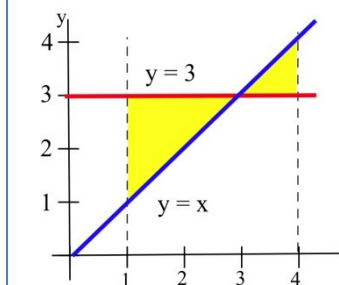
**The area between two curves  $f(x)$  and  $g(x)$ , where  $f(x) \geq g(x)$ , between  $x = a$  and  $x = b$  is**

$$\int_a^b (f(x) - g(x)) dx$$

The integrand is “top – bottom.” Make a graph to be sure which curve is which.

### Example 1

Find the area bounded between the graphs of  $f(x) = x$  and  $g(x) = 3$  for  $1 \leq x \leq 4$ .



Always start with a graph so you can see which graph is the top and which is the bottom. In this example, the two curves cross, and they change positions; we'll need to split the area into two pieces. Geometrically, we can see that the area is  $2 + \frac{1}{2} = 2.5$ .

Writing the area as a sum of definite integrals, we get:

$$\text{Area} = \int_1^3 (3-x)dx + \int_3^4 (x-3)dx$$

These integrals are easy to evaluate using antiderivatives:

$$\int_1^3 (3-x)dx = \left(3x - \frac{x^2}{2}\right)_1^3 = \left(\left(9 - \frac{9}{2}\right) - \left(3 - \frac{3}{2}\right)\right) = 2.$$

$$\int_3^4 (x-3)dx = \left(\frac{x^2}{2} - 3x\right)_3^4 = \left(\left(\frac{16}{2} - 12\right) - \left(\frac{9}{2} - 9\right)\right) = \frac{1}{2}.$$

The two integrals also tell us that the total area between  $f$  and  $g$  is 2.5 square units, which we already knew.

Note that the single integral  $\int_1^4 (3-x)dx = 1.5$  is not the **area** we want in the last example. The value of the **integral is 1.5**, and the value of the **area is 2.5**. That's because for the triangle on the right, the graph of  $y = x$  is above the graph of  $y = 3$ , so the integrand  $3 - x$  is negative; in the definite integral, the area of that triangle comes in with a negative sign.

In this example, it was easy to see exactly where the two curves crossed so we could break the region into the two pieces to figure separately. In other examples, you might need to solve an equation to find where the curves cross.

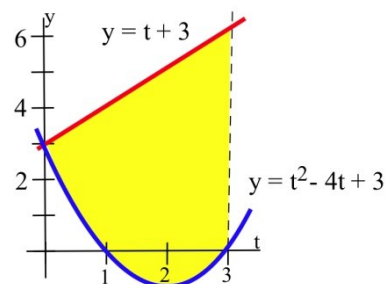
### Example 2

Two objects start from the same location and travel along the same path with velocities  $v_A(t) = t + 3$  and  $v_B(t) = t^2 - 4t + 3$  meters per second. How far ahead is A after 3 seconds?

Since  $v_A(t) \geq v_B(t)$ , the "area" between the graphs of  $v_A(t)$  and  $v_B(t)$  represents the distance between the objects.

After 3 seconds, the distance apart

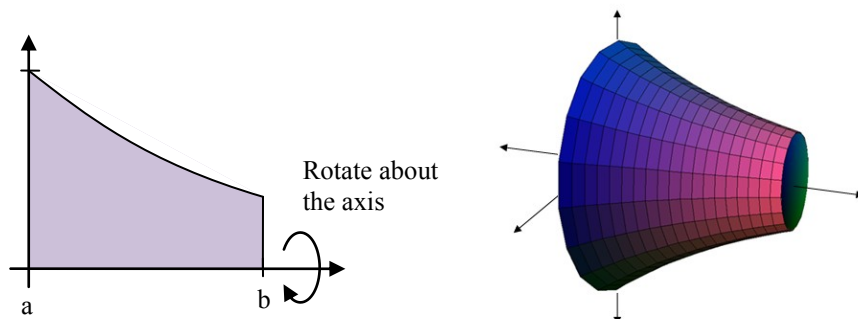
$$\begin{aligned} &= \int_0^3 (v_A(t) - v_B(t))dt = \int_0^3 ((t+3) - (t^2 - 4t + 3))dt = \int_0^3 (5t - t^2)dt \\ &= \left(\frac{5}{2}t^2 - \frac{t^3}{3}\right)_0^3 = \left(\frac{5}{2} \cdot 9 - \frac{27}{3}\right) - (0) = 13.5 \text{ meters.} \end{aligned}$$



## Volume

Just as we can partition an interval and imagine approximating an area with rectangles to find a formula for the area between curves, we can partition an interval and imagine approximating a volume with simple shapes to find a formula for the volume of a solid. While this approach works for a variety of shapes, our focus will be on shapes formed by revolving a curve around the horizontal axis.

We start with an area, the region below a function on the interval  $a \leq x \leq b$ . We are going to take that region, and rotate it around the  $x$  axis, creating the solid shape shown.



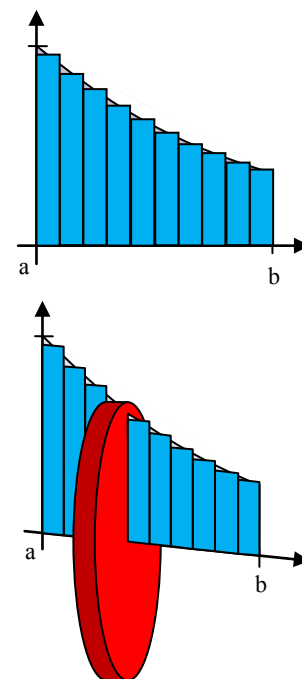
To find the volume of this solid, we can start by partitioning the interval  $[0,1]$  and approximating the area with rectangles. As before, the width of each rectangle would be  $\Delta x$  and the height  $f(c_i)$ .

If we took just one of these rectangles and rotated it about the horizontal axis, it would form a cylindrical shape. The radius of that cylinder would be  $f(c_i)$ , so the volume would be  $V = \pi r^2 h = \pi (f(c_i))^2 \Delta x$

The volume of the whole solid could be approximated by rotating each of the rectangles about the  $x$  axis. Adding up the volume of each of the little cylindrical discs gives an approximation of the total volume as

$\sum_{i=1}^n \pi (f(c_i))^2 \Delta x$ , a Riemann sum.

The limit of this sum as the width of the rectangles becomes small is the definite integral  $\int_a^b \pi (f(x))^2 dx$ .



**The volume of the solid obtained by rotating about the  $x$ -axis the area bounded by the curve  $f(x)$ , the  $x$ -axis,  $x = a$ , and  $x = b$  is**

$$\int_a^b \pi (f(x))^2 dx$$

**Example 3**

Find the volume of the solid formed by rotating the area under  $f(x) = e^{-x}$  on the interval  $[0,1]$  about the  $x$ -axis.

This is the region pictured in the earlier example. We substitute in the function and bounds into the formula we derived to set up the definite integral.

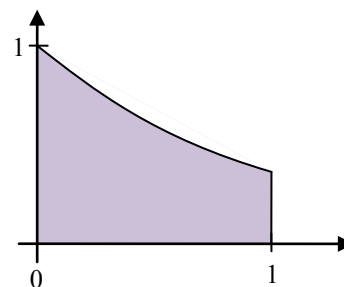
$$\text{Volume} = \int_0^1 \pi(e^{-x})^2 dx$$

Using exponent rules, the integrand can be simplified. The constant  $\pi$  can be pulled out of the integral.

$$\pi \int_0^1 e^{-2x} dx$$

Using the substitution  $u = -2x$ , we can integrate this function.

$$\pi \int_0^1 e^{-2x} dx = -\frac{1}{2} \pi e^{-2x} \Big|_0^1 = \left( -\frac{1}{2} \pi e^{-2(1)} \right) - \left( -\frac{1}{2} \pi e^{-2(0)} \right) \approx 1.358 \text{ cubic units}$$

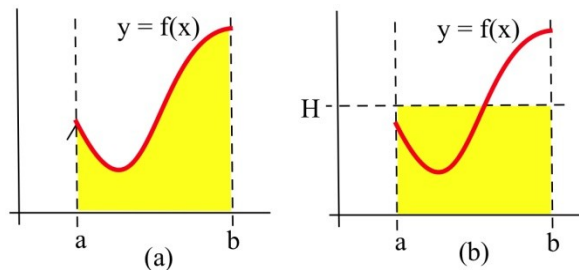
**Average Value**

We know the average of  $n$  numbers,  $a_1, a_2, \dots, a_n$ , is their sum divided by  $n$ . But what if we need to find the average temperature over a day's time -- there are too many possible temperatures to add them up. This is a job for the definite integral.

**The average value of a function  $f(x)$  on the interval  $[a, b]$  is given by**

$$\frac{1}{b-a} \int_a^b f(x) dx$$

The average value of a positive  $f$  has a nice geometric interpretation. Imagine that the area under  $f$  (Fig. a) is a liquid that can "leak" through the graph to form a rectangle with the same area (Fig. b).



If the height of the rectangle is  $H$ , then the area of the rectangle is  $H \cdot (b-a)$ . We know the area of the rectangle is the same as the area under  $f$  so  $H \cdot (b-a) = \int_a^b f(x) dx$ . Then  $H = \frac{1}{b-a} \int_a^b f(x) dx$ , the average value of  $f$  on  $[a,b]$ .

The average value of a positive function  $f$  is the height  $H$  of the rectangle whose area is the same as the area under  $f$ .

### Example 4

During a 9 hour work day, the production rate at time  $t$  hours after the start of the shift was given by the function  $r(t) = 5 + \sqrt{t}$  cars per hour. Find the average hourly production rate.

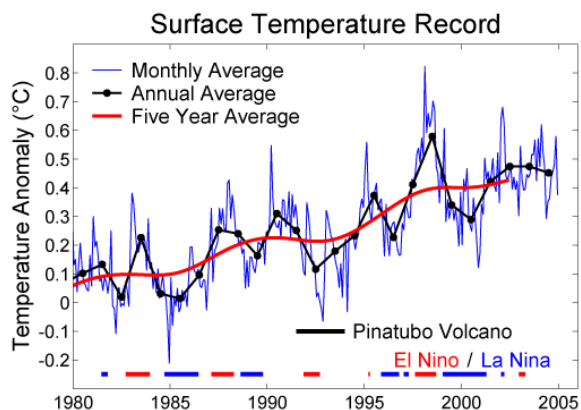
The average hourly production is  $\frac{1}{9-0} \int_0^9 (5 + \sqrt{t}) dt = 7$  cars per hour.

A note about the units – remember that the definite integral has units (cars per hour) · (hours) = cars. But the  $1/(b-a)$  in front has units 1/hours – the units of the average value are cars per hour, just what we expect an average rate to be.

**In general, the average value of a function will have the same units as the integrand.**

Function averages, involving means and more complicated averages, are used to "smooth" data so that underlying patterns are more obvious and to remove high frequency "noise" from signals. In these situations, the original function  $f$  is replaced by some "average of  $f$ ." If  $f$  is rather jagged time data, then the ten year average of  $f$  is the integral  $g(x) = \frac{1}{10} \int_{x-5}^{x+5} f(t) dt$ , an average

of  $f$  over 5 units on each side of  $x$ . For example, the figure here<sup>1</sup> shows the graphs of a Monthly Average (rather "noisy" data) of surface temperature data, an Annual Average (still rather "jagged"), and a Five Year Average (a much smoother function). Typically the average function reveals the pattern much more clearly than the original data. This use of a "moving average" value of "noisy" data (weather information, stock prices) is a very common.



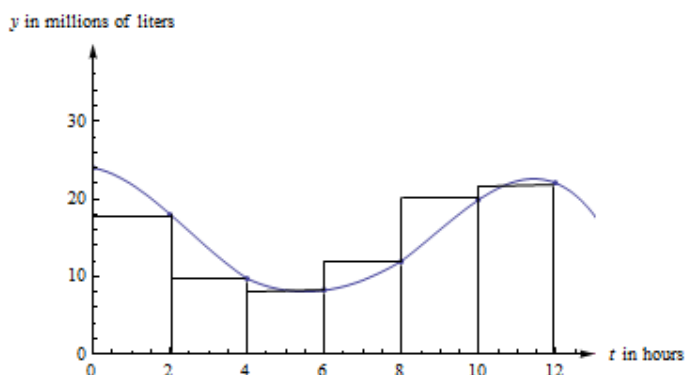
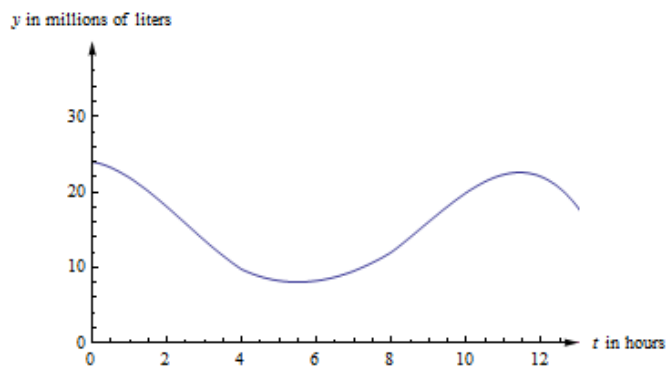
### Example 5

<sup>1</sup> [http://commons.wikimedia.org/wiki/File:Short\\_Instrumental\\_Temperature\\_Record.png](http://commons.wikimedia.org/wiki/File:Short_Instrumental_Temperature_Record.png), CC-BY

The graph to the right shows the amount of water in a reservoir over a 12 hour period. Estimate the average amount of water in the reservoir over this period.

If  $V(t)$  is the volume of the water (in millions of liters) after  $t$  hours, then the average amount is  $\frac{1}{12} \int_0^{12} V(t) dt$ . In order

to find the definite integral, we'll have to estimate. I'll use 6 rectangles, and I'll take the heights from their right edges.



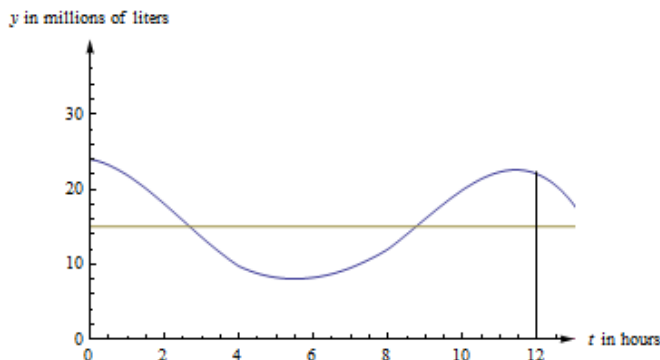
My estimate of the integral is

$$\int_0^{12} V(t) dt \cong (18)(2) + (9.7)(2) + (8.2)(2) + (12)(2) + (19.9)(2) + (22)(2) = 179.6.$$

The units of this integral are millions of liters  $\times$  feet. So my estimate of the average volume is  $\frac{179.6}{12} \cong 15$  millions of liters. Your estimate might be a little different.

In the figure below, you can see the same graph with the line  $y = 15$  drawn in. The area under the curve and the area under the rectangle are (approximately) the same.

In fact, that would be a different way to estimate the average value. We could have estimated the placement of the horizontal line so that the area under the curve and under the line were equal.



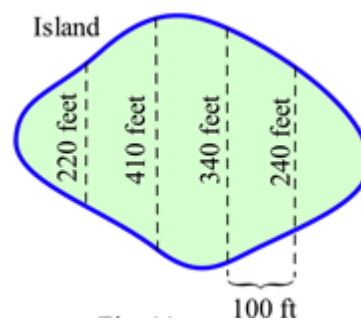
### 3.6 Exercises

In problems 1 – 4, use the values in the table to estimate the areas.

$x$	$f(x)$	$g(x)$	$h(x)$
0	5	2	5
1	6	1	6
2	6	2	8
3	4	2	6
4	3	3	5
5	2	4	4
6	2	0	2

1. Estimate the area between  $f$  and  $g$ , between  $x = 0$  and  $x = 4$ .
2. Estimate the area between  $g$  and  $h$ , between  $x = 0$  and  $x = 6$ .
3. Estimate the area between  $f$  and  $h$ , between  $x = 0$  and  $x = 4$ .
4. Estimate the area between  $f$  and  $g$ , between  $x = 0$  and  $x = 6$ .

5. Estimate the area of the island shown



In problems 6 – 15, find the **area** between the graphs of  $f$  and  $g$  for  $x$  in the given interval. Remember to draw the graph!

6.  $f(x) = x^2 + 3$ ,  $g(x) = 1$  and  $-1 \leq x \leq 2$ .
7.  $f(x) = x^2 + 3$ ,  $g(x) = 1 + x$  and  $0 \leq x \leq 3$ .
8.  $f(x) = x^2$ ,  $g(x) = x$  and  $0 \leq x \leq 2$ .
9.  $f(x) = (x - 1)^2$ ,  $g(x) = x + 1$  and  $0 \leq x \leq 3$ .
10.  $f(x) = \frac{1}{x}$ ,  $g(x) = x$  and  $1 \leq x \leq e$ .
11.  $f(x) = \sqrt{x}$ ,  $g(x) = x$  and  $0 \leq x \leq 4$ .
12.  $f(x) = 4 - x^2$ ,  $g(x) = x + 2$  and  $0 \leq x \leq 2$ .
13.  $f(x) = e^x$ ,  $g(x) = x$  and  $0 \leq x \leq 2$ .
14.  $f(x) = 3$ ,  $g(x) = \sqrt{1 - x^2}$  and  $0 \leq x \leq 1$ .
15.  $f(x) = 2$ ,  $g(x) = \sqrt{4 - x^2}$  and  $-2 \leq x \leq 2$ .

For problems 16-18, find the volume of the solid obtained by rotating the specified region about the  $x$  axis.

16. Region under  $f(x) = x^2 + 3$  for  $-1 \leq x \leq 2$ .

17. Region under  $f(x) = 4 - x^2$  for  $0 \leq x \leq 2$ .

18. Region under  $f(x) = \frac{1}{x}$  for  $1 \leq x \leq 2$ .

In problems 19 and 20 use the values in the table to estimate the average values.

$x$	$f(x)$	$g(x)$
0	5	2
1	6	1
2	6	2
3	4	2
4	3	3
5	2	4
6	2	0

19. Estimate the average value of  $f$  on the interval  $[0, 6]$ .

20. Estimate the average value of  $g$  on the interval  $[0, 6]$ .

In problems 21 – 26, find the **average value** of  $f$  on the given interval.

21.  $f(x)$  from the graph for  $0 \leq x \leq 2$ .

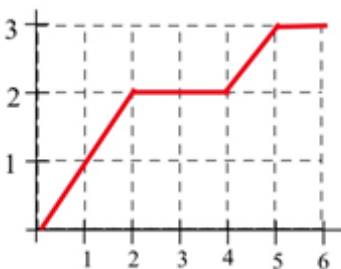
22.  $f(x)$  from the graph for  $0 \leq x \leq 4$ .

23.  $f(x)$  from the graph for  $1 \leq x \leq 6$ .

24.  $f(x)$  from the graph for  $4 \leq x \leq 6$ .

25.  $f(x) = 2x + 1$  for  $0 \leq x \leq 4$ .

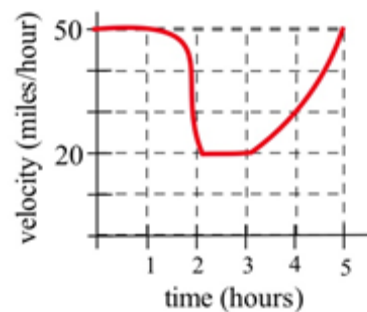
26.  $f(x) = x^2$  for  $0 \leq x \leq 2$ .



27. The graph shows the velocity of a car during a 5 hour trip.

(a) Estimate how far the car traveled during the 5 hours.

(b) At what **constant** velocity should you drive in order to travel the same distance in 5 hours?



28. The graph shows the number of telephone calls per minute at a large company. Estimate the average number of calls per minute

(a) From 8 am to 5 pm.

(b) From 9 am to 1 pm.

