Section 5: Additional Integration Techniques

Integration By Parts

Integration by parts is an integration method which enables us to find antiderivatives of some new functions such as $\ln(x)$ as well as antiderivatives of products of functions such as $x^2\ln(x)$ and xe^x.

If the function we're trying to integrate can be written as a product of two functions, u, and dv, then integration by parts lets us trade out a complicated integral for hopefully simpler one.

INTEGRATION BY PARTS FORMULA

$$\int u\,dv = uv - \int v\,du$$

For definite integrals,

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

Example 1

Integrate $\int xe^x dx$

To use the By Parts method, we break apart the product into two parts:
$u = x$ and $dv = e^x dx$

We now calculate du, the derivative of u, and v, the integral of dv.

$$du = \left(\frac{d}{dx}x\right)dx = 1dx \quad \text{and} \quad v = \int e^x dx = e^x.$$

Using the By Parts formula,

$$\int xe^x dx = uv - \int v\,du = xe^x - \int e^x dx$$

Notice the remaining integral is simpler that the original, and one we can easily evaluate.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

We could have chosen either x or e^x as our u in the last example, but had we chosen e^x, the second integral would have become messier, rather than simpler.

RULE OF THUMB

When selecting the u for By Parts, select a logarithmic expression if one is present. If not, select an algebraic expression (like x or dx).

Example 2

Integrate $\int_1^4 6x^2 \ln x \, dx$

Since this contains a logarithmic expression, we'll use it for our u.

$u = \ln x$ and $dv = 6x^2 dx$

We now calculate du and v.

$du = \frac{1}{x} dx$ and $v = \int 6x^2 dx = 6\frac{x^3}{3} = 2x^3$.

Using the By Parts formula,

$$\int_1^4 6x^2 \ln x \, dx = 2x^3 \ln x \Big]_1^4 - \int_1^4 6x^2 \frac{1}{x} dx$$

We can simplify the expression in the integral on the right:

$$\int_1^4 6x^2 \ln x \, dx = 2x^3 \ln x \Big]_1^4 - \int_1^4 6x dx$$

The remaining integral is a basic one we can now evaluate.

$$\int_1^4 6x^2 \ln x \, dx = 2x^3 \ln x \Big]_1^4 - 3x^2 \Big]_1^4$$

Finally, we can evaluate the expressions

$$\int_1^4 6x^2 \ln x \, dx = \left[\left(2 \cdot 4^3 \ln 4\right) - \left(2 \cdot 1^3 \ln 1\right)\right] - \left[(3 \cdot 4^2) - (3 \cdot 1^2)\right] = 128 \ln(4) - 45 \approx 132.446$$

Integration Using Tables of Integrals

There are many techniques of integration we will not be studying. Many of them lead to general formulas which can be compiled into a Table of Integrals - a type of cheat-sheet for integration.

For example, here are two entries you might find in a table of integrals:

TABLE OF INTEGRAL EXAMPLES

$$\int \frac{1}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$$

Example 3

Integrate $\int \frac{5}{x^2 - 9}dx$

This integral looks very similar to the form of the first integral in the examples table. By employing the rule that allows us to pull out constants, and by rewriting 9 as 3^2, we can better see the match.

$$\int \frac{5}{x^2 - 9}dx = 5\int \frac{1}{x^2 - 3^2}\,dx$$

Now we simply use the formula from the table, with $a = 3$.

$$\int \frac{5}{x^2 - 9}dx = 5\int \frac{1}{x^2 - 3^2}\,dx = 5\left(\frac{1}{2 \cdot 3}\ln\left|\frac{x-3}{x+3}\right|\right) = \frac{5}{6}\ln\left|\frac{x-3}{x+3}\right| + C$$

Sometimes we have to combine the table with other techniques we've learned, like substitution.

Example 4

Integrate $\int \frac{x^2}{\sqrt{x^6 + 16}}dx$

This integral looks somewhat like the second integral in the example table, but the power of x is incorrect, and there is an x^2 in the numerator which does not match. Trying to utilize this rule, we can try to rewrite the denominator to look like (something)^2. Luckily, $x^6 = \left(x^3\right)^2$

$$\int \frac{x^2}{\sqrt{x^6 + 16}}dx = \int \frac{x^2}{\sqrt{\left(x^3\right)^2 + 16}}dx$$

Now we can use substitution, letting $u = x^3$, so $du = 3x^2dx$.

Making the subsitution,

$$\int \frac{x^2}{\sqrt{\left(x^3\right)^2+16}}dx = \int \frac{1}{\sqrt{u^2+16}}\frac{du}{3} = \frac{1}{3}\int \frac{1}{\sqrt{u^2+16}}du$$

Now we can use the table entry.

$$\frac{1}{3}\int \frac{1}{\sqrt{u^2+16}}du = \frac{1}{3}\ln\left|u+\sqrt{u^2+16}\right|+C$$

Undoing the substitution,

$$\int \frac{x^2}{\sqrt{x^6+16}}dx = \frac{1}{3}\ln\left|x^3+\sqrt{x^6+16}\right|+C$$

3.5 Exericses

In problems 1–4, a function u or dv is given. Find the piece u or dv which is not given, calculate du and v, and apply the Integration by Parts Formula.

1. $\int 12x \cdot \ln(x)\, dx$ $u = \ln(x)$
2. $\int x \cdot e^{-x}\, dx$ $u = x$
3. $\int x^4 \ln(x)\, dx$ $dv = x^4\, dx$
4. $\int x \cdot (5x+1)^{19}\, dx$ $u = x$

In problems 5 - 10 evaluate the integrals

5. $\int_0^1 \frac{x}{e^{3x}}\, dx$
6. $\int_0^1 10x \cdot e^{3x}\, dx$
7. $\int_1^3 \ln(2x+5)\, dx$
8. $\int x^3 \ln(5x)\, dx$
9. $\int x \ln(x+1)\, dx$
10. $\int_1^2 \frac{\ln(x)}{x^2}\, dx$

For problems 11 - 14 integrate each function.

11. $\int \frac{1}{4-x^2}$
12. $\int \frac{2}{9-x^2}$
13. $\int \sqrt{4+x^2}$
14. $\int \sqrt{9+x^2}$