## Section 5: Quadratics

Quadratics

Quadratics are transformations of the $f(x) = x^2$ function. Quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4th side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the 4th side or backyard fence.
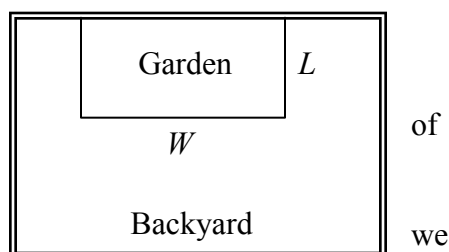
Since we know we only have 80 feet of fence available, we know that $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L: $W = 80 - 2L$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so
$$A = LW = L(80 - 2L)$$
$$A(L) = 80L - 2L^2$$
This formula represents the area of the fence in terms of the variable length L.

> **Forms of Quadratic Functions**
>
> The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$
>
> The **transformation form** of a quadratic function is $f(x) = a(x - h)^2 + k$
>
> The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

Example 2

Write an equation for the quadratic graphed below as a transformation of $f(x) = x^2$.
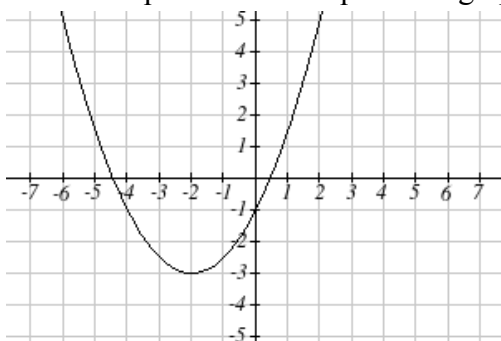
We can see the graph is the basic quadratic shifted to the left 2 and down 3, putting the vertex at (-2, 3), giving a formula in the form $g(x) = a(x+2)^2 - 3$. By plugging in a point that falls on the grid, such as (0,-1), we can solve for the stretch factor:

$-1 = a(0+2)^2 - 3$
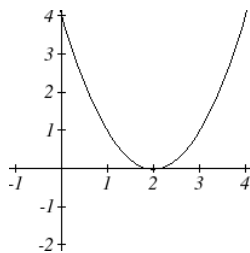
$2 = 4a$

$a = \dfrac{1}{2}$

The equation for this formula is $g(x) = \dfrac{1}{2}(x+2)^2 - 3$.
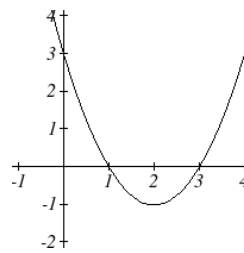
Short run Behavior: Intercepts

As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.
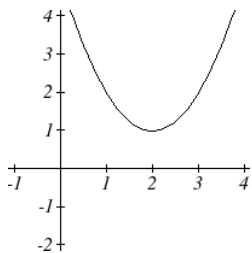
zero horizontal intercepts   one horizontal intercept   two horizontal intercepts

Notice that in the standard form of a quadratic, the constant term c reveals the vertical intercept of the graph, since $f(0) = a(0)^2 + b(0) + c = c$.

Example 3

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

We can find the vertical intercept by evaluating the function at an input of zero:
$f(0) = 3(0)^2 + 5(0) - 2 = -2$ Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero
$0 = 3x^2 + 5x - 2$

In this case, the quadratic can be factored easily, providing the simplest method for solution
$0 = (3x - 1)(x + 2)$
$0 = 3x - 1$
$ = x + 2$ wait

$0 = 3x - 1$
$x = \dfrac{1}{3}$ $\quad$ or $\quad$ $0 = x + 2$
$\phantom{x = \dfrac{1}{3}} \quad \quad x = -2$
$\qquad$ Horizontal intercepts at $\left(\dfrac{1}{3}, 0\right)$ and (-2,0)

When a quadratic is not factorable or is hard to factor, we can turn to the quadratic formula.

Quadratic Formula
For a quadratic function given in standard form $f(x) = ax^2 + bx + c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function.
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 4

A ball is thrown upwards from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation
$H(t) = -16t^2 + 80t + 40$. When does the ball hit the ground?

To find when the ball hits the ground, we need to determine when the height is zero – when
$H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:
$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$
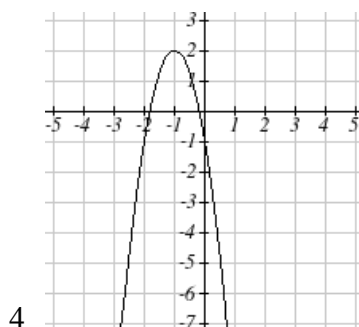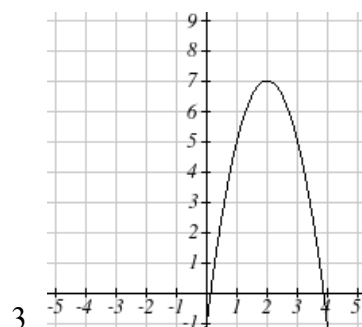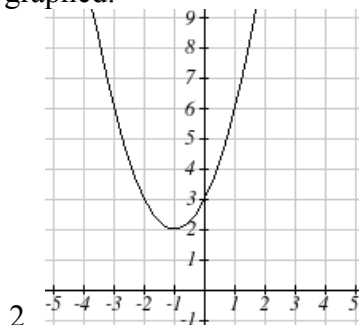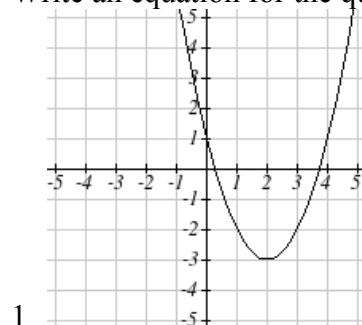
Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:
$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \qquad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$
The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

1.5 Exercises

Write an equation for the quadratic function graphed.

1.

2.

3.

4.

5.

6.
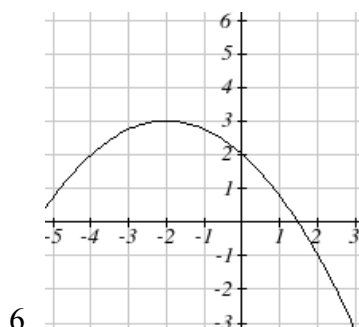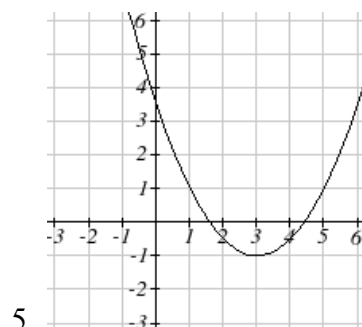
For each of the follow quadratic functions, find the vertical and horizontal intercepts.

7. $y(x) = 2x^2 + 10x + 12$ 8. $z(p) = 3x^2 + 6x - 9$

9. $f(x) = 2x^2 - 10x + 4$ 10. $g(x) = -2x^2 - 14x + 12$

11. $h(t) = -4t^2 + 6t - 1$ 12. $k(t) = 2x^2 + 4x - 15$

13. The height of a ball thrown in the air is given by $h(x) = -\dfrac{1}{12}x^2 + 6x + 3$, where x is the

   horizontal distance in feet from the point at which the ball is thrown.

   a. How high is the ball when it was thrown?
   b. What is the maximum height of the ball?
   c. How far from the thrower does the ball strike the ground?

14. A javelin is thrown in the air. Its height is given by $h(x) = -\dfrac{1}{20}x^2 + 8x + 6$, where x is the

   horizontal distance in feet from the point at which the javelin is thrown.

   a. How high is the javelin when it was thrown?
   b. What is the maximum height of the javelin?
   c. How far from the thrower does the javelin strike the ground?