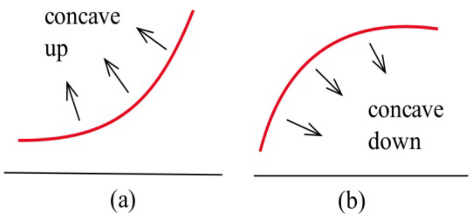
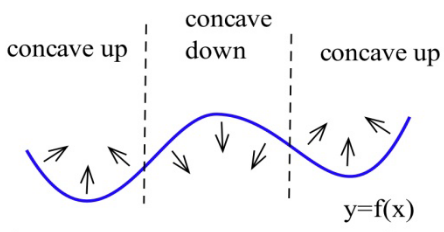
Section 6: Second Derivative and Concavity
Second Derivative and Concavity

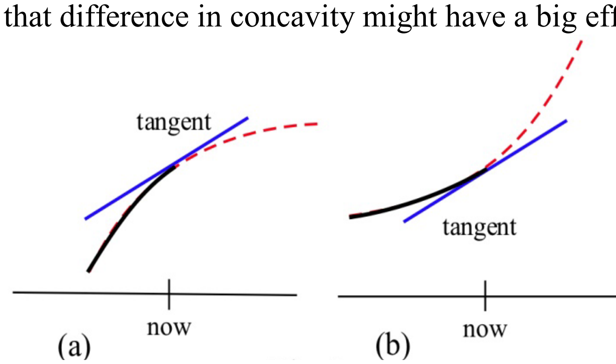
Graphically, a function is **concave up** if its graph is curved with the opening upward (a in the figure). Similarly, a function is **concave down** if its graph opens downward (b in the figure).



This figure shows the concavity of a function at several points. Notice that a function can be concave up regardless of whether it is increasing or decreasing.



For example, **An Epidemic:** Suppose an epidemic has started, and you, as a member of congress, must decide whether the current methods are effectively fighting the spread of the disease or whether more drastic measures and more money are needed. In the figure below, f(x) is the number of people who have the disease at time x, and two different situations are shown. In both (a) and (b), the number of people with the disease, f(now), and the rate at which new people are getting sick, f '(now), are the same. The difference in the two situations is the concavity of f, and that difference in concavity might have a big effect on your decision.



In (a), f is concave down at "now", the slopes are decreasing, and it looks as if it's tailing off. We can say "f is increasing at a decreasing rate." It appears that the current methods are starting to bring the epidemic under control.
In (b), f is concave up, the slopes are increasing, and it looks as if it will keep increasing faster and faster. It appears that the epidemic is still out of control.

The differences between the graphs come from whether the *derivative* is increasing or decreasing.

The derivative of a function f is a function that gives information about the slope of f. **The derivative tells us if the original function is increasing or decreasing.**

Because f ' is a function, we can take its derivative. This second derivative also gives us information about our original function f. The second derivative gives us a mathematical way to tell how the graph of a function is curved. **The second derivative tells us if the original function is concave up or down.**

Second Derivative Let $y = f(x)$
The **second derivative of f** is the derivative of $y' = f'(x)$.

Using prime notation, this is $f''(x)$ or y''. You can read this aloud as "y double prime."

Using Leibniz notation, the second derivative is written $\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$. This is read aloud as "the second derivative of f.

If $f''(x)$ is positive on an interval, the graph of $y = f(x)$ is **concave up** on that interval**.** We can say that *f* is increasing (or decreasing) **at an increasing rate**.
If $f''(x)$ is negative on an interval, the graph of $y = f(x)$ is **concave up** on that interval**.** We can say that *f* is increasing (or decreasing) **at a decreasing rate.**

Example 1

Find $f''(x)$ for $f(x) = 3x^7$

First, we need to find the first derivative:
$f'(x) = 21x^6$

Then we take the derivative of that function:
$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(21x^6) = 126x^5$$

If f(x) represents the position of a particle at time x, then v(x) = f '(x) will represent the velocity (rate of change of the position) of the particle and a(x) = v '(x) = f "(x) will represent the acceleration (the rate of change of the velocity) of the particle.

You are probably familiar with acceleration from driving or riding in a car. The speedometer tells you your velocity (speed). When you leave from a stop and press down on the accelerator, you are accelerating - increasing your speed.

Example 2

The height (feet) of a particle at time t seconds is $f(t) = t^3 - 4t^2 + 8t$. Find the height, velocity and acceleration of the particle when $t = 0, 1$, and 2 seconds.

$f(t) = t^3 - 4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet, and $f(2) = 8$ feet.

The velocity is $v(t) = f'(t) = 3t^2 - 8t + 8$ so $v(0) = 8$ ft/s, $v(1) = 3$ ft/s, and $v(2) = 4$ ft/s. At each of these times the velocity is positive and the particle is moving upward, increasing in height.

The acceleration is $a(t) = f''(t) = 6t - 8$ so $a(0) = -8$ ft/s^2, $a(1) = -2$ ft/s^2 and $a(2) = 4$ ft/s^2.

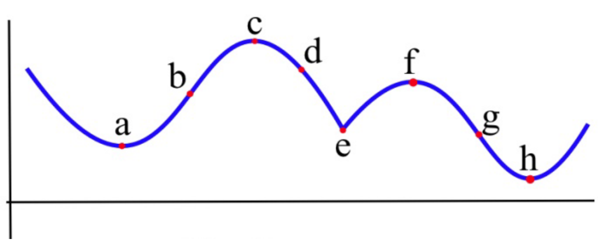
At time 0 and 1, the acceleration is negative, so the particle's velocity would be decreasing at those points - the particle was slowing down. At time 2, the velocity is positive, so the particle was increasing in speed.

Inflection Points

Definition: An **inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to down or from concave down to up.

Example 3

Which of the labeled points in the graph below are inflection points?



The concavity changes at points b and g. At points a and h, the graph is concave up on both sides, so the concavity does not change. At points c and f, the graph is concave down on both sides. At point e, even though the graph looks strange there, the graph is concave down on both sides – the concavity does not change.

Inflection points happen when the concavity changes. Because we know the connection between the concavity of a function and the sign of its second derivative, we can use this to find inflection points.

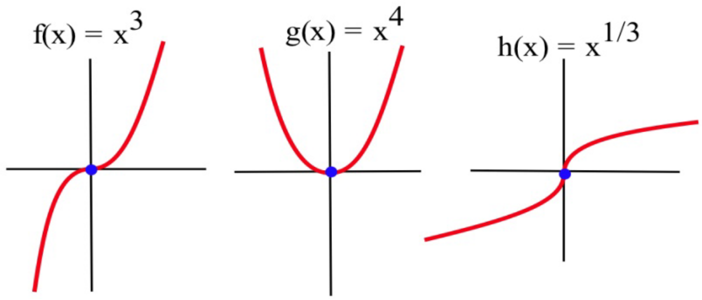
Working Definition: An **inflection point** is a point on the graph where the second derivative changes sign.

In order for the second derivative to change signs, it must either be zero or be undefined. So to find the inflection points of a function we only need to check the points where $f''(x)$ is 0 or undefined.

Note that it is not enough for the second derivative to be zero or undefined. We still need to check that the sign of f'' changes sign. The functions in the next example illustrate what can happen.

Example 4

Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$. For which of these functions is the point (0,0) an inflection point?



Graphically, it is clear that the concavity of $f(x) = x^3$ and $h(x) = x^{1/3}$ changes at (0,0), so (0,0) is an inflection point for f and h. The function $g(x) = x^4$ is concave up everywhere so (0,0) is not an inflection point of g.

We can also compute the second derivatives and check the sign change.

If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$. The only point at which $f''(x) = 0$ or is undefined (f' is not differentiable) is at $x = 0$. If $x < 0$, then $f''(x) < 0$ so f is concave down. If $x > 0$, then $f''(x) > 0$ so f is concave up. At $x = 0$ the concavity changes so the point $(0,f(0)) = (0,0)$ is an inflection point of x^3 .

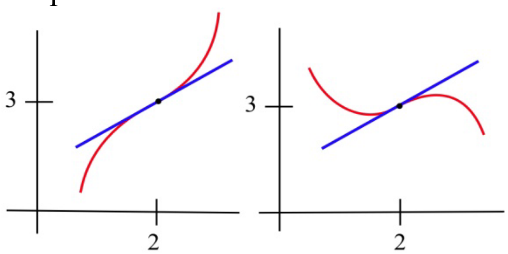
If $g(x) = x^4$, then $g'(x) = 4x^3$ and $g''(x) = 12x^2$. The only point at which $g''(x) = 0$ or is undefined is at $x = 0$. If $x < 0$, then $g''(x) > 0$ so g is concave up. If $x > 0$, then $g''(x) > 0$ so g is also concave up. At $x = 0$ the concavity **does not change** so the point $(0, g(0)) = (0,0)$ is **not an inflection point** of x^4 . Keep this example in mind!.

If $h(x) = x^{1/3}$, then $h'(x) = \frac{1}{3} x^{-2/3}$ and $h''(x) = -\frac{2}{9} x^{-5/3}$. h" is not defined if $x = 0$, but $h''(\text{negative number}) > 0$ and $h''(\text{positive number}) < 0$ so h changes concavity at (0,0) and (0,0) is an inflection point of h.

Example 5

Sketch the graph of a function with $f(2) = 3$, $f'(2) = 1$, and an inflection point at (2,3).

Two possible solutions are shown here.



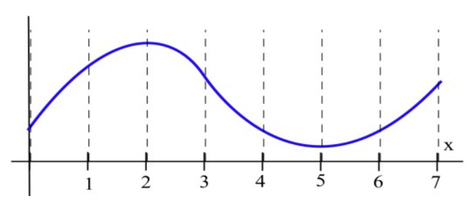
2.6 Exercises

In problems 1 and 2, each quotation is a statement about a quantity of something changing over time. Let f(t) represent the quantity at time t. For each quotation, tell what f represents and whether the first and second derivatives of f are positive or negative.

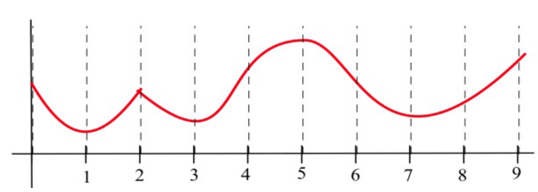
1. (a) "Unemployment rose again, but the rate of increase is smaller than last month."
   (b) "Our profits declined again, but at a slower rate than last month."
   (c) "The population is still rising and at a faster rate than last year."

2. (a) "The child's temperature is still rising, but slower than it was a few hours ago."
   (b) "The number of whales is decreasing, but at a slower rate than last year."
   (c) "The number of people with the flu is rising and at a faster rate than last month."

4. On which intervals is the function in the graph
   (a) concave up? (b) concave down?



5. On which intervals is the function in graph
   (a) concave up? (b) concave down?



6. Sketch the graphs of functions which are defined and concave up everywhere and which have (a) no roots. (b) exactly 1 root. (c) exactly 2 roots. (d) exactly 3 roots.

In problems 7 – 10, a function and values of x so that $f'(x) = 0$ are given. Use the Second Derivative Test to determine whether each point (x, f(x)) is a local maximum, a local minimum or neither

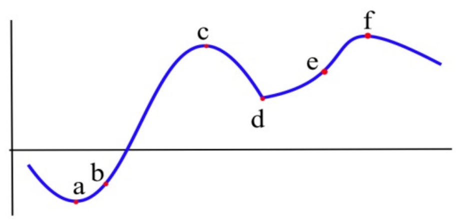
7. $f(x) = 2x^3 - 15x^2 + 6$, $x = 0, 5$.

8. $g(x) = x^3 - 3x^2 - 9x + 7$, $x = -1, 3$.

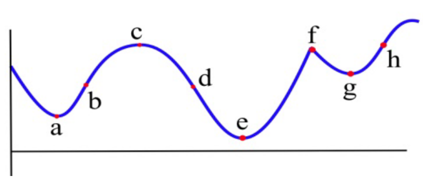
9. $h(x) = x^4 - 8x^2 - 2$, $x = -2, 0, 2$.

10. $f(x) = x \cdot \ln(x)$, $x = 1/e$.

11. Which of the labeled points in the graph are inflection points?



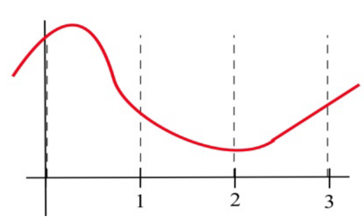
12. Which of the labeled points in the graph are inflection points?



13. How many inflection points can a (a) quadratic polynomial have? (b) cubic polynomial have? (c) polynomial of degree n have?

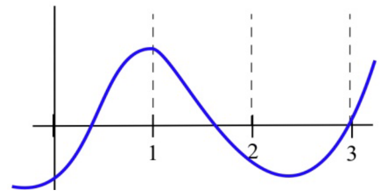
14. Fill in the table with "+", "–", or "0" for the function shown.

x	f(x)	f '(x)	f "(x)
0			
1			
2			
3			



15. Fill in the table with "+", "–", or "0" for the function shown.

x	g(x)	g '(x)	g "(x)
0			
1			
2			
3			



In problems 16 – 22 , find the derivative and second derivative of each function.

16. $f(x) = 7x^2 + 5x - 3$
17. $f(x) = (2x - 8)^5$
18. $f(x) = (6x - x^2)^{10}$
19. $f(x) = x \cdot (3x + 7)^5$
20. $f(x) = (2x^3 + 3)^6$
21. $f(x) = \sqrt{x^2 + 6x - 1}$
22. $f(x) = \ln(x^2 + 4)$