Section 11: Implicit Differentiation and Related Rates

In our work up until now, the functions we needed to differentiate were either given **explicitly,** such as $y = x^2 + e^x$, or it was possible to get an explicit formula for them, such as solving $y^3 - 3x^2 = 5$ to get $y = \sqrt[3]{5+3x^2}$. Sometimes, however, we will have an equation relating x and y which is either difficult or impossible to solve explicitly for y, such as $y + e^y = x^2$. In any case, we can still find y ' = f '(x) by using implicit differentiation.

The key idea behind implicit differentiation is to **assume that y is a function of x** even if we cannot explicitly solve for y. This assumption does not require any work, but we need to be very careful to treat y as a function when we differentiate and to use the Chain Rule.

Example 1

Assume that y is a function of x. Calculate

(a) $\frac{d}{dx}\left(y^3\right)$ (b) $\frac{d}{dx}\left(x^3y^2\right)$ and (c) $\frac{d}{dx}\ln(y)$

(a) We need the chain rule since y is a function of x:

$$\frac{d}{dx}\left(y^3\right) = 3y^2\frac{d}{dx}(y) = 3y^2y'$$

(b) We need to use the product rule and the Chain Rule:

$$\frac{d}{dx}\left(x^3y^2\right) = x^3\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^3) = x^3 2y\frac{dy}{dx} + y^2 3x^2 = 2x^3 2yy' + 3y^2x^2$$

(c) We know $\frac{d}{dx}\ln(x) = \frac{1}{x}$, so we use that and the Chain Rule:

$$\frac{d}{dx}\ln(y) = \frac{1}{y}\cdot y'$$

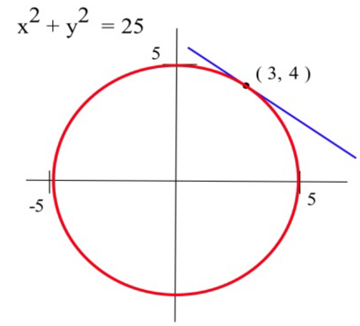
IMPLICIT DIFFERENTIATION:

To determine **y '** , differentiate each side of the defining equation, **treating y as a function of x** , and then algebraically solve for y '.

Example 2

Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3,4) using implicit differentiation.

We differentiate each side of the equation $x^2 + y^2 = 25$ and then solve for **y'**



$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2yy' = 0$$

Solving for **y'**, we have $y' = -\frac{2x}{2y} = -\frac{x}{y}$, and, at the point (3,4), **y'** = – 3/4.

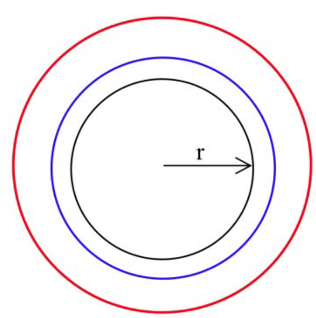
In the previous example, it would have been easy to explicitly solve for y, and then we could differentiate y to get y '. Because we could explicitly solve for y , we had a choice of methods for calculating y '. Sometimes, however, we can not explicitly solve for y , and the only way of determining y ' is implicit differentiation.

Related Rates

If several variables or quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other variables must be changing.

Example 3

Suppose the border of a town is roughly circular, and the radius of that circle has been increasing at a rate of 0.1 miles each year. Find how fast the area of the town has been increasing when the radius is 5 miles.



We could get an <u>approximate</u> answer by calculating the area of the circle when the radius is 5 miles ($A = \pi r^2 = \pi(5 \text{ miles})^2 \approx 78.6$ miles^2) and 1 year later when the radius is 0.1 feet larger than before ($A = \pi r^2 = \pi(5.1 \text{ miles})^2 \approx 81.7$ miles^2) and then finding ΔArea/Δtime = (81.7 mi^2 – 78.6 mi^2)/(1 year) = 3.1 mi^2/yr. This approximate answer represents the average change in area during the 1 year period when the radius increased from 5 miles to 5.1 miles, and would correspond to the secant slope on the area graph.

To find the exact answer, though, we need derivatives. In this case both radius and area are functions of time:

r(t) = radius at time *t* *A(t)* = area at time *t*

We know how fast the radius is changing, which is a statement about the derivative: $\frac{dr}{dt}=0.1\frac{mile}{year}$. We also know that $r = 5$ at our moment of interest.

We are looking for how fast the area is increasing, which is $\frac{dA}{dt}$.

Now we need an equation relating our variables, which is the area equation: $A=\pi r^2$.

Taking the derivative of both sides of that equation with respect to t, we can use implicit differentiation:

$$\frac{d}{dt}(A)=\frac{d}{dt}\left(\pi r^2\right)$$

$$\frac{dA}{dt}=\pi 2r\frac{dr}{dt}$$

Plugging in the values we know for r and dr/dt,

$$\frac{dA}{dt}=\pi 2(5\,miles)\left(.1\frac{miles}{year}\right)=3.14\frac{miles^2}{year}$$

The area of the town is increasing by 3.14 square miles per year when the radius is 5 miles.

Related Rates
When working with a related rates problem,
1. Identify the quantities that are changing, and assign them variables
2. Find an equation that relates those quantities
3. Differentiate both sides of that equation with respect to time
4. Plug in any known values for the variables or rates of change
5. Solve for the desired rate.

Example 4

A company has determined the demand curve for their product is $q=\sqrt{5000-p^2}$, where p is the price in dollars, and q is the quantity in millions. If weather conditions are driving the price up \$2 a week, find the rate at which demand is changing when the price is \$40.

The quantities changing are p and q, and we assume they are both functions of time, t, in weeks. We already have an equation relating the quantities, so we can implicitly differentiate it.

$$\frac{d}{dt}(q)=\frac{d}{dt}\left(\sqrt{5000-p^2}\right)$$

$$\frac{d}{dt}(q)=\frac{d}{dt}\left(5000-p^2\right)^{1/2}$$
$$\frac{dq}{dt}=\frac{1}{2}\left(5000-p^2\right)^{-1/2}\frac{d}{dt}\left(5000-p^2\right)$$
$$\frac{dq}{dt}=\frac{1}{2}\left(5000-p^2\right)^{-1/2}\left(-2p\frac{dp}{dt}\right)$$

Using the given information, we know the price is increasing by \$2/week when the price is \$40, giving $\frac{dp}{dt}=2$ when $p = 40$. Plugging in these values,

$$\frac{dq}{dt}=\frac{1}{2}\left(5000-40^2\right)^{-1/2}\left(-2\cdot 40\cdot 2\right)\approx -1.37$$

Demand is falling by 1.37 million items per week.

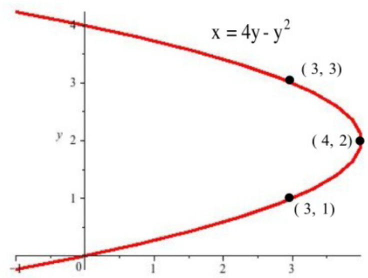
2.11 Exercises

In problems 1 – 10 find dy/dx by differentiating implicitly then find the value of dy/dx at the given point.

1. $x^2 + y^2 = 100$, point (6, 8)
2. $x^2 + 5y^2 = 45$, point (5, 2)
3. $x^2 - 3xy + 7y = 5$, point (2,1)
4. $\sqrt{x} + \sqrt{y} = 5$, point (4,9)
5. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point (0,4)
6. $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point (3,0)
7. $\ln(y) + 3x - 7 = 0$, point (2,e)
8. $x^2 - y^2 = 16$, point (5,3)
9. $x^2 - y^2 = 16$, point (5, –3)
10. $y^2 + 7x^3 - 3x = 8$, point (1,2)

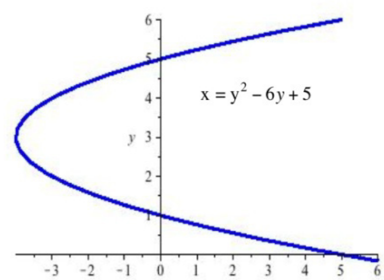
11. Find the slopes of the lines tangent to the graph in shown at the points (3,1), (3,3), and (4,2) .

12. Find the slopes of the lines tangent to the graph in shown where the graph crosses the y–axis.



13. Find the slopes of the lines tangent to the graph in graph shown at the points ((5,0), (5,6), and (–4,3).

14. Find the slopes of the lines tangent to the graph in the graph shown where the graph crosses the y–axis.

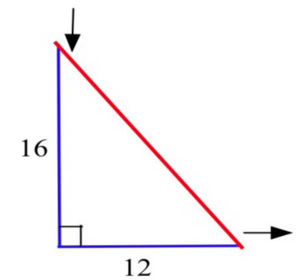


In problems 15 – 16 , find dy/dx using implicit differentiation and then find the slope of the line tangent to the graph of the equation at the given point.

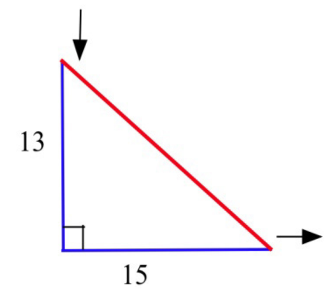
15. $y^3 - 5y = 5x^2 + 7$, point (1,3)
16. $y^2 - 5xy + x^2 + 21 = 0$, point (2,5)

17. An expandable sphere is being filled with liquid at a constant rate from a tap (imagine a water balloon connected to a faucet). When the radius of the sphere is 3 inches, the radius is increasing at 2 inches per minute. How fast is the liquid coming out of the tap? ($V = \frac{4}{3}\pi r^3$)

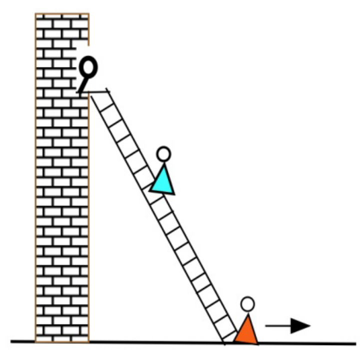
18. The 12 inch base of a right triangle is growing at 3 inches per hour, and the 16 inch height is shrinking at 3 inches per hour.
    (a) Is the area increasing or decreasing?
    (b) Is the perimeter increasing or decreasing?
    (c) Is the hypotenuse increasing or decreasing?



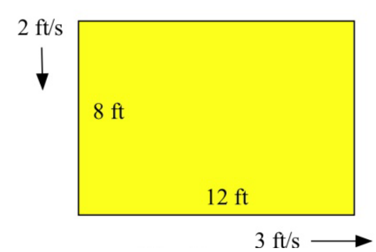
19. One hour later the right triangle in Problem 2 is 15 inches long and 13 inches high, and the base and height are changing at the same rate as in Problem 18.
    (a) Is the area increasing or decreasing now?
    (b) Is the hypotenuse increasing or decreasing now?
    (c) Is the perimeter increasing or decreasing now?



20. A young woman and her boyfriend plan to elope, but she must rescue him from his mother who has locked him in his room. The young woman has placed a 20 foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second. How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is
(a) 12 feet from the bottom of the wall?
(b) 16 feet from the bottom of the wall?
(c) 19 feet from the bottom of the wall?



21. The length of a 12 foot by 8 foot rectangle is increasing at a rate of 3 feet per second and the width is decreasing at 2 feet per second .
(a) How fast is the perimeter changing?
(b) How fast is the area changing?



22. An oil tanker in Puget Sound has sprung a leak, and a circular oil slick is forming. The oil slick is 4 inches thick everywhere, is 100 feet in diameter, and the diameter is increasing at 12 feet per hour. Your job, as the Coast Guard commander or the tanker's captain, is to determine how fast the oil is leaking from the tanker.